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ERGODIC THEOREMS FOR NON-LIPSCHITZIAN SEMIGROUPS WITHOUT CONVEXITY***

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Abstract

Let G be a semitopological semigroup. Let C be a nonempty subset of a Hilbert space and $\Im = \{T_t : t \in G\}$ be a representation of G as asymptotically nonexpansive type mappings of C into itself such that the common fixed point set $F(\Im)$ of $\Im$ in C is nonempty. It is proved that $\bigcap\limits_{s\in G} \overline{\text{co}}\{T_{ts}x : t \in G\} \bigcap F(\Im)$ is nonempty for each $x \in C$ if and only if there exists a nonexpansive retraction P of C onto $F(\Im)$ such that $PT_s = T_sP = P$ for all $s \in G$ and $P(x)$ is in the closed convex hull of $\{T_sx : s \in G\}$, $x \in C$. This result shows that many key conditions in [1–4, 9, 12–15] are not necessary.

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§1. Introduction

Let H be a Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$ and let C be a nonempty subset of H. A mapping $T : C \mapsto C$ is said to be Lipschitzian if there exists a nonnegative number k such that

$$\|Tx - Ty\| \le k\|x - y\| \text{ for every } x, y \in C \ ,$$

and nonexpansive in the case $k = 1$. Provided that C is closed and convex, Baillon[1] proved the first mean ergodic theorem for nonexpansive mappings in a Hilbere space : Let C be a nonempty convex closed subset of H, and let T be a nonexpansive mapping of C into itself and suppose that the set $F(T)$ of fixed points of T is nonempty, then the Cesàro means

$$s_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly as $n \to \infty$ to a fixed point y of T for each $x \in C$. In this case, by putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$. The analogous results are given for nonexpansive semigroups on C by Baillon[2] and Hirano and Takahashi[4]. Recently, in [9], Mizoguchi and Takahashi proved a nonlinear ergodic retraction theorem for Lipschitzian semigroups by using the notion of submean. However, it still remains open

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whether Baillon's theorem is valid without convexity. In 1992, Takahashi[15] proved the ergodic theorem for nonexpansive semigroups on condition that $\bigcap_{s\in G} \overline{\text{co}}\{T_{st}x : t \in G\} \subset C$ for some $x \in C$. This provided a partly affirmative answer to this question.

Without using the concept of submean, this paper attempts to prove the nonlinear ergodic theorem for semitopological semigroup of non-lipschitzian mappings without convexity in a Hilbert space. We prove that if C is a nonempty subset of H, G is a semitopological semigroup and $\Im = \{T_t : t \in G\}$ is a representation of G as asymptotically nonexpansive type mappings of C into itself such that the common fixed point set $F(\Im)$ of $\Im$ in C is nonempty, then $\bigcap_{s\in G} \overline{\text{co}}\{T_{ts}x : t \in G\} \bigcap F(\Im)$ is nonempty for each $x \in C$ if and only if there exists a nonexpansive retraction P of C onto $F(\Im)$ such that $PT_s = T_sP = P$ for all $s \in G$ and $P(x)$ is in the closed convex hull of $\{T_sx : s \in G\}$. This result seems to be new even if C is a convex closed subset. Our results generalize the previously known results of Brezis[1−2], Hirano and Takahashi[4], Mizoguchi and Takahashi[9], Takahashi and Zhang[12] and Takahashi[13−15]. Further, it would be safe to say that in their results[1−4,9,12−15] many key conditions are not necessary.

§2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space and G is a semitopological semigroup, i.e., a semigroup with a Hausdorff topology such that for each $s \in G$ the mappings $s \mapsto s \cdot t$ and $s \mapsto t \cdot s$ of G into itself are continuous. Let C be a nonempty subset of H, and let $\Im = \{T_t : t \in G\}$ be a semigroup on C, i.e. , $T_{ab}(x) = T_aT_b(x)$ for all $a, b \in G$ and $x \in C$.

