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THE ($\mathcal{U}$+ $\mathcal{K}$)-ORBITS OF ESSENTIALLY NORMAL OPERATORS AND THEIR STRONG IRREDUCIBILITY**

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Abstract

The authors characterize the $(\mathcal{U}+\mathcal{K})$-orbits of a class essentially normal operators and prove that some essentially normal operators with connected spectrum are strongly irreducible after a small compact perturbation. This partially answers a question of Domigo A. Herrero.

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§1. Introduction

Let T be a linear, bounded operator acting on a complex, separable, infinite dimensional Hibert space $\mathcal{H}$. We call

$$(\mathcal{U}+\mathcal{K})(T)=\{RTR^{-1},\ R\in(\mathcal{U}+\mathcal{K})(\mathcal{H})\}$$

the $(\mathcal{U}+\mathcal{K})$-oribit of T, where $(\mathcal{U}+\mathcal{K})(\mathcal{H})=\{R: R$ is an invertible operator of the form unitary plus compact$\}$. $T \underset{(\mathcal{U}+\mathcal{K})}{\cong} A$ denotes $A\in(\mathcal{U}+\mathcal{K})(T)$. $\underset{(\mathcal{U}+\mathcal{K})}{\cong}$ is an equivalence relation[1]. Al-Marcoux described the $(\mathcal{U}+\mathcal{K})$-orbits of normal operators and essentially normal operators with unit disk spectrum[1,2]. Ji, Y. Q.; Jiang, C. L.and Wang, Z. Y.[3] studied the $(\mathcal{U}+\mathcal{K})$-orbits of essentially normal operators whose spectra are closures of analytic Jordan domains. Using those results, we can prove that some essentially normal operators with connected spectrum are strongly irreducible after a small compact perturbation. An operator T is strongly irreducible, i.e., $T\in(SI)$, if it does not commute with any nontrivail idempotent. An operator is essentially normal if the self-commutator $[T,T^*]=T^*T-TT^*$ is compact.

In this article, we study a class of essentially normal operators whose spectrum pictures are more complicated, and prove that the essentially normal operator with connected spectrum in some bigger classes is strongly irreducible after a small compact perturbation. This partially answers a question of Domingo A. Herrero.

Question H (Herrero). Given an essentially normal operator T with connected spectrum $\sigma(T)$ and given $\varepsilon>0$, can we find a compact operator K with $\|K\|<\varepsilon$ such that $T+K\in(SI)$?

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The following theorems are our main results.

Theorem 1.1. *Let Ω be a bounded, simple connected open subset of $\mathbb{C}$, and let σ be a compact subset of Ω. Then for each natural number m, there exists an essentially normal operator A satisfying the following conditions:*

(i) $\sigma(A) = \overline{\Omega}, \quad \sigma_e(A) = \sigma \cup \partial\Omega$;

(ii) $\dim\ker(A-\lambda) = \mathrm{ind}(A-\lambda) = m(\lambda \in \Omega \backslash \sigma)$;

(iii) *Furthermore, for each $\varepsilon > 0$ and T essentially normal, if $\Lambda(T) = \Lambda(A)$, then there exists a K compact with $\|K\| < \varepsilon$ such that $(\mathcal{U}+\mathcal{K}) \underset{\mathcal{U}+\mathcal{K}}{\cong} A$, where $\Lambda(T)$ and $\Lambda(A)$ denote the spectral pictures of T and A respectively. $\sigma_e(A) = \{\lambda \in C; \lambda - A$ is not Fredholm$\}$.*

Theorem 1.2. *Given $T \in L(\mathcal{H})$ essentially normal satisfying the following condtions:*

(i) $\sigma(T) = \overline{\Omega}$ *and* $\mathrm{int}\overline{\Omega}$ *is a bounded, simply connected open set;*

(ii) $\sigma_e(T) = \overline{\Omega} \backslash \Omega$;

(iii) $\mathrm{ind}(T-\lambda) = n(\lambda \in \Omega)$.

Then for each $\varepsilon > 0$, there exists a K compact with $\|K\| < \varepsilon$ such that $T+K \in (SI)$, where Ω is a bounded connected open subset of $\mathbb{C}$.

Theroem 1.3. *Let Ω_0 be a bounded, simply connected open subset of $\mathbb{C}$ and let $\{\Omega_i\}_{i=1}^{l}$ $(1 \le l \le +\infty)$ be a sequence of disjoint simply connected open subsets of Ω_0. Given a sequence $\{n_i\}_{i=1}^{l}$ of integers such that $n_k \ge n_0$, $k = 1, 2, \cdots$, then there exists an essentially normal operator A satisfying*

(i) $\sigma(A) = \overline{\Omega}, \quad \sigma_e(A) = \overline{\bigcup\limits_{0 \le i \le l} \partial\Omega_i}$;

(ii) $\mathrm{ind}(A-\lambda) = \dim\ker(A-\lambda) = n_0(\lambda_0 \in \Omega_0 \backslash \bigcup\limits_{k=1}^{l} \overline{\Omega_k})$;

(iii) $\mathrm{ind}(A-\lambda) = \dim\ker(A-\lambda) = n_k(\lambda \in \Omega_k,\ k \ge 1)$;

(iv) *For each $\varepsilon > 0$ and each T essentially normal, if $\Lambda(T) = \Lambda(A)$, then there exists a K compact with $\|K\| < \varepsilon$ such that $T + K \underset{\mathcal{U}+\mathcal{K}}{\cong} A$.*

Theorem 1.4. *Given an essentially normal operator $T \in L(\mathcal{H})$ such that $\sigma(T) = \overline{\Omega}_0$, where $\overline{\Omega}_0$ is an analytic, simply connected, closed region, and let $\{\Omega_k\}_{k=1}^{l}$ be the set of connected components of $\rho_F(T) \cap \sigma(T)$ satisfying the following conditions:*

(i) $\Omega_0 \backslash \bigcup\limits_{k=1}^{l} \overline{\Omega}_k \ne \emptyset$ *and* $\{\Omega_k\}_{k=1}^{l}(\{\overline{\Omega}_k\}_{k=1}^{l})$ *is a sequence of disjoint, simply connected, open (closed) regions;*

(ii) $1 \le \mathrm{ind}(\lambda_0 - T) \le \mathrm{ind}(\lambda_k - T)$ $(\lambda \in \Omega_k,\ k = 1, 2, \cdots), (\lambda_0 \in \Omega_0 \backslash \bigcup\limits_{k=1}^{l} \overline{\Omega}_k)$, *then for each $\varepsilon > 0$, there exists a K compact with $\|K\| < \varepsilon$ such that $T+K \in (SI)$, where $\rho_F(T) = C \backslash \sigma_e(T)$.*

§2. Proof of Theorem 1.3 and Theorm 1.4

Let φ be a conformal one-to-one mapping of D (unit disc) onto Ω, a bounded simply connected open subset of $\mathbb{C}$. Then we say that T_φ is a Toeplitz operator whose symbol is φ.

