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SOME RESULTS ON ESTIMATION OF THE TAIL INDEX OF A DISTRIBUTION**

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Abstract

The author obtains the rate of strong convergence, mean squared error and optimal choice of the "smoothing parameter" (the sample fraction) of a tail index estimator which was proposed by the author from Pickands' estimator, and called modified Pickands' estimator. The similar results about Hill's estimator are also obtained, which generalize the corresponding results in [8, 9]. Besides, some comparisons between Hill's estimator and the modified Pickands' estimator are given.

Keywords Tail index, Parameter estimation, Strong convergence, Mean squared error, Comparisons of estimators

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§1. Introduction

Suppose that F is a distribution function such that, for any $x > 0$,

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-\frac{1}{\gamma}}, \quad \gamma > 0. \tag{1.1}$$

We call γ the tail index of F. (1.1) can be denoted by $1 - F \in RV_{-\frac{1}{\gamma}}$.

Let $X_1, X_2, \cdots$, be i.i.d observations from an unknown distribution F. Denote by $X_{1,n} \le \cdots \le X_{n,n}$ the order statistics of $X_1, \cdots, X_n$. The problem is to estimate γ by using $X_1, \cdots, X_n$. This problem has aroused enormous interest and has many applications in economics, finance, and hydrology (for an extensive survey see [4–6]).

An important estimator of γ is Hill's estimator[10]:

$$\widehat{H_n} = \frac{1}{k}\sum_{i=1}^{k} \log X_{n-i+1,n} - \log X_{n-k,n},$$

where k is such that

$$k \to \infty, \quad k/n \to 0, \tag{K}$$

Cheng and Pan[2] modified the Pickands' estimator (see [12]) and obtained the so-called modified Pickands' estimator:

$$\widehat{P}_{n,d} = \frac{1}{\log d}(\log X_{n-k_1+1,n} - \log X_{n-k_2+1,n}), \tag{1.2}$$

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where $d > 1$ is a real number, and $k_1 < k_2$ are integers such that

$$k_i \to \infty, \qquad \frac{k_i}{n} \to 0, \qquad i = 1, 2, \qquad \frac{k_2}{k_1} \to d. \tag{1.3}$$

In this paper, we discuss further properties of $\widehat{P}_{n,d}$ such as the rate of a.s convergence (Section 3), mean squared error and optimal choice of k_1 (Section 4). We also obtain the mean squared error of Hill's estimator and optimal choice of k in very general case, which generalize the corresponding results in [8, 9] (Section 5). How to compare the estimators of γ is an important problem, which has attracted much attention of statisticians. In the end of this paper, we give comparisons of $\widehat{H}_n$ and $\widehat{P}_{n,d}$ when F belongs to a well-known class of distributions (Section 6). But, first, we give some lemmas that we need for the proofs of our results (Section 2).

§2. Lemmas and Denotations

From now on, we denote

$$\begin{aligned} F^{\leftarrow}(t) &= \inf\{x : F(x) \ge t\}, \qquad 0 \le t \le 1, \\ U(t) &= \inf\{x : \frac{1}{1-F(x)} \ge t\}, \qquad 1 \le t \le \infty, \\ V(t) &= U(e^t), \qquad 0 \le t \le \infty. \end{aligned}$$

It has been proved that (1.1) is equivalent to $U(t) \in RV_\gamma$ (see [14]). Moreover, the following lemma was shown by De Haan and Stadtmuller[7].

Lemma 2.1. *If there is a positive and Borel measurable function $b(t)$ satisfying $\lim_{t\to\infty} b(t) = 0$ such that*

$$\lim_{t\to\infty} \frac{1}{b(t)} \left\{ \frac{U(tx)}{U(t)} - x^\gamma \right\} = K(x) \quad \textit{for all } x > 0 \tag{2.1}$$

for some function $K(x) \not\equiv 0$, then

(1) $b(t) \in RV_{-\alpha}$, *for some $\alpha \ge 0$,*

(2) $K(x) = Cx^\gamma \frac{1-x^{-\alpha}}{\alpha}$ *(when $\alpha = 0$, $K(x)$ is interpreted as $Cx^\gamma \log x$).*