Recall that $\Im$ is said to be

(a) nonexpansive if $\|T_tx - T_ty\| \le \|x - y\|$ for x, y in C and $t \in G$;

(b) asymptotically nonexpansive[6] if there exists a function $K : G \mapsto [0, \infty)$ with $\inf_{s\in G} \sup_{t\in G} K_{ts} \le 1$ such that $\|T_tx - T_ty\| \le K_t\|x - y\|$ for x, y in C and t in G;

(c) asymptotically nonexpansive type[6] if for each x in C, there is a function $r(\cdot, x) : G \mapsto [0, \infty)$ with $\inf_{s\in G} \sup_{t\in G} r(ts, x) = 0$ such that $\|T_tx - T_ty\| \le \|x - y\| + r(t, x)$ for all y in C and t in G.

It is easily seen that (a)⇒(b)⇒(c) and that both the inclusions are proper (cf. [6, p. 112]).

We denote by $F(\Im)$ the set $\{x \in C : T_s(x) = x \text{ for all } s \in G\}$ of common fixed points of $\Im$.

§3. Nonexpansive Retraction

In this section we shall prove erogodic retraction theorem for semitopoligical semigroup of asymptotically nonexpansive type mappings without convexity. We begin with the following theorem.

Theorem 3.1. *Let G be a semitopological semigroup, and let C be a nonempty subset of a Hilbert space H , and let $\Im = \{T_t : t \in G\}$ be a semitopological semigroup of asymptotically* *nonexpansive type mappings on C. Then for every $x \in C$ the set*

$$\bigcap_{s\in G} \overline{\text{co}}\{T_{ts}x : t \in G\} \bigcap F(\Im)$$

consists of at most one point.

Proof. Suppose $f \in F(\Im)$. For an arbitrary $\epsilon > 0$, since $\Im$ is asymptotically nonexpansive type, there exists $s_0 \in G$ such that for all $t \in G$, $r(ts_0, f) < \epsilon$, and hence for $a \in G$,

$$\inf_{s\in G}\sup_{t\in G} \|T_{ts}x - f\|^2 \le \sup_{t\in G} \|T_{ts_0a}x - f\|^2 \le \sup_{t\in G}(\|T_a x - f\| + r(ts_0, f))^2 \le (\|T_a x - f\| + \epsilon)^2.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\inf_{s\in G}\sup_{t\in G} \|T_{ts}x - f\|^2 \le \inf_{a\in G} \|T_a - f\|^2. \tag{3.1}$$

Let $u, v \in \bigcap_{s\in G} \overline{\text{co}}\{T_{ts}x : t \in G\} \bigcap F(\Im)$. Without loss of generality, we assume that

$$\inf_{t\in G} \|T_t x - u\|^2 \le \inf_{t\in G} \|T_t x - v\|^2.$$

Now, for each $t, s \in G$, since

$$\|u - v\|^2 + 2(T_{ts}x - u, u - v) = \|T_{ts}x - v\|^2 - \|T_{ts}x - u\|^2,$$

we have

$$\begin{aligned}\|u - v\|^2 + 2 \inf_{t\in G}(T_{ts}x - u, u - v) &\ge \inf_{t\in G} \|T_{ts}x - v\|^2 - \sup_{t\in G} \|T_{ts}x - u\|^2 \\ &\ge \inf_{t\in G} \|T_t x - v\|^2 - \sup_{t\in G} \|T_{ts}x - u\|^2.\end{aligned}$$

By (3.1), we have

$$\begin{aligned}\|u - v\|^2 + 2 \sup_{s\in G}\inf_{t\in G}(T_{ts}x - u, u - v) &\ge \inf_{t\in G} \|T_t x - v\|^2 - \inf_{s\in G}\sup_{t\in G} \|T_{ts}x - u\|^2 \\ &\ge \inf_{t\in G} \|T_{ts}x - v\|^2 - \inf_{t\in G} \|T_{ts}x - u\|^2 \ge 0.\end{aligned}$$