First, we need following lemmas.

Lemma 2.1. *Given a sequence $\{\Omega_k\}_{k=1}^{l}$ $(1 \le l \le \infty)$ of bounded simply connected open* *subsets of* $\mathbb{C}$; *for each* k, *let* φ_k *be a conformal one-to-one mapping of* D *onto* Ω_k,

$$T = \overset{l}{\underset{k=1}{\oplus}} T^*_{\varphi_k} \in \mathcal{L}\Big(\overset{l}{\underset{k=1}{\oplus}} H^2(\partial D)\Big),$$

$$\mathcal{M} = \overset{\delta_m}{\underset{k=1}{\oplus}} \ker(T^*_{\varphi_K} - \overline{\varphi_k(0)})^m \ (\delta_m = \min\{m, l\}).$$

Then

$$(I-P)T|_{\mathcal{M}^\perp} \underset{\text{unitary equivalence}}{\cong} T,$$

where P *is the orthogonal projection from* $\overset{l}{\underset{k=1}{\oplus}} H^2(\partial D)$ *onto* $\mathcal{M}$.

Proof. Note that $\mathcal{M}^\perp = \Big(\overset{\delta_m}{\underset{k=1}{\oplus}} H^2(\partial D) - \ker(T^*_{\varphi_k} - \overline{\varphi_k(0)})^m\Big) \oplus \Big(\overset{l}{\underset{k=\delta_m+1}{\oplus}} H^2(\partial D)\Big)$ and $(I-P)T|_{\mathcal{M}^\perp} = \Big(\overset{\delta_m}{\underset{k=1}{\oplus}} T_k\Big) \oplus \Big(\overset{i}{\underset{k=\delta_m+1}{\oplus}} T_{\varphi_k}\Big)$, where $T_k = (I-P_k)|_{\mathrm{run}(I-P_k)}$ and P_k is the orthogonal projection from $H^2(\partial D)$ onto $\ker(T^*_{\varphi_k} - \overline{\varphi_k(0)})^m$, $1 \le k \le \delta_m$. So, it is sufficient to prove that $T_k \cong T_k$, $1 \le k \le \delta_m$. Since φ_k is a conformal one-to-one mapping, $(\varphi_k(z) - \varphi_k(0))^m = z^m h(z)$, where h is invertible in $H^\infty(\partial D)$. Hence, $\ker(T^*_{\varphi_k} - \overline{\varphi_k(0)})^m = \ker T^*_{z^m}$ and $(\ker(T^*_{\varphi_k} - \overline{\varphi_k(0)})^m)^\perp = \mathrm{Ran} T^*_{z^m}$. Let U_k be the mapping T_{z^m} from $H^2(\partial D)$ to $\mathrm{Ran} T_{z^m}$. Then U_k is unitary, and $T_{\varphi_k} = U_k^* T_k U$, i.e., $T_{\varphi_k} \cong T_*$.

Lemma 2.2. *Let* φ *be a conformal one-to-one mapping of* D *onto simply connected region* Ω^* *and let* $\sigma \subsetneqq \Omega$ *with* $\mathrm{int}\bar{\sigma} = \emptyset$. *Assume that* $\{\lambda\}_{k=0}^\infty$ *is a dense subset of* σ *such that* $\mathrm{card}\{k : \lambda_n = \lambda_k\} = \infty$ $(n = 1, 2, \cdots)$, *and let* $Ee_n = \lambda_n e_n$ $(n = 1, 2, \cdots)$, *where* $\{e_n\}_{n=1}^\infty = \{e^{i(n-1)\theta}$ $(n = 1, 2, \cdots)\}$ *is an ONB of* $H^2(\partial D)$. *Then for each* $\varepsilon > 0$, *there exists a* K *compcact with* $\|K\| < \varepsilon$ *such that* $T^*_\varphi \oplus E + K (SI)$. *Furthermore, if there are no isolated points in* σ, K *can be a rank* 1 *operator* $x \otimes e_1$.

Proof. Suppose that $\{\mu_n\}_{n=1}^{l_1}$ are the isolated points of σ and $\{\lambda'_n\}_{n=1}^\infty = \{\lambda_k\}_{k=0}^\infty \setminus \{\mu_n\}_{n=1}^{l_1}$. Set $\overline{E} = E_1 \oplus E_2$, where $E_1 = \mathrm{diag}\{\lambda'_1, \lambda'_2, \cdots, \lambda'_n, \cdots\}$, and $E_2 = \overset{l_1}{\underset{n=1}{\oplus}} \mu_n I$, $\dim\ker(\mu_n - E_2) = \infty$. By Voiculesu's Theorem[5], there exists a $\overline{K}_1$ compact with $\|\overline{K}_1\| < \varepsilon$ such that $\overline{E} + \overline{K} \cong E$. Thus it is sufficient to prove that for each $\varepsilon > 0$ there exists a K compact with $\|K\| < \varepsilon$ such that $T^*_\varphi \oplus \overline{E} + K \in (SI)$. Set

$$B_n = \begin{bmatrix} 0 & 0 & & & \\ \alpha_1^n & 0 & & & 0 \\ & \alpha_2^n & 0 & & \\ & & & \ddots & \ddots \\ & & 0 & \ddots & \ddots \end{bmatrix},$$

$\alpha_k^n = \frac{\varepsilon}{(n+k)^{n+k}}$, $n = 1, 2, \cdots, l_1$. Then $B_n \in (SI)$ and $V_n = \mu_n + B_n \in (SI)$. Note that $T^*_\varphi(SI)$ and $T^*_\varphi \in \mathcal{B}_1(\Omega)$ (see [6,7]), where $\mathcal{B}(\Omega)$ is the set of Cowen-Douglas operators of index 1 in Ω and $\Omega = \{\lambda \in C;\ \overline{\lambda} \in \Omega^*\}$. From [6], there exists an analytic function $f(\lambda) : \Omega \to H^2(\partial D)$ such that $(T^* - \lambda)f(\lambda) \equiv 0$, and $f(\lambda) \ne 0$ $(\lambda \in \Omega)$. Let $\mathcal{H}_n$ $(n = 1, \cdots, l_1)$ denote the acting spaces of V_n and $\{e_k^n\}_{k=1}^\infty$ an ONB of $\mathcal{H}_n$. Let $\mathcal{H} = H^2(\partial D)$ and $\mathcal{H}_0$ denotes the acting sapce of E_1. $\{e_k = e^{i(k-1)\theta} :\ k = 1, 2, \cdots\}$ and $\{e_k^0\}_{k=1}^\infty$ are ONB of $\mathcal{H}_1$ and $\mathcal{H}_0$ respectively.

Set $a(\mu) = (f(\mu), e_1)$. Then $a(\mu)$ is analytic on Ω. Without loss of generality we can assume that $a(\lambda'_k) \neq 0$ $(k = 1, 2, \cdots)$.

