Chin. Ann. of Math.
19B: 2(1998),229-238.

A CRACK PROBLEM WITH A BROKEN LINE INTERFACE***

LU JIANKE* ERDOGAN FAZIL**

Abstract

The equilibrium problem for the infinite elastic plane consisting of two different media is considered, in which the interface is a broken line, there is a straight crack touching the vertex of the broken line with some symmetry and the same uniform pressures are applied to both of its sides. The problem is reduced to a uniquely solvable singular integral equation on the interface and the crack. The order of singularity at the vertex is considered, which may be determined by a transcendental equation.

Keywords Crack, Interface, Kolosov functions, Singular integral equation
1991 MR Subject Classification 73V35, 45E05
Chinese Library Classification O175.5, O343.1

§0. Introduction

Considerable plane crack problems of composite media for the case in which the interface is an infinite straight line were studied by various authors, for instance, [1,2]. The methods of solution are often by using integral transforms so as to reduce the problems to singular integral equations. In this paper we shall consider the problem for the case where the interface is a broken line consisting of two half-rays and there is a crack touching the interface at its vertex with some symmetry both in the elastic region and the boundary condition. We shall also reduce the problem to a singular integral equation along the crack and the interface directly by a method inspired by but different from the one originated by Sherman[3] for solving elasto-static problems without cracks. The method similar to our method used here was first developed in [4, 5] for solving problems of two bonded half-planes with cracks arbitrarily both in shape and number. The proposed method seems universally effective for general plane crack problems.

§1. Formulation of the Problem

Consider an elastic infinite plane consisting of two isotropic media S_1 and S_2 with elastic constants κ_1, μ_1 and κ_2, μ_2 respectively. The interface in the plane is a broken straight line consisting of two half-rays L, L' issued from the origin O and extending to infinity

Manuscript received April 4, 1995. Revised May 26, 1997.
*Institute of Mathematics, Wuhan University, Wuhan 430072, China.
**Department of Mechanical Engineering and Mechanics, Lehigh University, Bethlehem, PA 18015, U.S.A.
***Project supported by the National Natural Science Foundation of China.

respectively with angles of inclination $\pm\theta$ $(0 < \theta < \pi)$ with respect to the x-axis. There exists a crack γ in S_1 on the positive x-axis: $0 \le x \le a$ $(a > 0)$. Assume that a uniform tension p is applied to both sides of γ and that there is no stress and rotation at infinity. It is required to find the elastic equilibrium.

When $\theta = \frac{\pi}{2}$, the problem was solved in [7].

The stress $\sigma_x(z)$, $\sigma_y(z)$, $\tau_{xy}(z)$ and the displacement $u(z) + iv(z)$ $(z = x + iy)$ may be expressed in terms of Kolosov functions $\phi(z)$, $\psi(z)$ (or $\Phi(z) = \phi'(z)$, $\Psi(z) = \psi'(z)$) as (see [6])

$$\begin{aligned}\sigma_y + \sigma_x &= 4\mathrm{Re}\{\phi'(z)\} = 4\mathrm{Re}\{\Phi(z)\},\\ \sigma_y - \sigma_x + 2i\tau_{xy} &= 2[\bar{z}\phi''(z) + \psi'(z)] = 2[\bar{z}\Phi'(z) + \Psi(z)],\\ 2\mu(u + iv) &= \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)},\end{aligned} \tag{1.1}$$

where $\phi(z)$ and $\psi(z)$ are analytic functions in the elastic region. In our case, they are sectionally holomorphic in S_1 and S_2, and $\phi(\infty) = \psi(\infty) = 0$ may be assumed.

By the boundary condition on γ, we have

$$\phi^{\pm}(x) + x\overline{\phi'^{\pm}(x)} + \overline{\psi^{\pm}(x)} = -px + c, \quad x \in \gamma\ (0 \le x \le a), \tag{1.2}$$

where $\phi^{\pm}(x)$ and $\psi^{\pm}(x)$ are the boundary values of $\phi(z)$ and $\psi(z)$ when $z \to x \in \gamma$ from its upper (positive) and lower (negative) sides respectively.

By the condition of equivalence for the normal and the shearing stresses along the interface, we have

$$\phi^{+}(t) + t\overline{\phi'^{+}(t)} + \overline{\psi^{+}(t)} = \phi^{-}(t) + t\overline{\phi'^{-}(t)} + \overline{\psi^{-}(t)}, \quad t \in L + L', \tag{1.3}$$

where $\phi^{\pm}(t)$ and $\psi^{\pm}(t)$ are the boundary values of $\phi(z)$ and $\psi(z)$ when $z \to t \in L + L'$ from the left and the right sides of L, L' respectively.

Moreover, the continuity of the displacements along the interface gives

$$\alpha_2\phi^{+}(t) - \beta_2[t\overline{\phi'^{+}(t)} + \overline{\psi^{+}(t)}] = \alpha_1\phi^{-}(t) - \beta_1[t\overline{\phi'^{-}(t)} + \overline{\psi^{-}(t)}],\ t \in L, \tag{1.4}$$

$$\alpha_1\phi^{+}(t) - \beta_1[t\overline{\phi'^{+}(t)} + \overline{\psi^{+}(t)}] = \alpha_2\phi^{-}(t) - \beta_2[t\overline{\phi'^{-}(t)} + \overline{\psi^{-}(t)}],\ t \in L', \tag{1.5}$$

where we have put

$$\alpha_j = \kappa_j/\mu_j, \quad \beta_j = 1/\mu_j, \quad j = 1, 2. \tag{1.6}$$

Thus, our problem has been transferred to the boundary value problem (1.2)–(1.5) for sectionally holomorphic functions $\phi(z)$, $\psi(z)$ with $\Gamma = L + L' + \gamma$ as the jump curve and $\phi(\infty) = \psi(\infty) = 0$.