Let $U_1, \cdots, U_n$ be i.i.d r.v.s uniformly distributed on (0,1), and $e_1, \cdots, e_n$ be i.i.d r.v.s with standard exponential distribution. Denote by $U_{1,n} \le \cdots \le U_{n,n}$ the order statistics of $U_1, \cdots, U_n$, and $e_{1,n} \le \cdots \le e_{n,n}$ the order statistcs of $e_1, \cdots, e_n$

The following fact is obvious:

$$\begin{aligned} (X_1, X_2, \cdots) &=^d (F^{\leftarrow}(U_1), F^{\leftarrow}(U_2), \cdots) =^d \left(U\left(\frac{1}{1-U_1}\right), U\left(\frac{1}{1-U_2}\right), \cdots \right) \\ &=^d (V(e_1), V(e_2), \cdots). \end{aligned} \tag{2.2}$$

Denote by $\Phi(x)$ the standard normal d.f. and $\phi(x)$ its density.

Lemma 2.2. *If*

$$\frac{k_2}{k_1} - d = o({k_1}^{-3/2}), \tag{2.3}$$

then

$$P\left(\sqrt{\frac{d}{d-1}}{k_1}^{1/2}(e_{k_2-k_1,k_2-1}-\log d)\le x\right)$$
$$=\Phi(x)+{k_1}^{-1/2}\frac{1}{\sqrt{d(d-1)}}\int_{-\infty}^{x}(\frac{d+1}{6}t^3-t)\cdot\phi(t)dt+o({k_1}^{-1/2}) \qquad (2.4)$$

uniformly in $x\in R$.

Proof. From Corollary 4.2.7 in [13],

$$\sup_{B\in\mathcal{B}(R)}\left|P\left(\sqrt{\frac{k_1k_2}{k_2-k_1}}\left(e_{k_2-k_1,k_2-1}-\log\frac{k_2}{k_1}\right)\in B\right)-\int_B(1+L_{1,k_2-k_1,k_2-1}(t))\,d\Phi(t)\right|$$
$$=o({k_1}^{-1/2}),$$

where

$$L_{1,k_2-k_1,k_2-1}(t)=\frac{1}{\sqrt{(k_2-k_1)k_1k_2}}\left(\frac{k_1+k_2}{6}t^3-k_1t\right)$$
$$={k_1}^{-\frac12}\frac{1}{\sqrt{d(d-1)}}\left(\frac{d+1}{6}t^3-t\right)+o\left({k_1}^{-\frac12}\right)(|t|^3+|t|)\quad\text{(from (2.3))}.$$

Using (2.3) again, we obtain (2.4).

Suppose that k satisfies (K). Denote

$$a_n=(n+1)^{-\frac32}(k(n-k+1))^{\frac12},\quad b_n=1-\frac{k}{n+1},\quad u_n=a_nu+b_n, \qquad (2.5)$$

and denote by $\phi_n(u)$ the probability density of $(U_{n-k+1,n}-b_n)/a_n$, i.e.,

$$\phi_n(u)=\begin{cases}\dfrac{n!a_n}{(n-k)!(k-1)!}{u_n}^{n-k}(1-u_n)^{k-1}, & \text{if } -\frac{b_n}{a_n}<u<\frac{1-b_n}{a_n},\\ 0, & \text{otherwise.}\end{cases} \qquad (2.6)$$

We have the following result.

Lemma 2.3. *If* $T_n=o({k_n}^{1/6})$*, then for large* n *and* $|u|\le T_n$*,*

$$\phi_n(u)=\phi(u)-\frac{(u^3-3u)}{3k^{1/2}}+O\left(\frac1k\right)(1+|u|^6)\phi(u). \qquad (2.7)$$

Proof. (2.7) is easily derived from [13].

§3. Strong Convergence of $\widehat{P}_{n,d}$

Theorem 3.1. *If* (2.1) *and* (2.3) *hold, and moreover,*

$$k_1/\log\log n\to\infty, \qquad (3.1)$$
$$\left(\frac{k_1}{\log\log n}\right)^{1/2}b\left(\frac{n}{k_2}\right)\to\infty, \qquad (3.2)$$

then

$$\frac{\widehat{P}_{n,d}-\gamma}{b(n/k_2)}\longrightarrow\frac{C(1-d^{-\alpha})}{\alpha\log d},\quad a.s. \qquad (3.3)$$