Define

$$A = \begin{bmatrix} T_\varphi^* & & & & \\ C_0 & E_1 & & & \\ C_1 & 0 & V_1 & & \\ C_2 & 0 & 0 & V_2 & \\ \vdots & \vdots & & \ddots & \\ \vdots & \vdots & & \ddots & \end{bmatrix},$$

where C_n $(n = 0, 1, \cdots)$ is defined as follows:

$$C_0 e_1 = \sum_{k=1}^{\infty} \frac{\varepsilon}{k} e_k^0, \quad C_0 e_k = 0 \ (k = 2, 3, \cdots),$$

$$C_n\Big(\frac{f(\lambda'_n)}{\|f(\lambda'_n)\|}\Big) = \varepsilon 2^{-n} e_1^n, \quad C_n\Big(\Big[\frac{f(\lambda'_n)}{\|f(\lambda'_n)\|}\Big]^{\perp}\Big) = 0, \quad n = 1, 2, \cdots .$$

Set

$$\chi(\mu) = f(\mu) \oplus \Big(- a(\mu) \sum_{k=1}^{\infty} \frac{\varepsilon e_k^0}{k(\lambda'_k - \mu)} \Big) \mathop{\oplus}_{n=1}^{l} \varepsilon \frac{-(f(\mu), f(\mu_n))}{2^n \|f(\mu_n)\|}$$

$$\cdot \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \alpha_1^n \cdots \alpha_k^n}{(\mu_n - \mu)^k} e_k^n \quad (\mu \in \Omega \backslash \sigma).$$

Then computation shows that $0 \neq \chi(\mu) \in \ker(A - \mu)$. It is easy to see that $\dim\ker(A - \mu) = 1$. From the construction we can see that there exists a $\overline{K}$ compact with $\|\overline{K}\| < \varepsilon$ such that $T_\varphi^* \oplus \overline{E} + \overline{K} = A$.

In order to prove $A \in (SI)$, it is sufficient to prove that $A \in \mathcal{B}_1(\Omega\backslash\sigma)$ (because $\mathcal{B}_1(\Phi) \subset (SI)$, where Φ is a connected open subset of $\mathbb{C}$ (see [10]). If $g \oplus \mathop{\oplus}\limits_{k=0}^{l_1} y_k \perp \bigvee\limits_{\mu \in \Omega\backslash\sigma} \chi(\mu)$, i.e., $\Big(\chi(\mu), g \oplus \mathop{\oplus}\limits_{k=0}^{l_1} y_k\Big) = 0$, then

$$(f(\mu), g) = a(\mu) \sum_{k=1}^{\infty} \frac{\varepsilon b_k^0}{k(\lambda'_k - \mu)} + \sum_{n=1}^{l_1} \frac{\varepsilon(f(\mu), f(\mu_n))}{2^n \|f(\mu_n)\|} \cdot \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \alpha_1^n \cdots \alpha_{k-1}^n}{(\mu_n - \mu)^k} b_k^n,$$

where $y_n = \sum\limits_{k=1}^{\infty} \bar{b}_k^n e_k^n$ $(n = 0, 1, 2, \cdots)$. Since μ_n is an isolated point of σ and since $(f(\mu), g)$ is analytic in Ω, $b_k^n = 0$ $(k = 1, 2, \cdots, \ n = 1, 2, \cdots)$, we have

$$(f(\mu), g) = \varepsilon \sum_{k=1}^{\infty} \frac{a(\mu) b_k^0}{k(\lambda'_k - \mu)}.$$

In order to prove $b_k^0 = 0$ $(k = 1, 2, \cdots)$, it is sufficient to prove the following fact: assume that Φ is a connected open subset of $\mathbb{C}$; $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of distinct complex numbers in Φ with $\overline{\{\lambda_k\}}$ containing no interior; if $\sum\limits_{k=1}^{\infty} \frac{a_n}{(\lambda_n - \mu)}$ is anaytic in Φ, then $a_n = 0$ $(n = 1, 2, \cdots)$, where $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers in $\mathbb{C}$ such that $\sum\limits_{k=1}^{\infty} |a_n| < +\infty$.

If $a_k \neq 0$, then there exists an N such that $\sum\limits_{n=N+1}^{\infty} |a_n| < \frac{|a_k|}{2}$. Since $\overline{\{\lambda_k : \ k = 1, 2, \cdots\}}$ contains no interior points, there is $\{l_m\} \subset \Phi$ and $\{l_m\} \bigcap \{\lambda_n\} = \emptyset$ such that $\lim\limits_{m \to \infty} l_m = \lambda_k$

and $|l_m-\lambda_k|\le|l_m-\lambda_j|,\ \ m\ne j.$

Therefore

$$\Big|\sum_{k=1}^{\infty}\frac{a_n}{(\lambda_n-\lambda_m)}\Big|\ge\frac{|a_k|}{|\lambda_k-\lambda_m|}-\sum_{\substack{n\neq k\\ n\le N}}\frac{|a_n|}{|\lambda_n-\lambda_m|}-\sum_{n=N+1}^{\infty}\frac{|a_n|}{|\lambda_n-\lambda_m|}$$
$$\ge\frac{a_k}{2|\lambda_k-\lambda_m|}-\sum_{\substack{n\neq k\\ n\le N}}\frac{|a_n|}{|\lambda_n-l_m|}\to\infty\ \ (m\to\infty).$$

The contradiction implies that $a_n=0,\ n=1,2,\cdots$ and $\bigvee_{\mu\subset\Omega\backslash\sigma}\chi(\mu)=H^2(\partial D)\overset{l_1}{\underset{k=0}{\oplus}}\mathcal{H}_k$.

Recall that for natural number $n,\mathcal{B}_n(\Omega)$, the set of Cown-Douglas operators of index n, is the set of all operators B on $\mathcal{H}$ satisfying

(i) $\sigma(B)\supset\Omega$;

(ii) $\dim\ker(\lambda-B)=\mathrm{ind}(\lambda-B)=n\ (\lambda\in\Omega)$;

(iii) $\vee(\ker(\lambda-B):\lambda\in\Omega)=\mathcal{H}$.

By Theorem 1.2 of [8], $\mathcal{B}_1(\Omega)\subset(SI)$.

The remainder of the proof of $A\in\mathcal{B}_1(\Omega\backslash\sigma)$ is a routine work.

Just like the proof of Lemma 2.2, we have

Lemma 2.2.' *Let σ be a compact subset of simply connected region Ω with* $\mathrm{int}\sigma=\emptyset$ and let $\{\lambda_i\}_{i=1}^{\infty}$ be a dense sequence of σ. If $A\in\mathcal{B}_1(\Omega)$ and $D=\mathrm{diag}(\lambda_1,\lambda_2,\cdots)$, then for each $\varepsilon>0$ there exists a K compact with $\|K\|<\varepsilon$ such that $A\oplus D+K\in(SI)$.

Given a conformal one-to-one mapping φ from $D\to\Omega^*=\{\lambda:\overline{\lambda}\in\Omega\}$, then $T_\varphi^*\in\mathcal{B}_1(\Omega)$ and $\sigma(T_\varphi^*)=\overline{\Omega}$.