By the symmetry of the stresses and displacements, it is readily seen that

$$\overline{\phi(\bar{z})} = \phi(z), \quad \overline{\psi(\bar{z})} = \psi(z). \tag{1.7}$$

§2. Reduction to Singular Integral Equations

For solving our boundary value problem we introduce a new unknown function $\omega(\zeta)$, $\zeta \in \Gamma$, such that

$$\phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\zeta)}{\zeta - z}d\zeta, \quad \psi(z) = -\frac{1}{2\pi i}\int_\Gamma \frac{\overline{\omega(\zeta)} + \zeta\omega'(\zeta)}{\zeta - z}d\zeta, \quad z \notin \Gamma, \tag{2.1}$$

where we have assumed $\omega(\zeta) \in H_0$, $\omega'(\zeta) \in H^*$ (for notations, see [6]). Let $\omega(\zeta) = \omega(x)$ when $\zeta = x \in \gamma$, $\omega(\zeta) = \omega_L(t)$ and $\omega(\zeta) = \omega_{L'}(t)$ when $\zeta = t \in L$ and L' respectively. We assume

$$\omega(0) + \omega_L(0) + \omega_{L'}(0) = 0, \quad \omega(a) = 0, \quad \omega_L(\infty) = \omega_{L'}(\infty) = 0. \tag{2.2}$$

Of course, the existence of the function $\omega(z)$ satisfying (2.1) as well as (2.2) should be proved, but it is assumed for the time being.

By (1.7) and (2.1), it is easy to see that

$$\overline{\omega(\zeta)} = -\omega(\bar{\zeta}), \tag{2.3}$$

by which it follows that $\omega(x)$ is pure imaginary on γ.

Substituting (2.1) in (1.2), and by using the Plemelj formula for the boundary values of both sides on γ and integration by parts and noting that the terms not involving integrals cancel each other by (2.2), we get the same singular integral equation on γ:

$$\begin{aligned} \mathbf{K}_\gamma \omega \equiv \frac{1}{\pi i}\int_\Gamma \frac{\omega(\zeta)}{\zeta - x} d\xi - \frac{1}{2\pi i}\int_{L+L'} \omega(\zeta) d\log\frac{\zeta - x}{\bar{\zeta} - x} - \frac{1}{2\pi i}\int_{L+L'} \omega(\zeta) d\frac{\zeta - x}{\bar{\zeta} - x} \\ = -px + c, \quad 0 \le x \le a, \end{aligned} \tag{2.4}$$

which together with (2.3) gives

$$\begin{aligned} \mathbf{K}_\gamma \omega \equiv \frac{1}{\pi i}\int_0^a \frac{\omega(\xi)}{\xi - x} d\zeta + \mathrm{Re}\Big\{\frac{1}{\pi i}\int_L \frac{\omega(\tau)}{\tau - x} d\tau + \frac{1}{\pi i}\int_L \frac{\omega(\tau)}{\bar{\tau} - x} d\bar{\tau} - \frac{1}{\pi i}\int_L \overline{\omega(\tau)} d\frac{\tau - x}{\bar{\tau} - x}\Big\} \\ = -px + c, \quad 0 \le x \le a. \end{aligned} \tag{2.4$'$}$$

Condition (1.3) is found to be automatically satisfied by using (2.1). Similarly, condition (1.4) becomes a singular integral equation on L:

$$\mathbf{K}_L \omega \equiv A\omega(t) + \frac{B}{\pi i}\int_\Gamma \frac{\omega(\zeta)}{\zeta - t} d\zeta + \frac{D}{\pi i}\Big\{\int_{L'+\gamma} \omega(\zeta) d\log\frac{\zeta - t}{\bar{\zeta} - \bar{t}} + \int_{L'+\gamma} \overline{\omega(\zeta)} d\frac{\zeta - t}{\bar{\zeta} - \bar{t}}\Big\} = 0, \quad t \in L, \tag{2.5}$$

or,

$$\begin{aligned} \mathbf{K}_L \omega \equiv A\omega(t) + \frac{B}{\pi i}\Big\{\int_L \frac{\omega(\tau)}{\tau - t} d\tau - \int_L \frac{\overline{\omega(\tau)}}{\bar{\tau} - t} d\bar{\tau} + \int_0^a \frac{\omega(\xi)}{\xi - t} d\xi\Big\} \\ + \frac{D}{\pi i}\Big\{\int_0^a \omega(\xi) d\log\frac{\xi - t}{\xi - \bar{t}} + \int_0^a \overline{\omega(\xi)} d\frac{\xi - t}{\xi - \bar{t}} - \int_L \overline{\omega(\tau)} d\log\frac{\bar{\tau} - t}{\tau - \bar{t}} - \int_L \omega(\tau) d\frac{\bar{\tau} - t}{\tau - \bar{t}}\Big\} \\ = 0, \quad t \in L, \end{aligned} \tag{2.5$'$}$$

where we have set

$$A = \alpha_2 + \alpha_1 + \beta_2 + \beta_1, \quad B = \alpha_2 - \alpha_1 - \beta_2 + \beta_1, \quad D = \beta_2 - \beta_1, \tag{2.6}$$

and we shall denote $C = B + D = \alpha_2 - \alpha_1$ in the sequel.

(2.4) and (2.5) constitute a singular integral equation in $\omega(\zeta)$ on $L+\gamma$, which is of normal type since $A + B = 2(\alpha_2 + \beta_1)$ and $A - B = 2(\alpha_1 + \beta_2)$ are positive constants. Its solution to be found should belong to the most narrow class h, that is, it must be bounded at 0 and a as well as at ∞, and hence its index corresponding to this class is -1.