Let $\epsilon > 0$. Then there is $s_1 \in G$ such that $\|u - v\|^2 + 2(T_{ts_1}x - u, u - v) > -\epsilon$ for all $t \in G$. From $v \in \overline{\text{co}}\{T_{ts_1}x : t \in G\}$, we have

$$\|u - v\|^2 + 2(v - u, u - v) \ge -\epsilon.$$

This inequality implies $\|u - v\|^2 \le \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $u = v$. This completes the proof.

Remark 3.1. In the result of Takahashi and Zhang[12], it is assumed that C is a convex closed subset, G is a reversible semigroup and $\Im$ is asymptotically nonexpansive. The above theorem shows that those key conditions are not necessary.

By using Theorem 3.1, we now prove the ergodic retraction theorem.

Theorem 3.2. *Let C be a nonempty subset of a Hilbert space H, and let $\Im = \{T_t : t \in G\}$ be a semitopological semigroup of asymptotically nonexpansive type mappings on C, $F(\Im) \ne \emptyset$. Then the following are equivalent.*

(a) $\bigcap_{s\in G} \overline{\text{co}}\{T_{ts}x : t \in G\} \bigcap F(\Im) \ne \emptyset$ *for each* $x \in C$.

(b) *There is a nonexpansive retraction P of C onto $F(\Im)$ such that $PT_t = T_t P = P$ for every $t \in G$ and $Px \in \overline{\text{co}}\{T_t x : t \in G\}$ for every $x \in C$.*

Proof. (b)⇒(a): Let $x \in C$. Then $Px \in F(\Im)$. Also $Px \in \bigcap_{s\in G} \overline{\text{co}}\{T_t x : t \in G\}$. In fact, for each $s \in G$, $Px = PT_s x \in \overline{\text{co}}\{T_t T_s x : t \in G\} = \overline{\text{co}}\{T_{ts}x : t \in G\}$.

(a)⇒(b). Let $x \in C$. Then by Theorem 3.1 and (a), $\bigcap_{s\in G} \overline{\text{co}}\{T_{ts}x : t \in G\} \bigcap F(\Im)$ contains exactly one point Px. For each $a \in G$, we have

$$\{PT_a x\} = \bigcap_{s\in G} \overline{\text{co}}\{T_{tsa}x : t \in G\} \bigcap F(\Im) \sqsupseteq \bigcap_{s\in G} \overline{\text{co}}\{T_{ts}x : t \in G\} \bigcap F(\Im) = \{Px\},$$

and hence we have $PT_a = P$ for every $a \in G$. It is clear that $T_s P = P$ for every $s \in G$.

Finally, we show that P is nonexpansive. Let $x \in C, v \in F(\Im)$ and $0 < \lambda < 1$. Then we have, for each $t, s, a, b \in G$,

$$\begin{aligned} &\|\lambda T_{tsab}x + (1-\lambda)Px - v\|^2 \\ &= \|\lambda(T_{tsab}x - v) + (1-\lambda)(Px - v)\|^2 \\ &= \lambda\|T_{tsab}x - v\|^2 + (1-\lambda)\|Px - v\|^2 - \lambda(1-\lambda)\|T_{tsab}x - Px\|^2 \\ &\le \lambda(\|T_{ab}x - v\| + r(ts, x))^2 + (1-\lambda)\|Px - v\|^2 - \lambda(1-\lambda)\|T_{tsab}x - Px\|^2 \\ &\le \lambda(\|T_{ab}x - v\| + r(ts, x))^2 + (1-\lambda)\|Px - v\|^2 - \lambda(1-\lambda)\inf_{t\in G}\|T_t x - Px\|^2. \end{aligned}$$