Lemma 2.3. *Let T_φ^* be as above. Then for each $\lambda\in\Omega$ there exists an $f_\lambda\in H^2(\partial D)$ satisfying the following conditions:*

(i) $(T_\varphi^*)f_\lambda=0$.

(ii) $\lambda\to f_\lambda$ *is analytic on* Ω.

(iii) *For each* $g\in H^2(\partial D),\ (f_\lambda,g)=\overline{g(\varphi^{-1}(\bar\lambda))}$.

Proof. Since $H^2(\partial D)$ has the Szegö kernel $k_z\ (z\in D)$ such that $g(z)=(g,k_z)$ for each $g\in H^2(\partial D)$, let $f_\lambda=k_{\varphi^{-1}(\lambda)}$. Then, for each $g\in H^2(\partial D),\ (f_\lambda,g)=\overline{(g,f_\lambda)}=\overline{g(\varphi^{-1}(\bar\lambda))}$. Thus (iii) is satisfied. Futhermore, (f_λ,g) is analytic on Ω. So $\lambda\to f_\lambda$ is analytic on Ω, i.e., (ii) holds. Moreover, $(g,(T_\varphi^*-\lambda)f_\lambda)=((\varphi-\bar\lambda)g,f_\lambda)=(\varphi(\varphi^{-1}(\bar\lambda))-\bar\lambda)g(\varphi^{-1}(\bar\lambda))=0$ for each $g\in H^2(\partial D)$. Thus $(T_\varphi^*-\lambda)f_\lambda=0$.

Let T_φ^* be defined as above and let $\{D_n\}_{n=1}^{l}$ be a sequence of bounded operators satisfying the following:

(i) $\{\sigma(D_n)\}_{n=2}^{l}$ is pairwise disjoint and for each $\lambda\notin\overline{\bigcup_n\sigma(D_n)}$, $\|(\lambda-D_n)^{-1}\|$ is uniformly bounded;

(ii) $D_n(n\ge2)$ is rationally cyclic, x_n is its rationally cyclic vector, $\sum_{n=2}^{l}\|x_n\|^2<+\infty$ and $\overline{\bigcup_n\sigma(D_n)}\subsetneqq\Omega$;

(iii) D_1 is the operator $\overline{E}$ given in the proof of Lemma 2.2, i.e.,

$$D_1 = \operatorname{diag}(\lambda_1', \lambda_2', \cdots, \lambda_n', \cdots) \overset{l_1}{\underset{n=1}{\oplus}} \mu_n I$$

and int$\sigma(D_1) = \emptyset$, there are no isolated points in $\sigma(D_1)$ and $\sigma(D_1) \subsetneqq \Omega \backslash \bigcup\limits_{n=2}^{\infty} \sigma(D_n)$.

Assume that

$$T = \begin{bmatrix} T_\varphi^* & 0 & 0 \\ C & D_1 & 0 \\ C_2 & 0 & \overset{l}{\underset{n=2}{\oplus}} D_n \end{bmatrix},$$

and $\sigma = \bigcup\limits_{n=1}^{l} (D_n)$, where $C_1 = f_1 \oplus e_1$ is C_0 given in Lemma 2.2 and $C_2 = f_2 \otimes e_1$; $f_2 = \overset{l}{\underset{n=1}{\oplus}} x_n$, φ is a conformal one-to-one mapping of D onto Ω^*, $\sigma \subsetneqq \Omega$. Then we have

Lemma 2.4. (i) $T \in (SI)$, *and* $T \in \mathcal{B}_1(\Omega \backslash \sigma)$,

(ii) $\sigma(T) = \overline{\Omega}$ *and* $\sigma_e(T) = \sigma \cup \partial\Omega$.

Proof. We only need to prove $T \in \mathcal{B}_1(\Omega\backslash\sigma)$. Denote $\mathcal{H}_1 = \bigvee\limits_{\lambda \in (\Omega\backslash\sigma)} \ker(T - \lambda)$. If $f_\lambda \in \ker(T_\varphi^* - \lambda)$ is given in Lemma 2.3, and $x_\lambda = f_\lambda \oplus \sum\limits_{n=1}^{\infty} (\lambda - D_n)^{-1} x_n$, then $(x_\lambda, e_1) = 1$, where $e_1 = e^{i0} = 1$; and $(T - \lambda)x_\lambda = 0$. If $y \in \mathcal{H}$ and $y \perp \mathcal{H}_1$, then $y = g \oplus \overset{\infty}{\underset{n=1}{\oplus}} y_n$, where $g \in H^2(\partial D)$ and $y_n \in \mathcal{H}_n$ $(n = 1, 2, \cdots)$. From Lemma 2.3

$$0 = (x_\lambda, g) = g\overline{(\varphi^{-1}(\lambda))} + \sum_{n=1}^{\infty} ((\lambda - D_n)^{-1} x_n, y_n),$$

i.e., $g\overline{(\varphi^{-1}(\bar{\lambda}))} = -\sum\limits_{n=1}^{\infty} ((\lambda - D - n)^{-1} x_n, y_n)$.

The right side of the equality is analytic in $\mathbb{C} \backslash \overline{\Big(\bigcup\limits_{n=1}^{l} \sigma_n\Big)}$ and the left side is analytic in Ω. Thus both sides of equality can be extended to a function analytic in $\mathbb{C}$, since $\sigma = \bigcup\limits_{n=1}^{l} \sigma_n \subsetneqq \Omega$. Besides, $((\lambda - D_n)^{-1} x_n, y_n) \to 0$ $(\lambda \to \infty)$, thus $g\overline{(\varphi^{-1}(\bar{\lambda}))} = 0$ $(\lambda \in \Omega\backslash\sigma)$, i.e., $g \equiv 0$. When $|\lambda|$ is big enough, we have

$$((\lambda - D_n)^{-1} x_n, y_n) = \sum_{n=1}^{\infty} \frac{1}{\lambda^{k+1}} (D_n^k x_n, y_n) \to 0.$$

Thus $(D^k x_n, y_) = 0$, $k \geq 0$, i.e., if $\lambda \notin \sigma)D_n)$, $((\lambda - D_n)^{-1} x_n, y_n) = 0$. Since x_n is a rationally cyclic vector of $D_n, y_n = 0$ $(n = 2, 3, \cdots)$. By the arguments used in the proof of Lemma 2.2, $y_1 = 0$. Thus $y \equiv 0$ and $\mathcal{H}_1 = \mathcal{H}$.

Lemma 2.5. *Suppose that operator* $B_k \in \mathcal{B}_1(\Omega_k)$ $(k = 1, 2, \cdots)$ *satisfies*

(i) *If* $\Omega_k = \Omega_1$, *then* $B_k = B_1$; (ii) *If* $\Omega_k \neq \Omega_1$, *then* $\Omega_k \subsetneqq \Omega_1$; (iii) $\sigma(B_k) = \overline{\Omega_k}$.