Proof. From (2.2),

$$\{(X_{1,n},\cdots,X_{n,n})\}=^d\{(V(e_{1,n}),\cdots,V(e_{n,n}))\},\quad n\ge1.$$

Hence, the same functions of the two sides of the above equality have the same a.s. convergence properties. Therefore, without loss of generality, we write

$$\widehat{P}_{n,d} = \frac{1}{\log d}\{\log V(e_{n-k_1+1,n}) - \log V(e_{n-k_2+1,n})\}. \tag{3.4}$$

Then, by the condition (2.1) and (2.3),

$$\begin{aligned}\widehat{P}_{n,d} &= \frac{1}{\log d}\log\{d^\gamma + V(e_{n-k_1+1,n})/V(e_{n-k_2+1,n}) - d^\gamma\}\\ &= \gamma + \frac{1}{\log d}\log\{1 + [V(e_{n-k_1+1,n})/(d^\gamma V(e_{n-k_2+1,n})) - 1]\}\\ &= \gamma + \Lambda_n + \Lambda_{n,1},\end{aligned}$$

where

$$\begin{aligned}\Lambda_n &= \frac{1}{d^\gamma \log d}\Big\{\frac{V(e_{n-k_1+1,n}) - V(e_{n-k_2+1,n}+\log d)}{V(e_{n-k_2+1,n})} + \frac{V(e_{n-k_2+1,n}+\log d)}{V(e_{n-k_2+1,n})} - d^\gamma\Big\}\\ &= \frac{1}{d^\gamma \log d}\Big\{\frac{V(e_{n-k_2+1,n}+\log d)}{V(e_{n-k_2+1,n})}\\ &\quad\cdot\Big[\frac{V(e_{n-k_2+1,n}+\log d + (e_{n-k_1+1,n} - e_{n-k_2+1,n} - \log d))}{V(e_{n-k_2+1,n}+\log d)}\\ &\quad - \exp\{\gamma(e_{n-k_1+1,n} - e_{n-k_2+1,n} - \log d)\}\\ &\quad + \exp\{\gamma(e_{n-k_1+1,n} - e_{n-k_2+1,n} - \log d)\} - 1\Big]\\ &\quad + b(\exp\{e_{n-k_2+1,n}\})K(d) + o(b(\exp\{e_{n-k_2+1,n}\}))\Big\}\end{aligned}$$

and $\Lambda_{n,1}$ is the remainder term which has higher order than Λ_n. Notice that

$$\{e_{n-k_1+1,n} - e_{n-k_2+1,n} - \log d\} =^d \{-\log U_{k_1,n} + \log U_{k_2,n} - \log d\}, \quad n \ge 1,$$

and from [15],

$$\limsup_{n\to\infty}\Big(\frac{k_i}{\log\log n}\Big)^{1/2}\Big|\frac{n}{k_i}U_{k_i,n} - 1\Big| \le 1,\ a.s.\ \ i = 1, 2.$$

Conbining the above two formulas and the condition (2.3), we have

$$\limsup_{n\to\infty}\left(\frac{k_i}{\log\log n}\right)^{1/2}|e_{n-k_1+1,n} - e_{n-k_2+1,n} - \log d| \le 2, \quad \text{a.s.}\ \ i = 1, 2.$$

Then from the condition (3.2), we have

$$\lim_{n\to\infty}\frac{e_{n-k_1+1,n} - e_{n-k_2+1,n} - \log d}{b(n/k_2)} = 0, \quad \text{a.s.}$$

Therefore

$$\begin{aligned}&\lim_{n\to\infty}\frac{\widehat{P}_{n,d} - \gamma}{b(\frac{n}{k_2})} = \lim_{n\to\infty}\frac{\Lambda_n}{b(n/k_2)}\\ &= \frac{1}{d^\gamma \log d}\lim_{n\to\infty}\frac{V(e_{n-k_2+1,n}+\log d)}{V(e_{n-k_2+1,n})}\\ &\quad\cdot\Big[\lim_{n\to\infty}K(\exp\{e_{n-k_1+1,n} - e_{n-k_2+1,n} - \log d\})\lim_{n\to\infty}b(d\exp\{e_{n-k_2+1,n}\})/b\Big(\frac{n}{k_2}\Big)\end{aligned}$$

$$+\lim_{n\to\infty}\frac{\exp\{\gamma(e_{n-k_1+1,n}-e_{n-k_2+1,n}-\log d)\}-1}{b(n/k_2)}\Big]$$
$$+\frac{1}{d^\gamma\log d}K(d)\lim_{n\to\infty}b(\exp\{e_{n-k_2+1,n}\})/b\Big(\frac{n}{k_2}\Big)$$
$$=\frac{K(d)}{d^\gamma\log d},\qquad \text{a.s.}$$