Since $\Im$ is an asymptotically nonexpansive type, it follows that

$$\begin{aligned} &\inf_{s\in G}\sup_{t\in G}\|\lambda T_{ts}x + (1-\lambda)Px - v\|^2 \\ &\le \inf_{s\in G}\sup_{t\in G}\|\lambda T_{tsab}x + (1-\lambda)Px - v\|^2 \\ &\le \lambda\|T_{ab}x - v\|^2 + (1-\lambda)\|Px - v\|^2 - \lambda(1-\lambda)\inf_{t\in G}\|T_t x - Px\|^2 \\ &= \|\lambda T_{ab}x + (1-\lambda)Px - v\|^2 + \lambda(1-\lambda)\|T_{ab}x - Px\|^2 - \lambda(1-\lambda)\inf_{t\in G}\|T_t x - Px\|^2. \end{aligned}$$

It is then easily seen that

$$\begin{aligned} &\inf_{s\in G}\sup_{t\in G}\|\lambda T_{ts}x + (1-\lambda)Px - v\|^2 - \lambda(1-\lambda)\inf_{b\in G}\sup_{a\in G}\|T_{ab}x - Px\|^2 \\ &\le \sup_{b\in G}\inf_{a\in G}\|\lambda T_{ab}x + (1-\lambda)Px - v\|^2 - \lambda(1-\lambda)\inf_{t\in G}\|T_t x - Px\|^2. \end{aligned}$$

From (3.1), we have

$$\inf_{s\in G}\sup_{t\in G}\|\lambda T_{ts}x + (1-\lambda)Px - v\|^2 \le \sup_{s\in G}\inf_{t\in G}\|\lambda T_{ts}x + (1-\lambda)Px - v\|^2. \tag{3.2}$$

Let

$$h(\lambda) = \inf_{s\in G}\sup_{t\in G}\|\lambda T_{ts}x + (1-\lambda)Px - v\|^2.$$

Then for any $\epsilon > 0$, there exists $s_0 \in G$ such that for all $t \in G$,

$$\|\lambda T_{ts_0}x + (1-\lambda)Px - v\|^2 \le h(\lambda) + \epsilon,$$

and hence

$$(\lambda T_{ts_0}x + (1-\lambda)Px - v, Px - v) \le (h(\lambda) + \epsilon)^{\frac{1}{2}}\|Px - v\| \quad \text{for all } t \in G.$$

From $Px \in \overline{\text{co}}\{T_{ts_0}x : t \in G\}$, we have

$$(\lambda Px + (1-\lambda)Px - v, Px - v) \le (h(\lambda) + \epsilon)^{\frac{1}{2}}\|Px - v\|.$$

Since $\epsilon > 0$ is arbitrary, this yields $\|Px - v\|^2 \le h(\lambda)$, that is,

$$\|Px - v\|^2 \le \inf_{s\in G}\sup_{t\in G}\|\lambda T_{ts}x + (1-\lambda)Px - v\|^2. \tag{3.3}$$

Now one can choose an $s_1 \in G$ such that $r(ts_1, x) < 1$ for all $t \in G$. Put $M = 1 + \|x - Px\|$. Then $\|T_{ts_1}x - Px\| \le M$ for all $t \in G$. It thus follows that

$$\begin{aligned} &\|\lambda T_{tss_1}x + (1-\lambda)Px - v\|^2 = \|\lambda(T_{tss_1}x - Px) + (Px - v)\|^2 \\ &= \lambda^2\|T_{tss_1}x - Px\|^2 + \|Px - v\|^2 + 2\lambda(T_{tss_1}x - Px, Px - v) \\ &\le M^2\lambda^2 + \|Px - v\|^2 + 2\lambda(T_{tss_1}x - Px, Px - v). \end{aligned}$$