Then for each K *comacpt and each* $\varepsilon > 0$, *there exists* G_1 *compact such that* $\|B_1 G_1 - G_1 A - K\| < \varepsilon$, *where* $A = \overset{\infty}{\underset{k=1}{\oplus}} B_k$.

Proof. Write $B = \overset{+\infty}{\underset{k=1}{\oplus}} B_1$. Let $\tau = \tau_{B,A|\mathcal{K}(\mathcal{H})}$. From Lemma 3.5 of [3], $\tau^* = -\tau_{A,B}|\mathcal{C}_1(\mathcal{H})$, where $\mathcal{C}_1(\mathcal{H})$ denote the trace class operators. Thus we need only to prove that $\ker\tau^* = \{0\}$.

If there is an X in the trace class satisfying that

$$AX = XB,$$

and suppose $X=[x_{ij}]_{ij}$, then $B_kX_{k,j}=X_{k,j}B_1$ $(k=1,2,\cdots)$. If $B_k=B_1$ $(\Omega_k=\Omega_1)$, by Proposition 1.21 of [6] and X in $\mathcal{L}_1(\mathcal{H})$, $X_{kj}=0$. If $B_k\neq B_1$ $(\Omega_k\subsetneqq\Omega_1)$, there is a neibourghhord O_y of $y\in\Omega_1\backslash\Omega_k$ such that $O_y\subset\Omega_1\backslash\Omega_k$. Since $B_1\in\mathcal{B}\in(\Omega_1)$, $\bigvee\limits_{\lambda\in O_y}\ker(B_1-\lambda)=H^2(\partial\Omega_1)$. From Lemma 2 of [9], $X_{kj}=0$. Therefore $X=0$.

Proof of Theorem 1.3. Let $\{n_k\}_{k=0}^{\infty}$ be a sequence of natural numbers. Assume that there is a subsequence $\{n_{p_k}\}_{k=1}^{\infty}$ of $\{n_k\}_{k=0}^{\infty}$ such that $n_{p_k}=n_0$ $(k=1,2,\cdots)$. Suppose that $\{n_k\}_{k=1}^{\infty}\backslash\{n_{p_k}\}_{k=1}^{\infty}=\{n_{l_k}\}_{k=1}^{\infty}$. Denote $d_k=n_{l_k}-n_0$ $(k=1,2,\cdots)$, $d_0=n_0$ and $\sigma=\cup\overline{\partial\Omega_{p_k}}$. Let φ_k $(k=0,1,2,\cdots)$ be a conformal one-to-one mapping of D onto $\Omega_{l_k}^*=\{\lambda:\bar\lambda\in\Omega_{l_k}\}$ $(l_0=0)$.

Denote $T_1=\bigoplus\limits_{k=0}^{\infty}\bigoplus\limits_{1}^{d_k}T_{\varphi_k}^*$. Given a dense subset $\{\lambda_n\}_{n=1}^{\infty}$ of σ such that

$$\mathrm{Card}\{k:\lambda_n=\lambda_k\}=\infty\ (n=1,2,\cdots).$$

Set $D=\mathrm{diag}(\lambda_1,\lambda_2,\cdots)$, i.e., $De_n=\lambda_ne_n$ $(n=1,2,\cdots)$, where $\{e_n\}_{n=1}^{\infty}$ is an ONB of $\mathcal{H}_1$. Note that the acting sapce of T_1 is $\bigoplus\limits_{k=0}^{\infty}\bigoplus\limits_{n=1}^{d_k}\mathcal{H}_n^k$, where $\mathcal{H}_n^k=H^2(\partial D)$. Set

$$A=\begin{bmatrix}T_1&0\\0&D\end{bmatrix}\begin{matrix}\bigoplus\limits_{k=0}^{\infty}\bigoplus\limits_{n=1}^{d_k}\mathcal{H}_n^k\\ \mathcal{H}_1\end{matrix}.$$

Then A satisfies (i), (ii), (iii) of Theorem 1.3. Since $\Lambda(T)=\Lambda(A)$, there exist K_0 compact and U unitary such that $U^*TU=A+K_0$ by BDF Theorem[10]. Assume that $\{e_m^{n_k}=e^{i(m-1)\theta}\}_{m=1}^{\infty}$ is the ONB of $\mathcal{H}_n^k$. Let P_{L_1} and P_{L_2} be the orthogonal projection onto $\bigvee\{e_m^{m_k}: 0\le m\le L;\ n_k\le L\}$ and $\bigvee\{e_m:0\le m\le L\}$ respectively, where L is a natural number. Then $P_L=P_{L_1}+P_{L_2}\overset{\mathrm{SOT}}{\to}I$ $(L\to\infty)$. Therefore there exists L_0 such that $\|P_{L_0}K_0P_{L_0}-K_0\|<\frac{\varepsilon}{8}$. Denote $K_1=P_{L_0}K_0P_{L_0}-K_0$. Then

$$\begin{aligned}A+K_0+K_1&=A+P_{L_0}K_0P_{L_0}\\&=\begin{bmatrix}P_{L_1}T_1P_{L_1}+P_{L_1}K_0P_{L_1}&P_{L_1}T_1P_{L_1}^{\perp}&P_{L_1}K_0P_{L_2}&0\\0&P_{L_1}^{\perp}T_1P_{L_1}^{\perp}&0&0\\P_{L_2}K_0P_{L_1}&0&P_{L_2}K_0P_{L_2}+P_{L_2}DP_{L_2}&0\\0&0&0&P_{L_2}^{\perp}DP_{L_2}^{\perp}\end{bmatrix}\\&\cong\begin{bmatrix}K_{11}&K_{12}&K_{13}&0\\K_{21}&K_{22}&0&0\\0&0&P_{L_1}^{\perp}T_1P_{L_1}^{\perp}&0\\0&0&0&P_{L_2}^{\perp}DP_{L_2}^{\perp}\end{bmatrix},\end{aligned}$$

where $K_{11}=P_{L_1}T_0P_{L_1}+P_{L_1}K_0P_{L_1}$, $K_{21}=P_{L_2}K_0P_{L_1}$, $K_{L_2}=P_{L_1}K_0P_{L_2}$, $K_{13}=P_{L_1}T_1P_{L_1}^{\perp}$, $K_{22}=P_{L_2}K_0P_{L_2}+P_{L_2}DP_{L_2}$. From Lemma 2.1, $P_{L_1}^{\perp}T_1P_{L_1}^{\perp}\cong T_1$. Since $D=\mathrm{diag}\{\lambda_1,\lambda_2,\cdots\}$ and $\mathrm{Card}\{n,\lambda_n=\lambda_k\}=\infty$, $P_{L_2}^{\perp}DP_{L_2}^{\perp}\cong D$. Thus

$$A+K_0+K_1\cong\begin{bmatrix}L_{11}&L_{12}&0\\0&T_1&0\\0&0&D\end{bmatrix},$$

where $L_{11}=\begin{bmatrix}K_{11}&K_{12}\\K_{21}&K_{22}\end{bmatrix}$ and $L_{12}=\begin{bmatrix}K_{13}\\0\end{bmatrix}$ are finite rank operators.