§4. Asymptotic Mean Squared Error of $\widehat{P}_{n,d}$ and Optimal Choice of k_1 in $\widehat{P}_{n,d}$

Denote

$$m_{1,n}(t)=b^2(t)\frac{C^2(1-d^{-\alpha})^2}{(\alpha\log d)^2}+\frac{\gamma^2(d-1)}{(n+1)(\log d)^2}t \tag{4.1}$$

where $b(t), C, \alpha$ are the same as in Lemma 2.1

Theorem 4.1. *Suppose that* (2.1) *and* (2.3) *hold. Then*

$$\lim_{n\to\infty}\frac{E(\widehat{P}_{n,d}-\gamma)^2}{m_{1,n}(\frac{n+1}{dk_1})}=1. \tag{4.2}$$

Furthermore, if $b(t)$ *is differentiable, the best choice of* k_1 *which minimizes the mean squared error of* $\widehat{P}_{n,d}$ *is*

$$k_{1,0}=\frac{n+1}{d\cdot f(\frac{\gamma^2(d-1)\alpha^2}{2C^2(1-d^{-\alpha})^2(n+1)})}, \tag{4.3}$$

where

$$f(t)=:(-b\cdot b')^{\to}(t)=\sup\{x:-b(x)b'(x)>t\}. \tag{4.4}$$

Proof. We have

$$E(\widehat{P}_{n,d}-\gamma)^2=E\Big(\frac{1}{\log d}\log\frac{F^{\leftarrow}(U_{n-k_1+1,n})}{F^{\leftarrow}(U_{n-k_2+1,n})}-\gamma\Big)^2$$
$$=\Big(\frac{1}{\log d}\Big)^2\int\int_{0<t<s<1}\Big[\log\frac{F^{\leftarrow}(s)}{F^{\leftarrow}(t)}-\log d^\gamma\Big]^2$$
$$\cdot\frac{n!}{(n-k_2)!(k_2-k_1-1)!(k_1-1)!}t^{n-k_2}(s-t)^{k_2-k_1-1}(1-s)^{k_1-1}dtds$$
$$=\Big(\frac{1}{\log d}\Big)^2\frac{n!}{(n-k_2)!(k_2-k_1-1)!(k_1-1)!}\int_0^1 t^{n-k_2}(1-t)^{k_2-1}dt$$
$$\cdot\int_0^1\Big[\log\frac{F^{\leftarrow}(t+(1-t)s')}{F^{\leftarrow}(t)}-\log d^\gamma\Big]^2(s')^{k_2-k_1-1}(1-s')^{k_1-1}ds'$$
$$=\Big(\frac{1}{\log d}\Big)^2\int_0^1 E\Big[\log\frac{F^{\leftarrow}(t+(1-t)U_{k_2-k_1,k_2-1})}{F^{\leftarrow}(t)}-\log d^\gamma\Big]^2$$
$$\cdot\frac{n!}{(n-k_2)!(k_2-1)!}t^{n-k_2}(1-t)^{k_2-1}dt$$
$$=\Big(\frac{1}{\log d}\Big)^2\int_{-\infty}^{\infty}\Delta_n(u)\phi_n(u)du, \tag{4.5}$$

where

$$\Delta_n(u)=E\Big\{\log\frac{U((1-u_n)^{-1}(1-U_{k_2-k_1,k_2-1})^{-1})}{U((1-u_n)^{-1})}-\log d^\gamma\Big\}^2,$$

and $\phi_n(u)$ is defined as (2.6) when k is replaced by k_2. Denote

$$Z_n(u) = \log \frac{U((1-u_n)^{-1}(1-U_{k_2-k_1,k_2-1})^{-1})}{U((1-u_n)^{-1})} - \gamma \log(1-U_{k_2-k_1,k_2-1})^{-1},$$
$$W_n = \gamma \log(1-U_{k_2-k_1,k_2-1})^{-1} - \gamma \log d.$$