From (3.2) and (3.3),

$$\begin{aligned} &2\lambda \sup_{s\in G}\inf_{t\in G}(T_{ts}x - Px, Px - v) \\ &\ge 2\lambda \sup_{s\in G}\inf_{t\in G}(T_{tss_1}x - Px, Px - v) \\ &\ge \sup_{s\in G}\inf_{t\in G}\|\lambda T_{tss_1}x + (1-\lambda)Px - v\|^2 - \|Px - v\|^2 - M^2\lambda^2 \\ &= \sup_{s\in G}\inf_{t\in G}\|\lambda T_{ts}T_{s_1}x + (1-\lambda)PT_{s_1}x - v\|^2 - \|Px - v\|^2 - M^2\lambda^2 \\ &\ge \|PT_{s_1}x - v\|^2 - \|Px - v\|^2 - M^2\lambda^2 = -M^2\lambda^2, \end{aligned}$$

and hence

$$\sup_{s\in G}\inf_{t\in G}(T_{ts}x - Px, Px - v) \ge -\frac{1}{2}M^2\lambda.$$

Letting $\lambda \to 0$, we have

$$\sup_{s\in G}\inf_{t\in G}(T_{ts}x - Px, Px - v) \ge 0.$$

Therefore, for $y \in C$, we have

$$\sup_{s\in G}\inf_{t\in G}(T_{ts}x - Px, Px - Py) \ge 0. \tag{3.4}$$

Let $\epsilon > 0$. Then there is $s_2 \in G$ such that $r(ts_2, x) < \epsilon$ for all $t \in G$. For such an $s_2 \in G$, from (3.4), we have

$$\sup_{s\in G}\inf_{t\in G}(T_{ts}T_{s_2}x - PT_{s_2}x, PT_{s_2}x - Py) \ge 0,$$

and hence there is $s_3 \in G$ such that

$$\inf_{t\in G}(T_{ts_3}T_{s_2}x - PT_{s_2}x, PT_{s_2}x - Py) > -\epsilon.$$

Then, from $PT_{s_2}x = Px$, we have

$$\inf_{t\in G}(T_{ts_3s_2}x - Px, Px - Py) > -\epsilon. \tag{3.5}$$

Similarly, from (3.4), we also have

$$\sup_{s\in G}\inf_{t\in G}(T_{ts}T_{s_3s_2}y - PT_{s_3s_2}y, PT_{s_3s_2}y - Px) \ge 0,$$

and there exists $s_4 \in G$ such that

$$\inf_{t\in G}(T_{ts_4s_3s_2}y - PT_{s_4s_3s_2}y, PT_{s_4s_3s_2}y - Px) \ge -\epsilon,$$

that is,

$$\inf_{t\in G}(Py - T_{ts_4s_3s_2}y, Px - Py) \ge -\epsilon. \tag{3.6}$$

On the other hand, from (3.5)

$$\inf_{t\in G}(T_{ts_4s_3s_2}x - Px, Px - Py) > -\epsilon. \tag{3.7}$$

Combining (3.6) and (3.7), we have

$$\begin{aligned}-2\epsilon &< (T_{ts_4s_3s_2}x - T_{ts_4s_3s_2}y, Px - Py) - \|Px - Py\|^2 \\ &\le \|T_{ts_4s_3s_2}x - T_{ts_4s_3s_2}y\| \cdot \|Px - Py\| - \|Px - Py\|^2 \\ &\le (r(ts_4s_3s_2, x) + \|x - y\|) \cdot \|Px - Py\| - \|Px - Py\|^2 \\ &\le (\epsilon + \|x - y\|) \cdot \|Px - Py\| - \|Px - Py\|^2.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this implies $\|Px - Py\| \le \|x - y\|$.

The following corollaries are immediately deduced from this theorem.