We prove that, if (2.4)–(2.5) has a solution $\omega(\zeta) \in h$ (for certain constant c), then (2.2) is fulfilled. In fact, if the first equation in (2.2) is not fulfilled, then the left side of (2.5) would have a logarithmic singularity at $t = 0$ while each integral in the braces of (2.5) tends to zero evidently; this is impossible since the right-hand side of (2.5) is zero. $\omega(a) = 0$ is evident as readily seen by (2.4). Let $\omega(t) \to \omega(\infty)$ as $t \to \infty$ on L. It is easy to prove that the third term on the left-hand side of (2.5) tends to zero, and also $\int_{L'+\gamma} \frac{\omega(\zeta)}{\zeta - t} d\zeta \to 0$, while

$$\frac{1}{\pi i} \int_L \frac{\omega(\tau)}{\tau - t} d\tau = \frac{1}{\pi i} \int_L \frac{\omega(\tau) - \omega(\infty)}{\tau - t} d\tau + \frac{\omega(\infty)}{\pi i} \int_L \frac{d\tau}{\tau - t},$$

where the first term of the right-hand member is convergent when $t \to \infty$ since $\omega(\tau) \in H$ on L, and the second term is divergent if $\omega(\infty) \neq 0$. Therefore $\omega(t) \to 0$ as $t \to \infty$ along L.

By (2.3), we see that $\overline{\omega_L(0)} = -\omega_{L'}(0)$ so that

$$\omega(0) + 2i\mathrm{Im}\omega_L(0) = 0. \tag{2.7}$$

§3. Unique Solvability for the Integral Equation

We shall prove that (2.4)–(2.5) has a unique solution in class h for certain (uniquely) suitably chosen constant c.

First, we show that the corresponding homogeneous equation $\mathbf{K}_\gamma \omega = 0$, $\mathbf{K}_L \omega = 0$ has only the trivial solution in h, and more generally, $\mathbf{K}_\gamma \omega = c$, $\mathbf{K}_L \omega = 0$ is solvable only when $c = 0$ and the solution is zero. Assume $\omega_0(\zeta)$ is a solution of them for certain c. When substituting it back into (2.1), we get two sectionally holomorphic functions $\phi_0(z)$ and $\psi_0(z)$ which satisfy (1.2)–(1.5) with $p = 0$. This is the natural equilibrium state for the elastic body without any external loads on the crack and stress or rotation at infinity. By the uniqueness theorem in elasticity, we must have $\phi_0(z) \equiv 0$, $\psi(z) \equiv 0$ since $\phi_0(\infty) = \psi_0(\infty) = 0$. Then, by (1.2), we know that $c = 0$ and

$$\frac{1}{2\pi i} \int_\Gamma \frac{\omega_0(\zeta)}{\zeta - z} d\zeta = 0, \quad \frac{1}{2\pi i} \int_\Gamma \frac{\overline{\omega_0(\zeta)} + \zeta \omega_0'(\zeta)}{\zeta - z} d\zeta = 0, \quad z \notin \Gamma.$$

Applying the Plemelj formula to the first equation of above, we obtain

$$0 = \pm \frac{1}{2} \omega_0(\zeta_0) + \frac{1}{2\pi i} \int_\Gamma \frac{\omega_0(\zeta)}{\zeta - \zeta_0} d\zeta, \quad \zeta_0 \in \Gamma,$$

from which it follows that $\omega_0(\zeta_0) = 0$, $\zeta_0 \in \Gamma$, what is to be proved. Thus, the number of linearly independent solutions for $\mathbf{K}_\gamma \omega = 0$, $\mathbf{K}_L \omega = 0$ in class h is $l = 0$.

By the Noether theorem, the number of linearly independent solutions for the adjoint equation $\mathbf{K'}_\gamma \omega = 0$, $\mathbf{K'}_L \omega = 0$ in the adjoint class h_0 is $l' = 1$ since $\kappa = -1$. Here class h_0 (the widest class) means that the solutions of the equation may have integrable singularities at $\zeta = 0$, a and ∞. Assume $\chi_0(\zeta)$ ($\neq 0$) is the unique solution (up to a nonzero constant coefficient factor). Then (2.4)–(2.5) is (uniquely) solvable if and only if

$$c \int_0^a \chi_0(x) dx = p \int_0^a \chi_0(x) dx - \int_L \chi_0(\tau) d\tau \tag{3.1}$$

is fulfilled. We show that $\int_0^a \chi_0(x) dx \neq 0$. In fact, we consider again the case $p = 0$. We have proved that (2.4)–(2.5) is solvable only when $c = 0$. It means that the equation

$$c \int_0^a \chi_0(x) dx = - \int_L \chi_0(\tau) d\tau$$

holds only when $c = 0$, so that $\int_0^a \chi_0(x) dx \neq 0$ (and $\int_L \chi_0(\tau) d\tau = 0$). Thus, (3.1) is uniquely solvable for c.

Note that $\omega(x)$ is pure imaginary on γ. Hence, if we denote

$$\omega(x) = \omega_0(x)i, \quad 0 \le x \le a, \tag{3.2}$$

then $\omega_0(x) \in H$ is real with $\omega_0(a) = 0$ and satisfies (by denoting $\omega_L(t) = \omega(t)$ on L)

$$\omega_0(0) + 2\mathrm{Im}\omega(0) = 0; \tag*{(2.7)$'$}$$

(2.4)$'$ and (2.5)$'$ become respectively

$$\begin{aligned}\mathbf{K}_\gamma\omega \equiv{}& \frac{1}{\pi i}\int_0^a \frac{\omega(\xi)}{\xi - x}d\xi + \mathrm{Re}\Big\{\frac{1}{\pi i}\int_L \frac{\omega(\tau)}{\tau - x}d\tau + \frac{1}{\pi i}\int_L \frac{\omega(\tau)}{\bar\tau - x}d\bar\tau + \frac{1}{\pi i}\int_L \omega(\tau)d\frac{\bar\tau - x}{\tau - x}\Big\} \\ ={}& -px + c, \quad 0 \le x \le a,\end{aligned} \tag{3.3}$$