Then $\Delta_n(u) = E\{Z_n(u)+W_n\}^2$. By straight computation, when (2.3) is satisfied, we obtain

$$EW_n{}^2 = E\gamma^2\{e_{k_2-k_1,k_2-1} - \log d\}^2 = \gamma^2(1-d^{-1})k_1{}^{-1} + o(k_1{}^{-1}),$$
$$E\{K((1-U_{k_2-k_1,k_2-1})^{-1})W_n\} = O(k_1{}^{-1}),$$
$$EK^2((1-U_{k_2-k_1,k_2-1})^{-1}) = K^2(d) + o(1) = \frac{c^2(1-d^{-\alpha})^2}{\alpha^2} + o(1).$$

But, from Lemma 2.1, Cheng[1] and the dominated convergence theorem, we have

$$\sup_{|u|\le T_n} \left| E\left\{\frac{Z_n(u)}{b((1-u_n)^{-1})}\right\} - EK^2((1-U_{k_2-k_1,k_2-1})^{-1})\right| \longrightarrow 0,$$
$$\sup_{|u|\le T_n} \left| E\left\{\frac{Z_n(u)}{b((1-u_n)^{-1})}W_n\right\} - EK((1-U_{k_2-k_1,k_2-1})^{-1})W_n\right| \longrightarrow 0,$$

for $T_n = o(k_1{}^{1/6})$.

Hence

$$\begin{aligned}&\Delta_n(u)\\ &= b^2((1-u_n)^{-1})E\Big\{\frac{Z_n(u)}{b((1-u_n)^{-1})}\Big\}^2 + 2b((1-u_n)^{-1})E\Big\{\frac{Z_n(u)}{b((1-u_n)^{-1})}W_n\Big\} + EW_n^2\\ &= b^2((1-u_n)^{-1})\frac{c^2(1-d^{-\alpha})^2\alpha^2}{\alpha^2}(1+o(1))\\ &\quad + \frac{\gamma^2(d-1)}{dk_1}(1+o(1)) + O(k_1{}^{-1})b((1-u_n)^{-1})\end{aligned} \tag{4.6}$$

holds uniformly for $|u| \le T_n$. Notice that

$$\frac{1-u_n}{k_2/(n+1)} = 1 - \left(\frac{n-k_2+1}{n+1}\right)^{1/2} k_2{}^{-1/2}u,$$

and this implies that

$$(1-u_n)^{-1} = \frac{(n+1)}{k_2}\left(1 + \Big(\frac{n-k_2+1}{n+1}\Big)^{1/2} k_2{}^{-1/2}u + O(k_2{}^{-1})u^2\right).$$

Hence, for any $x > 0$, when $T_n \le u \le (1-b_n)/a_n = O(1)k^{-1/2}$,

$$\frac{U(\frac{(n+1)}{k_2}x)}{U(\frac{(n+1)}{k_2}(1+O(1)))} \le \frac{U((1-u_n)^{-1}x)}{U((1-u_n)^{-1})} \le \frac{U(\frac{(n+1)}{k_2}(1+O(1))x}{U(\frac{(n+1)}{k_2})}$$

Therefore, at this time, $\Delta_n(u)$ is bounded. Similarly, it can be proved that $\Delta_n(u)$ is also bounded when $-b_n/a_n \le u \le T_n$. Combining with (4.6) we see that $\Delta_n(u)$ is a bounded function of $u \in (-b_n/a_n, (1-b_n)/a_n)$.

But

$$\sup_{|u|\le T_n} \left| b((1-u_n)^{-1}) - b\Big(\frac{(n+1)}{k_2}\Big)\right| = o\Big(b\Big(\frac{(n+1)}{k_2}\Big)\Big).$$

From (4.5) and Lemma 2.3,

$$\lim_{n\to\infty}\frac{E(\widehat{P}_{n,d}-\gamma)^2}{m_{1,n}(\frac{n+1}{dk_1})}=\lim_{n\to\infty}\frac{E(\widehat{P}_{n,d}-\gamma)^2}{m_{1,n}(\frac{n+1}{k_2})}=\lim_{n\to\infty}\frac{E(\widehat{P}_{n,d}-\gamma)^2}{\frac{1}{(\log d)^2}\int_{-T_n}^{T_n}\Delta_n(u)\phi(u)du}$$
$$=\lim_{n\to\infty}E(\widehat{P}_{n,d}-\gamma)^2/\frac{1}{(\log d)^2}\int_{-T_n}^{T_n}\Delta_n(u)\phi_{n,k_2}(u)du=1.$$