Corollary 3.1.[9] *Let C be a closed convex subset of H and X be an r_s-invariant subspace of $B(G)$ containing constants which has a right invariant submean. Let $\Im = \{T_t : t \in G\}$ be a Lipschitzian semigroup on C with $\inf_s \sup_t k_{ts}^2 \le 1$ and $F(\Im) \ne \emptyset$. If for each $x, y \in C$, the function f on G defided by*

$$f(t) = \|T_t x - y\|^2 \text{ for all } t \in G ,$$

and the function g on G defined by $g(t) = k_t^2$ for all $t \in G$ belong to X, then the following are equivalent:

(a) $\bigcap_{s\in G} \overline{\text{co}}\{T_{ts}x : t \in G\} \bigcap F(\Im) \ne \emptyset$ *for each $x \in C$.*

(b) *There is a nonexpansive retraction P of C onto $F(\Im)$ such that $PT_t = T_t P = P$ for every $t \in G$ and $Px \in \overline{\text{co}}\{T_t x : t \in G\}$ for every $x \in C$.*

Remark 3.2. By Theorem 1.2, in the result of Mizoguchi and Takahashi, many key conditions are not necessary.

§4. Fixed Point Theorem

In this section, we prove a fixed point theorem for asymptotically nonexpansive type semigroups without convexity. Let $m(G)$ be the Banach space of all bounded real valued function on G with supremum norm and let X be a subspace of $m(G)$ containing constants. Then, according to Mizoguchi and Takahashi[9], a real valued function μ on X is called a submean on X if the following conditions are satisfied:

(1) $\mu(f + g) \le \mu(f) + \mu(g)$ for every $f, g \in X$;

(2) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in G$ and $\alpha \ge 0$;

(3) $\mu(c) = c$ for every constant c.

For a submean μ on X and $f \in X$, according to time and circumstances, we also use $\mu_t(f(t))$ instead of $\mu(f)$.

Lemma 4.1.[9] *Let G be a semitopological semigroup, let X be a subspace of $m(G)$ containing constants and let μ be a submean on X. Let $\{x_t : t \in G\}$ be a bounded subset of a Hilbert space H and let D be a closed convex subset of H. Suppose that for each $x \in D$, the real-valued function f on G defined by*

$$f(t) = \|x_t - x\|^2 \quad \text{for all } t \in G$$

belongs to X. If $g(x) = \mu_t\|x_t - x\|^2$ for all $x \in D$ and $r = \inf\{g(x) : x \in D\}$, then there exists a unique element $z \in D$ such that $g(z) = r$. Further the following inequality holds:

$$r + \|z - x\|^2 \le g(x) \quad \text{for every } x \in D.$$

Let X be a subspace of $m(G)$ containing constants which is l_s-invariant, i.e., $l_s(X) \subset X$ for each $s \in G$. Then a submean μ on X is said to be left invariant if $\mu(f) = \mu(l_s f)$ for all $s \in G$ and $F \in X$. Now, we can prove a fixed point theorem for asymptotically nonexpansive type semigroups with convexity in a Hilbert space.

Theorem 4.1. *Let C be a nonempty subset of a Hilbert space H and let X be an l_s-invariant subspace of $m(G)$ containing constants which has a left invariant submean μ on X. Let $\Im = \{T_t : t \in G\}$ be an asymptotically nonexpansive semigroup on C such that each T_t is continuous. Suppose that $\{T_t x : t \in G\}$ is bounded and $\bigcap_{s \in G} \overline{\text{co}}\{T_{st}x : t \in G\} \subset C$ for some $x \in C$. If for each $u \in C$ and $v \in H$, the real valued function f on G defined by*

$$f(t) = \|T_t u - v\|^2 \text{ for all } t \in G$$

belongs to X, then there exists an element $z \in C$ such that $T_t z = z$ for all $t \in G$.