$$\begin{aligned}\mathbf{K}_L\omega \equiv{}& A\omega(t) + B\Big\{\frac{1}{\pi i}\int_L \frac{\omega(\tau)}{\tau - t}d\tau - \frac{1}{\pi i}\int_L \frac{\overline{\omega(\tau)}}{\bar\tau - t}d\bar\tau + \frac{1}{\pi}\int_0^a \frac{\omega_0(\xi)}{\xi - t}d\xi\Big\} \\ &+ \frac{D}{\pi}\Big\{\int_0^a \omega_0(\xi)d\log\frac{\xi - t}{\xi - \bar t} + \int_0^a \overline{\omega_0(\xi)}d\frac{\xi - t}{\xi - \bar t}\Big\} \\ &- \frac{D}{\pi i}\Big\{\int_L \overline{\omega(\tau)}d\log\frac{\bar\tau - t}{\tau - \bar t} + \int_L \omega(\tau)d\frac{\bar\tau - t}{\tau - \bar t}\Big\} = 0, \quad t \in L.\end{aligned} \tag{3.4}$$

Thus our problem is reduced to solving (3.3)–(3.4) in class h, where c is an undetermined constant; it is hence uniquely solvable when c is chosen to fulfil (3.1), where χ_0 is the unique solution of the adjoint equation $\mathbf{K'}_\gamma\omega = 0$, $\mathbf{K'}_L\omega = 0$ in class h_0.

§4. Simplification of the Equations

It is rather complicated to solve (3.3)–(3.4) since the determination of c involves to solve the adjoint equation on $L + \gamma$. But in practice it is often sufficient to determine the stress distribution which depends on $\Phi(z)$ and $\Psi(z)$ and so indirectly on $\Omega(\zeta) = \omega'(\zeta)$. Hence it is sufficient for us to get $\Omega(\zeta)$ instead of $\omega(\zeta)$ itself. The equation satisfied by $\Omega(\zeta)$ may be obtained by differentiating (3.3)–(3.4), where the undetermined constant c disappears. Let

$$\Omega_0(x) = \omega_0'(x),\ x \in \gamma; \quad \Omega(\tau) = \omega'(\tau),\ \tau \in L.$$

By (2.3) we have

$$\overline{\Omega(\zeta)} = -\Omega(\bar\zeta). \tag{4.1}$$

By diffrentiation and integration by parts in (3.3)–(3.4), we obtain respectively

$$\begin{aligned}\mathbf{K}_\gamma'\Omega \equiv{}& \frac{1}{\pi}\int_0^a \frac{\Omega_0(\xi)}{\xi - x}d\xi + \mathrm{Re}\Big\{\frac{1}{\pi i}\int_L \frac{\Omega(\tau)}{\tau - x}d\tau + \frac{1}{\pi i}\int_L \frac{\Omega(\tau)}{\bar\tau - x}d\tau + \frac{1}{\pi i}\int_L \frac{\tau - \bar\tau}{(\tau - x)^2}\Omega(\tau)d\tau\Big\} \\ ={}& -p, \quad 0 < x < a,\end{aligned} \tag{4.2}$$

$$\begin{aligned}\mathbf{K}_L'\Omega \equiv{}& A\Omega(t) + B\Big\{\frac{1}{\pi i}\int_L \frac{\Omega(\tau)}{\tau - t}d\tau - \frac{1}{\pi i}\int_L \frac{\overline{\Omega(\tau)}}{\bar\tau - t}d\bar\tau + \frac{1}{\pi}\int_0^a \frac{\Omega_0(\xi)}{\xi - t}d\xi\Big\} \\ &+ D\Big\{\frac{1}{\pi}\int_0^a \frac{\Omega_0(\xi)}{\xi - t}d\xi - \frac{e^{-2i\theta}}{\pi}\int_0^a \frac{\Omega_0(\xi)}{\xi - \bar t}d\xi + \frac{(1 - e^{-2i\theta})}{\pi}\int_0^a \frac{\xi\Omega_0(\xi)}{(\xi - \bar t)^2}d\xi \\ &- \frac{1}{\pi i}\int_L \frac{\overline{\Omega(\tau)}}{\bar\tau - t}d\bar\tau + \frac{e^{-2i\theta}}{\pi i}\int_L \frac{\overline{\Omega(\tau)}}{\tau - \bar t}d\bar\tau + \frac{1}{\pi i}\int_L \frac{-\tau + \bar\tau e^{-2i\theta}}{(\tau - \bar t)^2}\Omega(\tau)d\tau\Big\} = 0, \quad t \in L,\end{aligned}$$

or

$$
\begin{aligned}
\mathbf{K}'_L\Omega \equiv\ & A\Omega(t) + \frac{B}{\pi i}\int_L \frac{\Omega(\tau)}{\tau - t}d\tau - \frac{C}{\pi i}\int_L \frac{\overline{\Omega(\tau)}}{\bar\tau - t}d\bar\tau + \frac{C}{\pi}\int_0^a \frac{\Omega_0(\xi)}{\xi - t}d\xi \\
& - \frac{De^{-2i\theta}}{\pi}\int_0^a \frac{\Omega_0(\xi)}{\xi - \bar t}d\xi + \frac{D(1-e^{-2i\theta})}{\pi}\int_0^a \frac{\xi\Omega_0(\xi)}{(\xi - \bar t)^2}d\xi \\
& + \frac{De^{-2i\theta}}{\pi i}\int_L \frac{\overline{\Omega(\tau)}}{\tau - \bar t}d\bar\tau + \frac{D}{\pi i}\int_L \frac{-\tau + \bar\tau e^{-2i\theta}}{(\tau - \bar t)^2}\Omega(\tau)d\tau = 0, \quad t \in L. \qquad (4.3)
\end{aligned}
$$