Thus there exists a U_1 unitary such that

$$U_1(A+K_0+K_1)U_1^* = \begin{bmatrix} L_{11} & L_{12} & 0 \\ 0 & T_1 & 0 \\ 0 & 0 & D \end{bmatrix} = T_2.$$

From the upper semi-continuity of spectrum $\sigma(L_{11}) \subset (\Omega_0)_{\sigma/8} = \{z : \mathrm{disc}(z, \Omega_0) < \varepsilon/8\}$. Therefore there exists L with $\|L\| < \varepsilon/4$ such that $\sigma(L_{11}+L) \subset \Omega_0$ and the eigenvalues of $E = (L_{11} - L)$ are pairwise distinct, i.e., there exists K_2 compact with $\|K_2\| < \varepsilon/4$ such that

$$T_2 + K_2 = \begin{bmatrix} E & L_{12} & 0 \\ 0 & T_1 & 0 \\ 0 & 0 & D \end{bmatrix}.$$

Denote $A_1 = \Big(\overset{d_0-1}{\underset{1}{\oplus}} T^*_{\varphi_0}\Big)\Big(\overset{\infty}{\underset{k=1}{\oplus}}\overset{d_k}{\underset{1}{\oplus}} T^*_{\varphi_j}\Big)$. Then

$$T_2 + K_2 = \begin{bmatrix} E & L'_{12} & L''_{12} & 0 \\ 0 & T^*_{\varphi_0} & 0 & 0 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & D \end{bmatrix},$$

where $L_{12} = (L'_{12}, L'_{12})$.

From Lemma 3.4 of [3], there exist $\overline{K}_2$ compact with $\|K_2\| < \varepsilon/8$ and $\overline{X}_1 \in (\mathcal{U} + \mathcal{K})(H^2(\partial D)$ such that

$$\overline{X}_1 \left[\begin{bmatrix} E & L'_{12} \\ 0 & T^*_{\varphi_0} \end{bmatrix} + \overline{K}_2\right] \overline{X}_1^{-1} = T^*_{\varphi_0}.$$

Therefore we can find K_3 compact with $\|K_3\| < \varepsilon/8$ and $X_1 \in (\mathcal{M} + \mathcal{K})((\oplus H^2(\partial D)) \oplus \mathcal{H}_1)$ such that

$$X_1(T_2 + K_3 + K_2)X_1^{-1} = \begin{bmatrix} T^*_{\varphi_0} & C_{12} & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & D \end{bmatrix},$$

where C_{12} is still a finite rank operator. From Lemma 2.5, we can find G_1, G_2 compact with $\|G_1\| < \frac{\varepsilon}{8\|X_1\|\,\|X_1^{-1}\|}$ such that $T^*_{\varphi_0}G_2 - G_2A_1 = G_1 + C_{12}$. Thus there are K_4 compact with $\|K_4\| < \varepsilon/8\|X_1\|\,\|X_1^{-1}\|$ and $X_2 \in (\mathcal{U}+\mathcal{K})\big(H^2(\partial D)\big) \oplus \mathcal{H}_1$ such that

$$X_2X_1(T + K_3 + K_2 + X_1^{-1}K_4X_1)X_1^{-1}X_2^{-1} = \begin{bmatrix} T^*_{\varphi_0} & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & D \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & D \end{bmatrix} = A.$$

The proof of Theorem 1.3 is now complete.

In order to prove Theorem 1.4, we need the following Lemmas. Let $A_j = \mathcal{L}(H^2(\partial D))$ $(j = 1, 2, \cdots)$ be defined as $A_je_{n+1} = \left(\frac{n+2}{n+1}\right)^{1/j} e_n$ $(n = 1, 2, \cdots)$, where $\{e_n = e^{i(n-1)\theta}\}_{n=1}^{\infty}$ is the ONB of $H^2(\partial D)$. Then $C_j = A_j - T_z^*$ is compact and for each $\varepsilon > 0$ we can find $j \in N$ such that $\|A_j - T_z^*\| = \|C_j\| < \varepsilon$, where $T_ze_{n+1} = e_n$. Then, for each $\varepsilon > 0$, there exists D_0 compact with $\|D_0\| < \varepsilon$ such that for each $f \in H^2(\partial D)$ we have

Lemma 2.6. (i) $D_0 + f \otimes e_1 \notin \mathrm{ran}\tau_{(T_z^*+C_j)T_z^*} = \mathrm{ran}\tau_{A_1,T_z^*}$.

(ii) $\ker\tau \underset{T_z^*A_j}{=} \{0\}$ *and* $\ker\tau \underset{A_{j_1},A_{j_2}}{=} \{0\}$ $(j_1 > j_2)$.

(iii) *There is a* $D_j (k \geq 1)$ *compact with* $\|D_j\| < \varepsilon/2$ *and* $D_j \notin \mathrm{ran}\tau_{A_j,A_{j+1}}$.

Proof. If $j_1 > j_2$, we have

$$A_{j_k} = \begin{bmatrix} 0 & (4/3)^{1/j_k} & & \\ 0 & 0 & (5/4)^{1/j_k} & \\ 0 & 0 & 0 & \\ & & & \ddots \end{bmatrix} \quad (k = 1, 2, \cdots).$$

If $X = \begin{bmatrix} x_{11} & x_{12} & \cdots \\ x_{21} & x_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \in \ker\tau_{A^*_{j_1}, A_{j_2}}$, i.e., $A_{j_1}X = XA_{j_2}$, compuations show that $(4/3)1/j_1 x_{21} = 0$, or $x_{21} = 0, \cdots, \left(\frac{n+2}{n+1}\right)^{1/j} x_{n1} = 0$ or $x_{n1} = 0, \cdots$. Similarly, $x_{n,k} = 0$ $(n > k)$, i.e., X has an upper triangular representation with respect to the ONB $(e_n)_{n=1}^{\infty}$ of the sapce. Also, computations indicate that

$$(4/3)^{1/j_1} x_{22} = (4/3)^{1/j_2} x_{11}, \quad \left(\frac{n+2}{n+1}\right)^{1/j_1} x_{nn} = \left(\frac{n+2}{n+1}\right)^{1/j_2} x_{n-1,n-1}.$$

Thus $x_{nn} = (n + 2/3)^{1/j_2 - 1/j_1} x_{11} \to \infty$ $(n \to \infty)$, if $x_{11} \neq 0$. Therefore $x_{11} = 0$ and $x_{nn} = 0$ $(n = 2, 3, \cdots)$. The same arguments indicate that $x_{ij} = 0$ $(j > i)$, i.e., $X = 0$ and $\ker\tau_{A_{j_1}, A_{j_2}} = \{0\}$. Similarly, we can show that $\ker\tau \underset{T_z^*, A_j}{=} \{0\}$ $(j = 1, 2, \cdots)$.