Proof. Define a real valued function g on H by

$$g(y) = \mu_t \|T_t x - y\|^2 \quad \text{for each } y \in H.$$

If $r = \inf\{g(y) : y \in H\}$, then by Lemma 4.1 there exists a unique element $z \in H$ such that $g(z) = r$. Further, we know that

$$r + \|z - y\|^2 \leq g(y) \quad \text{for every } y \in H.$$

For each $s \in G$, let Q_s be the metric projection of H onto $\overline{co}\{T_{st}x : t \in G\}$. Then by [10], Q_s is nonexpansive and for each $t \in G$,

$$\|T_{st}x - Q_s z\|^2 = \|Q_s T_{st}x - Q_s z\|^2 \leq \|T_{st}x - z\|^2.$$

So, we have

$$\mu_t \|T_t x - Q_s z\|^2 = \mu_t \|T_{st}x - Q_s z\|^2 \leq \mu_t \|T_{st}x - z\|^2 = \mu_t \|T_t x - z\|^2,$$

and thus $Q_s z = z$. This implies

$$z \in \overline{\text{co}}\{T_{st}x : t \in G\} \quad \text{for all } s \in G\ ,$$

and hence

$$z \in \bigcap_{s \in G} \overline{\text{co}}\{T_{st}x : t \in G\} \subset C.$$

Now, for each $h \in G$, since T_h is continuous at $z \in C$, for any $\epsilon \in (0, 1)$, there exists $0 < \delta < \epsilon$ such that $\|T_h y - T_h z\| < \epsilon$ whenever $y \in C$ and $\|y - z\| < \delta$. Since $\Im$ is an asymptotically nonexpansive type semigroup, there is an $s_0 \in G$ such that $r(ts_0, z) < \frac{1}{2(M+1)}\delta^2$ for all $t \in G$, where $M = \sup_{t \in G} \|T_t x - z\|$. On the other hand, from Lemma 4.1

$$\|z - y\|^2 \leq \mu_t \|T_t x - y\|^2 - \mu_t \|T_t x - z\|^2 \quad \text{for all } y \in H.$$

Putting $y = T_{ss_0} z$ for each $s \in G$, we have

$$\begin{aligned}
\|T_{ss_0} z - z\|^2 &\leq \mu_t \|T_t x - T_{ss_0} z\|^2 - \mu_t \|T_t x - z\|^2 \\
&= \mu_t \|T_{ss_0 t} x - T_{ss_0} z\|^2 - \mu_t \|T_t x - z\|^2 \\
&\leq \mu_t (\frac{1}{2}\delta^2 + \|T_t x - z\|)^2 - \mu_t \|T_t x - z\|^2 < \delta^2,
\end{aligned}$$

and hence $\|T_{ss_0} z - z\| < \delta$ for all $s \in G$. This implies

$$\|T_h z - z\| \leq \|T_h z - T_h T_{ss_0} z\| + \|T_{hss_0} z - z\| < 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $T_h z = z$ for every $h \in G$.

As a direct consequence of Theorem 4.1, we can prove the following fixed point theorem which extends Ishihara's result[5] to non-Lipschitzian semigroups setting. A semitopological semigroup G is left reversible if and only if any two closed right ideals of G have nonvoid intersection. In this case, $(G, \leq)$ is a directed system when the binary relation $\leq$ on G is defined by $a \leq b$ if and only if $\{a\} \bigcup \overline{aG} \supset \{b\} \bigcup \overline{bG}$.

Theorem 4.2. *Let C be a nonempty subset of a Hilbert space H and let G be a left reversible semigroup. Let $\Im = \{T_t : t \in G\}$ be an asymptotically nonexpansive type semigroup on C such that each T_t is continuous. If $\{T_t x : t \in G\}$ is bounded and $\bigcap_{s \in G} \overline{\text{co}}\{T_{st} x : t \in G\} \subset C$ for some $x \in C$, then there exists an element $z \in C$ such that $T_t z = z$ for all $t \in G$.*

Proof. Define a real valued function μ on $m(G)$ by

$$\mu(f) = \limsup_{t \in G} f(t) \quad \text{for every } f \in m(G),$$

μ is a left invariant submean on $m(G)$. By using Theorem 4.1, the proof is completed.

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