Let $m'_{1,n}(t)=0$, i.e.

$$-b(t)b'(t)=\frac{\gamma^2(d-1)\alpha^2}{2C^2(1-d^{-\alpha})^2(n+1)}.$$

This is equivalent to

$$t=f\left(\frac{\gamma^2(d-1)\alpha^2}{2C^2(1-d^{-\alpha})^2(n+1)}\right).$$

Taking $t=\frac{n+1}{dk_1}$, we obtain (4.3).

§5. Asymptotic Mean Squared Error of $\widehat{H}_n$ and Optimal Choice of k in $\widehat{H}_n$

Denote

$$m_{2,n}(t)=\frac{\gamma^2}{n+1}t+b^2(t)\left(\frac{C}{\alpha+1}\right)^2, \tag{5.1}$$

where $b(t),\alpha,C$ are the same as in Lemma 2.1.

Theorem 5.1. *If* (2.1) *holds, then*

$$\lim_{n\to\infty}\frac{E(\widehat{H}_n-\gamma)^2}{m_{2,n}(\frac{n+1}{k})}=1. \tag{5.2}$$

Furthermore, if $b(t)$ is differentiable, then the best choice of k which minimizes the mean squared error of $\widehat{H}_n$ is

$$k_0=(n+1)/f\left(\frac{(\alpha+1)^2\gamma^2}{2(n+1)C^2}\right), \tag{5.3}$$

where $f(t)$ is defined by (4.4).

Proof. Note that

$$E(\widehat{H}_n-\gamma)^2$$
$$=\frac{n!}{(n-k-1)!}\int\cdots\int_{0<y<y_1<\cdots<y_k<1}\left\{\frac{1}{k}\sum_{i=1}^{k}\log F^{\leftarrow}(y_i)-\log F^{\leftarrow}(y)-\gamma\right\}^2y^{n-k-1}dydy_1\cdots dy_k$$
$$=\frac{n!}{(n-k-1)!k!}\int_0^1y^{n-k-1}(1-y)^kdy$$
$$\cdot\int\cdots\int_{0<t_1<\cdots<t_k<1}k!\left\{\frac{1}{k}\sum_{i=1}^{k}\log F^{\leftarrow}(y+(1-y)t_i)-\log F^{\leftarrow}(y)-\gamma\right\}^2dt_1\cdots dt_k$$
$$=\int_0^1E\left\{\frac{1}{k}\sum_{i=1}^{k}\log F^{\leftarrow}(y+(1-y)U_{i,k})-\log F^{\leftarrow}(y)-\gamma\right\}^2\frac{n!}{(n-k-1)!k!}y^{n-k-1}(1-y)^kdy$$
$$=\int_{-\infty}^{\infty}l_n(u_n)\phi_n(u)du,$$

where $\phi_n(u)$ is defined by (2.6), u_n, a_n, b_n are defined by (2.5), and

$$\begin{aligned} l_n(u_n) &= E\Big\{\frac{1}{k}\sum_{i=1}^{k}\log F^{\leftarrow}(u_n+(1-u_n)U_i)-\log F^{\leftarrow}(u_n)-\gamma\Big\}^2 \\ &= E\Big\{\frac{1}{k}\sum_{i=1}^{k}\log U((1-u_n)^{-1}(1-U_i)^{-1})-\log U((1-u_n)^{-1})-\gamma\Big\}^2. \end{aligned}$$

If we denote

$$\begin{aligned} R_{n,i}(u) &= \log U((1-u_n)^{-1}(1-U_i)^{-1})-\log U((1-u_n)^{-1}) \\ &\quad -\log(1-U_i)^{-\gamma},\quad i=1,\cdots,k, \\ R_n(u) &= \frac{1}{k}\sum_{i=1}^{k}R_{n,i}(u),\quad P_n^0=\frac{1}{k}\sum_{i=1}^{k}\left(\log(1-U_i)^{-1}-1\right), \end{aligned}$$

then

$$l_n(u_n)=E\{R_n(u)+\gamma P_n^0\}^2=\gamma^2E(P_n^0)^2+2\gamma ER_n(u)P_n^0+ER_n^2(u),$$

and

$$\begin{aligned} ER_n^2(u) &= (ER_n(u))^2+Var(R_n(u))=(ER_{n,1}(u))^2+\frac{1}{k}Var(R_{n,1}(u)), \\ ER_n(u)P_n^0 &= \frac{1}{k}E\{(R_{n,1}(u))(\log(1-U_1)^{-1}-1\}. \end{aligned}$$