Now, $\Omega(\zeta) \in H$ must satisfy, by (2.7),

$$
\int_0^a \Omega_0(x)dx + 2\mathrm{Im}\int_L \Omega(t)dt = 0. \qquad (4.4)
$$

We may change complex variables to real ones by letting $\tau = \rho e^{i\theta}$, $t = re^{i\theta}(0 \le \rho, r < +\infty)$, and denote $\Omega(\tau) = \Omega(\rho)$ on γ. Then (4.2)–(4.3) becomes

$$
\begin{aligned}
\mathbf{K}'_\gamma\Omega \equiv\ & \mathrm{Re}\Big\{\frac{1}{\pi i}\int_0^\infty \frac{\Omega(\rho)d\rho}{\rho - xe^{-i\theta}} + \frac{e^{2i\theta}}{\pi i}\int_0^\infty \frac{\Omega(\rho)d\rho}{\rho - xe^{i\theta}} + \frac{1-e^{-2i\theta}}{\pi i}\int_0^\infty \frac{\rho\Omega(\rho)d\rho}{(\rho - xe^{-i\theta})^2}\Big\} \\
& + \frac{1}{\pi}\int_0^a \frac{\Omega_0(\xi)}{\xi - x}d\xi = -p, \quad 0 < x < a, \qquad (4.5)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{K}'_L\Omega \equiv\ & A\Omega(t) + \frac{B}{\pi i}\int_0^\infty \frac{\Omega(\rho)}{\rho - r}d\rho - \frac{C}{\pi i}\int_0^\infty \frac{\overline{\Omega(\rho)}d\rho}{\rho - re^{2i\theta}} + \frac{C}{\pi}\int_0^a \frac{\Omega_0(\xi)}{\xi - re^{i\theta}}d\xi \\
& - \frac{De^{-2i\theta}}{\pi}\int_0^a \frac{\Omega_0(\xi)d\xi}{\xi - re^{-i\theta}}d\xi + \frac{D(1-e^{-2i\theta})}{\pi}\int_0^a \frac{\xi\Omega_0(\xi)d\xi}{(\xi - re^{-i\theta})^2} \\
& + \frac{De^{-4i\theta}}{\pi i}\int_0^\infty \frac{\overline{\Omega(\rho)}d\rho}{\rho - re^{-2i\theta}} + \frac{D(e^{-4i\theta}-1)}{\pi i}\int_0^\infty \frac{\rho\Omega(\rho)d\rho}{(\rho - re^{-2i\theta})^2} \\
=\ & 0, \quad t = re^{i\theta} \in L, \qquad (4.6)
\end{aligned}
$$

while (4.4) becomes

$$
\int_0^a \Omega_0(x)dx + 2\mathrm{Im}\Big\{e^{i\theta}\int_0^\infty \Omega(r)dr\Big\} = 0,
$$

or, if we denote $\Omega(r) = \Omega_1(r) + i\Omega_2(r)$,

$$
\int_0^a \Omega_0(x)dx + 2\Big\{\sin\theta\int_0^\infty \Omega_1(r)dr + \cos\theta\int_0^\infty \Omega_2(r)dr\Big\} = 0. \qquad (4.7)
$$

Thus our problem is reduced to solving (4.5)–(4.6) for the real $\Omega_0(x)$ $(0 < x < a)$ and the complex $\Omega(r) = \Omega_1(r) + i\Omega_2(r)$ $(0 < r < \infty)$ in class h_0 with the additional requirement (4.7), of which the solution is unique as proved.

§5. Determination of the Order of Singularity for the Solution

From the practical point of view it is very important to determine the orders of the singularities for the solution of the problem at the tips of the crack so as to know the behavior of the stresses near them. It is well-known that at the tip a the singularity is of order $\frac{1}{2}$. For determining the order of singularity at the tip O, we assume

$$
\Omega_0(x) = \frac{\Omega_0}{x^\alpha} + o(x^{-\alpha}) \ (x \to +0), \quad \Omega(r) = \frac{\Omega}{r^\alpha} + o(r^{-\alpha}) \ (r \to +0), \qquad (5.1)
$$

where α is the undetermined order which must be real in our problem and $0 < \alpha < 1$, while Ω_0 (real) and $\Omega = \Omega_1 + \Omega_2 i$ are undermined constants.

In the sequel, the function z^α is chosen as that continuous branch in the plane cut along the positive real axis which takes positive real values on its upper side. Then $z^\alpha = x^\alpha e^{\alpha i\theta}$ when $z = xe^{i\theta}$ and $z^\alpha = x^\alpha e^{\alpha i(2\pi-\theta)}$ when $z = xe^{-i\theta}$ $(x > 0)$. Similar equalities are valid for r (> 0) in place of x as well as 2θ in place of θ $(0 < \theta < \pi)$.