Let D_0 be given by $D_0 e_{n+1} = \frac{\varepsilon}{\sqrt{n+1}} e_n$ $(n = 1, 2, \cdots)$. Suppose $X \in \mathcal{L}(H^2(\partial D))$ satisfies $A_j X - X T_z^* = f \otimes e + D_0$.

Note

$$f \oplus e_1 + D_0 = \begin{bmatrix} a_0 & \varepsilon/\sqrt{2} & 0 & \\ a_1 & 0 & & \varepsilon/\sqrt{3} \\ \cdots & \cdots & \cdots & \ddots \\ a_k & 0 & & \varepsilon/\sqrt{k+1} \\ & & & \ddots \end{bmatrix},$$

where $f = \sum\limits_{k=1}^{\infty} a_{k-1} e^{i(k-1)\theta}$.

Set $X = \begin{bmatrix} x_{11} & x_{12} & \cdots \\ x_{21} & x_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$. Then comparing the $(k, k+1)$ entries of both sides of the equality, we have

$$(4/3)^{1/j} x_{22} - x_{11} = \varepsilon/\sqrt{2}, \quad (5/4)^{1/j} x_{22} - x_{22} = \varepsilon/\sqrt{3},$$
$$\left(\frac{n+2}{n+1}\right)^{1/j} x_{n+1,n+1} - x_{nn} = \varepsilon/\sqrt{n} \cdots.$$

Since

$$\begin{aligned}\left(\frac{n+2}{n+1}\right)^{1/j} &= \left(1 + \frac{1}{n+1}\right)^{1/j} \\ &= 1 + \frac{1}{j}\left(\frac{1}{n+1}\right) + O\left(\left(\frac{1}{n+1}\right)^2\right)\left(x_{n+1,n+1}\left(\frac{n+2}{n+1}\right)^{1/j} x_{n+1,n+1}\right) \\ &= 1/j\left(\frac{1}{n+1}\right)x_{n+1,n+1} + O\left(\left(\frac{1}{n+1}\right)^2\right)x_{n+1,n+1}\end{aligned}$$

and

$$\lim_{x\to\infty} \frac{\left|x_{n+1,n+1}\left(\frac{n+2}{n+1}\right)^{1/j} x_{n+1,n+1}\right|}{\varepsilon/\sqrt{n}} = \lim_{x\to\infty} \frac{1/j\left(\frac{1}{n+1}\right)^2}{\varepsilon/\sqrt{n}} |x_{n+1,n+1}| = 0,$$

we see that

$$\lim_{x\to\infty}\frac{\left(\frac{n+2}{n+1}\right)x_{n+1,n+1}-x_{nn}}{\varepsilon/\sqrt{n}}=1$$

implies $\lim\limits_{x\to\infty}\frac{x_{n+1,n+1}-x_{nn}}{\varepsilon/\sqrt{n}}=1$. Therefore, when n is big enough,

$$\mathrm{Re}(x_{n+1,n+1}-x_{nn})>\varepsilon/2\sqrt{n},\quad\text{and}\quad\|X\|\geq\lim_{x\to\infty}\mathrm{Re}x_{nn}=\infty.$$

This contradiction implies that $f\oplus e_1+D_0\notin\mathrm{ran}\tau_{A_j,T_z^*}$. Set

$$D_j=\begin{bmatrix}0&\delta_j/\sqrt{2}&&\\0&0&\delta_j/\sqrt{3}&\\&&\ddots&\ddots\\&&&\ddots\end{bmatrix}\quad(j=1,2,\cdots)$$

with respect to the ONB $\{e_n\}_{n=1}^{\infty}$, where $\delta_j=\varepsilon/2^j$. Then by the similar argument we can prove that $D_j\notin\mathrm{ran}\tau_{A_j,A_{j+1}}\quad(j=2,3,\cdots)$.

Lemma 2.7. *If Ω_0 given in Theorem 1.4 is the unit disc, then for the operator $A=\bigoplus\limits_{k=0}^{l}\bigoplus\limits_{1}^{\delta_m}T_{\varphi_k}^*\oplus D$ defined in the proof of Theorem 1.3 and for each $\varepsilon>0$ there exists K compact with $\|K\|<\varepsilon$ such that $A+K\in(SI)$.*

Proof. Note that $T_{\varphi_0}^*=T_z^*$. Set $B_k=\bigoplus\limits_{1}^{\delta_m}T_{\varphi_k}^*$. From the condition (i) of Theorem 1.4 and from the choice of $\sigma(D)$ in the proof of Theorem 1.3,

$$\sigma(D)\bigcap\sigma(B_k)\bigcap\rho_F(B_k)=\emptyset\ (k=1,2,\cdots)\mathrm{int}(D)=\emptyset,$$

and $\sigma(D)$ contains no isolated points. From Theorem 1 of [10] and $B_k\in\mathcal{B}_{d_k}(\Omega_k)$, B_k has cyclic vector x_k $(k=1,2,\cdots)$. Without loss of generality, we can assume that $\sum\limits_{k=1}^{l}\|x_k\|^2<+\infty$. Thus by Lemma 2.4, there is a K_1 compact with $\|K_1\|<\varepsilon/4$ such that

$$A_0=\begin{bmatrix}T_z^*&0&0\\0&\underset{k}{\oplus}B_k&0\\0&0&D\end{bmatrix}+K_1=\begin{bmatrix}T_z^*&0&0&\\L_1&&\underset{k}{\oplus}B_k&0\\L_2&0&D&\end{bmatrix}\in\mathcal{B}_1\Big(\Omega_0\backslash\Big(\overline{\bigcup\sigma(B_k)}\bigcup X\sigma(D)\Big)\Big),$$

where $L_1=f_1\otimes e_1,\ L_2=f_2\otimes e_1$. Thus $A_0\in(SI)$. Use Lemma 2.6 to construct compact operators $D_0,\ D_1,\cdots D_{d_0-2},\ C_1,C_2,C_{d_0-1}$, and $A_1=T_z^*+C_1,\cdots,A_{d_0-1}=T_z^*+C_{d_0-1}$ such that

(i) $\ker\tau_{A_{j_1},A_{j_2}}=\{0\}\quad j_1<j_2$ and $\ker\tau_{T_z^*,A_j}=\{0\}$;

(ii) For each $x\in H^2(\partial D),\ D_0+x\otimes e_1\notin\mathrm{ran}\tau_{A_j,T_z^*}$;

(iii) $D_j\notin\mathrm{ran}\tau_{A_j,A_{j-1}}$, and $\|D_j\|<\varepsilon/100d_0,\ \|C_j\|<\varepsilon/100d_0\ (j=0,1,2,\cdots)$.

Define $\overline{D}_0=(D_0,0,0)$ and

$$G=\begin{bmatrix}A_0&&&&&0\\\overline{D}_0&A_1&&&&\\&D_1&A_2&&&\\&&D_2&\ddots&&\\&&&\ddots&&\\&&&&D_{d_0-2}&A_{d_0-1}\end{bmatrix},$$

and $K_2=G-A$. Then K_2 is compact with $\|K_2\|<\varepsilon$. It is sufficent to prove $G\in(SI)$.