By the properties of Cauchy-type integrals and Cauchy principal value integrals near the end of the path of integration (see [6]), we have the following equalities:

$$\frac{1}{\pi}\int_0^a \frac{\Omega_0(\xi)}{\xi - x}d\xi = \frac{\cos\alpha\pi}{x^\alpha \sin\alpha\pi}\Omega_0 + \cdots ,$$

$$\mathrm{Re}\Big\{\frac{1}{\pi i}\int_0^\infty \frac{\Omega(\rho)d\rho}{\rho - xe^{-i\theta}}\Big\} = \frac{1}{x^\alpha \sin\alpha\pi}\{\cos\alpha(\pi-\theta)\Omega_2 - \sin\alpha(\pi-\theta)\Omega_1\} + \cdots ,$$

$$\mathrm{Re}\Big\{\frac{e^{2i\theta}}{\pi i}\int_0^\infty \frac{\Omega(\rho)d\rho}{\rho - xe^{-i\theta}}\Big\} = \frac{1}{x^\alpha \sin\alpha\pi}\{\cos[\alpha(\pi-\theta)+2\theta]\Omega_2 + \sin[\alpha(\pi-\theta)+2\theta]\Omega_1\} + \cdots ,$$

$$\begin{aligned}
&\mathrm{Re}\Big\{\frac{1-e^{-2i\theta}}{\pi i}\int_0^\infty \frac{\rho\Omega(\rho)d\rho}{(\rho - xe^{-i\theta})^2}\Big\}\\
&= \mathrm{Re}\frac{1-e^{-2i\theta}}{\pi i}\Big\{\int_0^\infty \frac{\Omega(\rho)d\rho}{\rho - xe^{-i\theta}} + x\frac{d}{dx}\int_0^\infty \frac{\Omega(\rho)d\rho}{\rho - xe^{-i\theta}}\Big\}\\
&= \frac{2(1-\alpha)\sin\theta}{\sin\alpha\pi x^\alpha}\{\cos[\alpha(\pi-\theta)-\theta]\Omega_1 + \sin[\alpha(\pi-\theta)-\theta]\Omega_2\} + \cdots ,
\end{aligned}$$

where the omitted terms are of orders less than α.

Substituting these equalities in (4.5) which is bounded on its right-side and multiplying the resulting equation by $\sin\alpha\pi x^\alpha$, we obtain

$$a_{00}(\alpha)\Omega_0 + a_{01}(\alpha)\Omega_1 + a_{02}(\alpha)\Omega_2 = 0, \tag{5.2}$$

where

$$\begin{aligned}
a_{00}(\alpha) &= \cos\alpha\pi,\\
a_{01}(\alpha) &= -\sin\alpha(\pi-\theta) + \sin[\alpha(\pi-\theta)+2\theta] + 2(1-\alpha)\sin\theta\cos[\alpha(\pi-\theta)+\theta],\\
a_{02}(\alpha) &= \cos\alpha(\pi-\theta) + \cos[\alpha(\pi-\theta)+2\theta] + 2(1-\alpha)\sin\theta\cos[\alpha(\pi-\theta)+\theta].
\end{aligned}$$

We also have

$$\begin{aligned}
\frac{1}{\pi i}\int_0^\infty \frac{\Omega(\rho)}{\rho - r}d\rho &= \frac{\cos\alpha\pi}{r^\alpha\sin\alpha\pi}(\Omega_2 - i\Omega_1) + \cdots ,\\
-\frac{1}{\pi i}\int_0^\infty \frac{\overline{\Omega(\rho)}d\rho}{\rho - re^{2i\theta}} &= \frac{e^{i\alpha(\pi-2\theta)}(\Omega_2 + i\Omega_1)}{ir^\alpha\sin\alpha\pi} + \cdots ,\\
\frac{1}{\pi}\int_0^a \frac{\Omega_0(\xi)}{\xi - re^{i\theta}}d\xi &= \frac{e^{i\alpha(\pi-\theta)}\Omega_0}{r^\alpha\sin\alpha\pi} + \cdots ,\\
-\frac{e^{-2i\theta}}{\pi}\int_0^a \frac{\Omega_0(\xi)d\xi}{\xi - re^{-i\theta}} &= \frac{e^{-i[\alpha(\pi-\theta)+2\theta]}\Omega_0}{r^\alpha\sin\alpha\pi} + \cdots ,\\
\frac{1-e^{-2i\theta}}{\pi}\int_0^a \frac{\xi\Omega_0(\xi)d\xi}{(\xi - re^{-i\theta})^2} &= (1-e^{-2i\theta})(1-\alpha)\frac{1}{\pi}\int_0^a \frac{\Omega_0(\xi)}{\xi - re^{-i\theta}}d\xi\\
&= (1-\alpha)\frac{2i\sin\theta e^{-i[\alpha(\pi-\theta)+\theta]}\Omega_0}{r^\alpha\sin\alpha\pi} + \cdots ,\\
\frac{e^{-4i\theta}}{\pi i}\int_0^\infty \frac{\overline{\Omega(\rho)}d\rho}{\rho - re^{-2i\theta}} &= \frac{e^{-i[\alpha(\pi-2\theta)+4\theta]}(-i\Omega_1 - \Omega_2)}{r^\alpha\sin\alpha\pi} + \cdots ,
\end{aligned}$$

$$
\begin{aligned}
&\frac{e^{-4i\theta}-1}{\pi i}\int_0^\infty \frac{\rho\Omega(\rho)d\rho}{(\rho-re^{-2i\theta})^2}\\
&=(e^{-4i\theta}-1)\Big\{\frac{1}{\pi i}\int_0^\infty \frac{\Omega(\rho)d\rho}{\rho-re^{-2i\theta}}+r\frac{d}{dr}\frac{1}{\pi i}\int_0^\infty \frac{\Omega(\rho)d\rho}{\rho-re^{-2i\theta}}\Big\}+\cdots,\\
&=-2(1-\alpha)\sin 2\theta\frac{e^{-i[\alpha(\pi-2\theta)+2\theta]}(\Omega_1+i\Omega_2)}{r^\alpha\sin\alpha\pi}+\cdots.
\end{aligned}
$$

Substituting these equalities in (4.6), multiplying the resulting equation by $r^\alpha\sin\alpha\pi$ and separating the real and imaginary parts, we get

$$a_{10}(\alpha)\Omega_0+a_{11}(\alpha)\Omega_1+a_{12}(\alpha)\Omega_2=0, \tag{5.3}$$

$$a_{20}(\alpha)\Omega_0+a_{21}(\alpha)\Omega_1+a_{22}(\alpha)\Omega_2=0, \tag{5.4}$$

where

$$
\begin{aligned}
a_{10}&=C\cos\alpha(\pi-\theta)-D\cos[\alpha(\pi-\theta)+2\theta]+2D(1-\alpha)\sin\theta\sin[\alpha(\pi-\theta)+\theta],\\
a_{11}&=A\sin\alpha\pi-C\sin\alpha(\pi-2\theta)-D\sin[\alpha(\pi-2\theta)+4\theta]\\
&\quad-2D(1-\alpha)\sin 2\theta\cos[\alpha(\pi-2\theta)+2\theta],\\
a_{12}&=B\cos\alpha\pi+C\cos\alpha(\pi-2\theta)-D\cos[\alpha(\pi-2\theta)+4\theta]\\
&\quad-2D(1-\alpha)\sin 2\theta\sin[\alpha(\pi-2\theta)+2\theta],\\
a_{20}&=C\sin\alpha(\pi-\theta)+D\sin[\alpha(\pi-\theta)+2\theta]+2D(1-\alpha)\sin\theta\cos[\alpha(\pi-\theta)+\theta],\\
a_{21}&=-B\cos\alpha\pi+C\cos\alpha(\pi-2\theta)-D\cos[\alpha(\pi-2\theta)+4\theta]\\
&\quad+2D(1-\alpha)\sin 2\theta\sin[\alpha(\pi-2\theta)+2\theta],\\
a_{22}&=A\sin\alpha\pi+C\sin\alpha(\pi-2\theta)+D\sin[\alpha(\pi-2\theta)+4\theta]\\
&\quad-2D(1-\alpha)\sin 2\theta\cos[\alpha(\pi-2\theta)+2\theta].
\end{aligned}
$$

Thus α is the root of the determinant equation

$$\Delta(\alpha)=|a_{jk}(\alpha)|=0,\quad j,k=0,1,2; \tag{5.5}$$

α exists in the interval (0,1) by the general theory.

Remark 5.1. It is worthy to note that Ω_0, Ω_1 and Ω_2 must be proportional to the values Δ_0, Δ_1 and Δ_2 of the algebraic complements of the elements in the first row of the determinant $\Delta(\alpha)$ for the root α obtained above.

§6. The Kolosov Functions

After the solution of (5.5) in $0<\alpha<1$ is obtained, equations (4.5)–(4.7) in class h_0 may be solved numerically by various methods. When $\Omega_0(x)$ and $\Omega(\zeta)$ are obtained, the Kolosov functions $\Phi(z)$ and $\Psi(z)$ are easily calculated. In fact, by (2.1) and (4.1), we obtain

$$\Phi(z)=\frac{1}{2\pi i}\int_\Gamma\frac{\Omega(\zeta)}{\zeta-z}d\zeta=\frac{1}{2\pi i}\int_0^\infty\frac{\Omega(\rho)d\rho}{\rho-ze^{-i\theta}}-\frac{1}{2\pi i}\int_0^\infty\frac{\overline{\Omega(\rho)}d\rho}{\rho-ze^{-i\theta}}+\frac{1}{2\pi}\int_0^a\frac{\Omega_0(\xi)}{\xi-z}d\xi, \tag{6.1}$$

$$\Psi(z)=-\frac{1}{2\pi i}\int_L\frac{\overline{\Omega(\zeta)}}{\zeta-z}d\bar\zeta-\frac{1}{2\pi i}\int_\Gamma\frac{\zeta\Omega(\zeta)}{(\zeta-z)^2}d\zeta. \tag{6.2}$$

For the stress distribution, by (1.1), it is more important for us to have

$$
\begin{aligned}
&\bar{z}\Phi'(z)+\Psi(z)\\
&=-\frac{1}{2\pi i}\int_L \frac{\overline{\Omega(\zeta)}}{\zeta-z}d\bar{\zeta}-\frac{1}{2\pi i}\int_\Gamma \frac{(\zeta-\bar{z})\Omega(\zeta)}{(\zeta-z)^2}d\zeta\\
&=-\frac{e^{-2i\theta}}{2\pi i}\int_0^\infty \frac{\overline{\Omega(\rho)}d\rho}{\rho-ze^{-i\theta}}+\frac{e^{2i\theta}}{2\pi i}\int_0^\infty \frac{\Omega(\rho)d\rho}{\rho-ze^{i\theta}}-\frac{1}{2\pi i}\int_0^\infty \frac{\rho-\bar{z}e^{-i\theta}}{(\rho-ze^{i\theta})^2}\Omega(\rho)d\rho\\
&\quad+\frac{1}{2\pi}\int_0^a \frac{\Omega_0(\zeta)}{\zeta-z}d\zeta+\frac{1}{2\pi i}\int_0^\infty \frac{\rho-\bar{z}e^{i\theta}}{(\rho-ze^{i\theta})^2}\overline{\Omega(\rho)}d\rho-\frac{1}{2\pi}\int_0^a \frac{\xi-\bar{z}}{(\xi-z)^2}\Omega_0(\xi)d\xi\\
&=\frac{e^{2i\theta}}{2\pi i}\int_0^\infty \frac{\Omega(\rho)d\rho}{\rho-ze^{i\theta}}-\frac{1}{2\pi i}\int_0^\infty \frac{\Omega(\rho)d\rho}{\rho-ze^{-i\theta}}-\frac{(z-\bar{z})e^{-i\theta}}{2\pi i}\int_0^\infty \frac{\Omega(\rho)d\rho}{(\rho-ze^{-i\theta})^2}\\
&\quad-\frac{e^{-2i\theta}}{2\pi i}\int_0^\infty \frac{\overline{\Omega(\rho)}d\rho}{\rho-ze^{-i\theta}}+\frac{1}{2\pi i}\int_0^\infty \frac{\overline{\Omega(\rho)}d\rho}{\rho-ze^{i\theta}}\\
&\quad+\frac{(z-\bar{z})e^{i\theta}}{2\pi i}\int_0^\infty \frac{\overline{\Omega(\rho)}d\rho}{(\rho-ze^{i\theta})^2}-\frac{z-\bar{z}}{2\pi}\int_0^a \frac{\Omega_0(\xi)}{(\xi-z)^2}d\xi.
\end{aligned}
$$