Suppose that $P \in \mathcal{A}(G)$ is idempotent and

$$P = \begin{bmatrix} P_{00} & P_{01} & \cdots & P_{0d_0-1} \\ \vdots & & & \\ \vdots & & & \\ P_{d_0-1,0} & P_{d_0-1,1} & \cdots & P_{d_0-1,d_0-1} \end{bmatrix}.$$

Then calculation indicates that $P_{0d_0-1}A_{d_0-1} = A_0P_{0d_0-1}$. Assume that $P_{0d_0-1}\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}$. Then

$$M_1A_{d_0-1} = T_z^*M_1, \quad M_2A_{d_0-1} = L_1M_1 + \Big(\mathop{\oplus}_k B_k\Big)M_2, \quad M_3A_{d_0-1} = L_2M_1 + DM_3.$$

Since $\ker\tau_{T_z^*,A_j} = \{0\}$ $(j \geq 1)$, $M_1 = 0$. Since $\sigma(B_k) \subsetneqq D, = A_{d_0-1}\mathcal{B}_1(D)$, $\sigma(D) \subsetneqq D$ and from Lemma 4 of [9], $M_2 = M_3 = 0$, i.e., $P_{0d_0-1} = 0$. Similarly, from $A_0P_{0d_0-2} = P_{0d_0-2}A_{d_0-2}$, we can prove that $P_{0d_0-2} = 0$. Inductively, $P_{0,k} = 0$, $k = 1, 2, \cdots, d_0-1$. Similarly, $P_{ij} = 0$ $(i < j)$. Thus P admits a lower matrix form. Since $P_iA_i = A_iP_i$ and since $A_i \in (SI)$, $P_i = \delta_iI$, $\delta_i = 0$ or 1 $(i = 0, 1, \cdots, d_0-1)$.

Suppose that $\delta_0 = 0$ (if $\delta_0 = 1$, consider $\oplus I - P$). Since $PG = GP$, calculation indicates that

$$P_{10}A_0 + \delta_1\overline{D}_0 = A_1P_{10}.$$

If $P_{10} = (M_1, M_2, M_3)$, then

$$(M_1T_z^* + M_2L_1 + M_3L_2, M_2 \mathop{\oplus}_k B_k, M_3D) + (\delta_1D_0, 0, 0) = (A_1M_1, A_1M_2, A_1M_3),$$
$$A_1M_1 - M_1T_z^* = \delta_1D_0 + M_2L_1 + M_3L_2 = \delta_1D_0 + ((M_2f_1 + M_3f_2) \otimes e_1).$$

Thus $\delta_1 = 0$ (Lemma 2.6).

Also, since $P_{21}A_1 + \delta_2D_1 = A_2P_{21}$ and since $D_1 \notin \mathrm{ran}\tau_{A_2,A_1}$, $\delta_2 = 0$. Inductively, $\delta_k = 0$ $(1, 2, \cdots, d_0-1)$, and $P = 0$, i.e., $G \in (SI)$.

Now we are in a position to prove Theorem 1.4.

Since Ω_0 is an analytic, simply connected region, there exists a conformal one-to-one mapping $\psi_0(\Omega_0)_{\delta_1} \to (D)_{\delta_2}$, where $(\Omega_0)_{\delta_1} = \{z, \mathrm{dist}(z, \Omega_0) < \delta_1\}$, $\delta_1 > 0$, $(D)_{\delta_2} = \{z, \mathrm{dist}(z, d) < \delta_2\}$, $\delta_2 > 0$. Suppose that φ_0 is the conformal one-to-one mapping of $(D)_{\delta_2}$, i.e., $\varphi_0 = \psi_0^{-1}$.

Set

$$A = \begin{bmatrix} \mathop{\oplus}\limits_{i=1}^{d_0} T_{\varphi_0}^* & 0 & 0 \\ 0 & \oplus_k \mathop{\oplus}\limits_{i=1}^{d_k} T_{\varphi_k}^* & 0 \\ 0 & 0 & D \end{bmatrix},$$

where A is given in the proof of Theorem 1.3. Then

$$\psi_0(A) = \begin{bmatrix} \mathop{\oplus}\limits_{i=1}^{d_0} T_z^* & 0 & 0 \\ 0 & \mathop{\oplus}\limits_k \mathop{\oplus}\limits_{i=1}^{d_k} T_{\psi_0(\varphi_k)}^* & 0 \\ 0 & 0 & \psi_0(D) \end{bmatrix}.$$

From Theorem 1.3, for each $\varepsilon > 0$, there exists a K_0 compact with $\|K_0\| < \varepsilon/4$ such that

$\psi_0(T)+K_0 \underset{\mathcal{U}+\mathcal{K}}{\cong} \psi_0(A)$, where T is given by Theorem 1.4, i.e., there is an invertible operator X of the form unitary plus compact such that $X(\psi_0(T)+K_0)X^{-1} = \psi_0(A)$.

By Lemma 2.3, we can find a K_1 compact with $\|K_1\| < \varepsilon/\|X\|\|X^{-1}\|$ such that

$$X(\psi_0(T)+K_0+X^{-1}K_1X)X^{-1} = \psi_0(A)+K_1 \in (SI).$$

This means that there exists a K compact with $\|K\| < \varepsilon/2$ such that $\psi_0(T)+K \in (SI)$. Note that $\sigma(\psi_0(T)+K) \subset \overline{D}$. Thus we can deduce that $\varphi_0(\psi_0(T)+K) = T+\widetilde{K} \in (SI)$ by using the following Theorem.

Theorem J.[11, Theorem 2.8] *If $T \in (SI)$ and φ is a conformal one-to-one mapping in a neighbourhood of $\sigma(T)$, then $\varphi(T) \in (SI)$.*

It is easy to see that $\overline{K} = \varphi_0(\psi_0(T)+K) - T$ is compact and $\|\overline{K}\| \to 0$.

§3. Proof of Theorem 1.1 and Theorem 1.2

Proof of Thoerem 1.1 Let operator T, open set Ω and compact set σ be given in Theorem 1.1. And let φ be a conformal one-to-one mapping of D onto Ω^*. $D = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n)$, where $\{\lambda_n\}_{n=1}^{\infty}$ is dense in σ.

Set

$$A = \begin{bmatrix} \overset{m}{\underset{1}{\oplus}} T_\varphi^* & 0 \\ 0 & D \end{bmatrix}.$$

Directly using the proof of Thorem 1.3, we conclude that if $\Lambda(T) = \Lambda(A)$, then for each $\varepsilon > 0$, there exists a K compact with $\|K\| < \varepsilon$ such that $T+K \underset{\mathcal{U}+\mathcal{K}}{\cong} A$.

Simlarly, we can prove Theorem 1.2 by using the arguments in Theorem 1.4.

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