Remark 6.1. For $z=x$ $(-\infty<x<0$ or $a<x<+\infty)$, by (6.1) and (6.3), both $\Phi(z)$ and $\bar{z}\Phi'(z)+\Psi(z)$ are real so that $\tau_{xy}=0$, as expected.

§7. Suggestions for Numerical Method of Solution for the Problem

For numerical methods of solution for the proposed problem, or equivalently, for the integral equation (4.5)–(4.6) together with (4.7), we suggest two different methods.

As the first one, we may solve (4.5) for $\Omega_0(x)$ $(0<x<a)$ in class h_0 by regarding $\Omega(r)$ as known for the time being, the solution of which may be obtained in explicit form by the inversion formula for Cauchy principal value integrals. There is one arbitrary real constant in its general solution, which may be determined by the additional condition (4.7). Then, by substituting it in (4.6), a singular integral equation for $\Omega(r)$ $(0<r<\infty)$ is obtained. By a suitable change of variable, it is then transformed to an equation on a finite interval, which may be solved numerically by usual methods with application of hypergeometric functions.

The equation obtained as above is rather complicated. We suggest another method of solution by using "quasi-linear splines".

After the order of singularity α at the origin O is determined, taking a division of the interval $[0,a]: x_0=0,\ x_1=\delta,\dots,x_n=n\delta=a$, where $\delta=\frac{a}{n}$, we interpolate $\Omega_0(x)$ linearly in each subinterval $[x_{j-1},x_j]$ $(j=2,\cdots,n-1)$, of the form $c_0\Delta_0/x^\alpha+B_1^0$ in $(0,x_1]$ and of the form $c_n/x^{1/2}+B_n^0$ in $[x_{n-1},a)$ since it is well-known that $\Omega_0(x)$ has a singularity of order 1/2 at $x=a$, where c_0, c_n, B_1^0, B_n^0 are undetermined real constants. At the same time, taking a division of $[0,\infty): r_0=0,\ r_1=\delta,\ r_2=2\delta,\cdots$, we interpolate $\Omega(r)$ linearly on $[r_{k-1},r_k]$ $(k=2,3,\cdots)$ and of the form $c_0(\Delta_1+i\Delta_2)/r^\alpha+B_1$ on $(0,r_1]$, where B_1 is an undetermined complex constant. It must be noted that the form of the interpolatory function on $(0,x_1]$ and that on $(0,r_1]$ mentioned above are due to the remark at the end of §5. Then, if we replace $\Omega_0(x)$ and $\Omega(r)$ in (4.5) and (4.6) as well as in (4.7) by these interpolatory functions and let $x=x_j^*=\frac{2j-1}{2n}\delta$ $(j=1,\cdots,n)$, $r=r_k^*=\frac{2k-1}{2n}\delta$ $(k=1,2,\dots)$ in turn, then we obtain an infinite system of real linear equations in infinite number of unknowns after the real and imaginary parts in (4.6) are separated. The coefficients of this system of equations involve convergent improper intergrals or Cauchy principal value integrals with kernel density $x^{-\alpha}$, $(a-x)^{-1/2}$ or $r^{-\alpha}$, which may be easily evaluated approximately. Because $\Omega(r)$ has a zero of order greater than 1 at infinity in general, we may round off this system to a finite system of real linear equations in finite (sufficiently large) number of unknowns, which may be solved at once. If $\delta > 0$ is small enough, then its solution will be very close to the required one.

The idea for evaluating singular integrals approximately by linear splines was due to K. Atkinson[8] in case the path of integration L is a closed contour and was extended in [9] to the case where L may be open, though only when the weight function identical to one was assumed.

References

[1] Erdogan, F., Aksogan, O., Bounded half planes containing an arbitrarily oriented crack, *Internat. J. Solids and Structure,* **10**(1974), 569–585.

[2] Ashbaugh, N., Stress solution for a crack at an arbitrary angle to the interface, *Internat. J. Fracture,* **11**(1975), 205–219.

[3] Sherman, D. I., On the solution of the plane static problem of the theory of elasticity for given external forces, *Dokl. A.N. SSSR.,* **28**(1940), 25–28.

[4] Lu Jianke, The mathematical problems of bounded plane materials with cracks, *J. Wuhan Univ.,* 2(1982), 1–10 (in Chinese with English abstract).

[5] Lu Chien-ke (Lu Jianke), Yew Ching H., Bounded half-planes containing two arbitrarily oriented cracks, *J. Soc. of Petroleum Engrg., Feb.,* (1985), 55–66.

[6] Muskhelishvili, N. I., Some basic problems of the mathematical theory of elasticity, Noordhoff, 1953.

[7] Cook, T. S., Erdogan, F., Stresses on bounded materials with a crack perpendicular to the interface, *Internat. J. Engrg. Sci.,* **10**(1972), 677–697.

[8] Atkinson, K., The numerical evalution of the Cauchy transform on simple closed curves, *SIAM J. Numer. Anal.,* **9**(1972), 284–299.

[9] Lu Chien-ke, The approximation of Cauchy-type integrals by some kinds of interpolatory splines, *J. Approxi. Theory,* **36**(1982), 197–212.