But, from Lemma 2.1, [1] and the dominated convergence theorem, it can be seen that for $T_n=o({k_n}^{1/6})$,

$$\sup_{|u|\le T_n}\left|E\frac{R_{n,1}(u)}{b((1-u_n)^{-1})}-\frac{c}{\alpha+1}\right|\longrightarrow 0;\quad \sup_{|u|\le T_n}\frac{Var(R_{n,1}(u))}{b^2((1-u_n)^{-1})}=O(1);$$

$$\sup_{|u|\le T_n}\left|E\frac{R_n(u)P_n^0}{k^{-1}b((1-u_n)^{-1})}\right|=O(1).$$

Hence, for $|u|\le T_n$ uniformly,

$$l_n(u_n)=\frac{\gamma^2}{k}+b^2((1-u_n)^{-1})\Big(\frac{C}{\alpha+1}\Big)^2+o(b^2(1-u_n)^{-1})+O(k^{-1}b(1-u_n)^{-1}).$$

But,

$$\frac{1-u_n}{k/(n+1)}=1-\Big(\frac{n-k+1}{n+1}\Big)^{1/2}k^{-1/2}u$$

implies that

$$\sup_{|u||\le T_n}\left|b\left((1-u_n)^{-1}\right)-b\left(\frac{n+1}{k}\right)\right|=o\left(b\left(\frac{n+1}{k}\right)\right)$$

and $l_n(u_n)$ is a bounded function of $u\in(-b_n/a_n,(1-b_n)/a_n)$ (by the similar proof as that in Section 4). Therefore,

$$\begin{aligned} \lim_{n\to\infty}\frac{E(\widehat{H}_n-\gamma)^2}{m_{2,n}(\frac{n+1}{k})} &= \lim_{n\to\infty}\frac{E(\widehat{H}_n-\gamma)^2}{\int_{|u|\le T_n}l_n(u_n)\phi(u)du} \\ &= \lim_{n\to\infty}\frac{E(\widehat{P}_n-\gamma)^2}{\int_{|u|\le T_n}l_n(u_n)\phi_{n,k_n}(u)du}=1. \end{aligned}$$

Because $m'_{2,n}(t)=0$ is equivalent to

$$t=f\left(\frac{\gamma^2(\alpha+1)^2}{2(n+1)C^2}\right),$$

we have (5.3).

Corollary 5.1. *If*

$$1-F(x)=C_1x^{-1/\gamma}\{1+C_2x^{-1/\beta}(1+o(1))\},\quad x\to\infty, \tag{5.4}$$

where $C_1\neq 0, C_2\neq 0, \gamma>0, \beta>0$ are constants[8]*, then*

$$E(\widehat{H}_n-\gamma)^2=\Big\{\frac{\gamma^2}{k}+\Big(\frac{k}{n+1}\Big)^{2\alpha}\frac{\gamma^2{C_2}^2}{{C_1}^{2\alpha}}\Big(\frac{\alpha}{\alpha+1}\Big)^2\Big\}(1+o(1)), \tag{5.5}$$

where $\alpha=\gamma/\beta$. In particular, when $\gamma=\beta$, i.e. $\alpha=1$, we have the result

$$E(\widehat{H}_n-\gamma)^2=\Big\{\frac{\gamma^2}{k}+\frac{1}{4}\Big(\frac{k}{n+1}\Big)^2{C_1}^{-2}{C_2}^2\gamma^2\Big\}(1+o(1)), \tag{5.6}$$

which was obtained in [9].

Proof. It is easy to see that (5.4) is equivalent to

$$U(t)={C_1}^{\gamma}t^{\gamma}\{1+\gamma C_2{C_1}^{-\alpha}t^{-\alpha}(1+o(1))\},\quad t\to\infty. \tag{5.7}$$

Hence, the condition (2.1) is satisfied for $b(t)=t^{-\alpha}, K(x)=-\gamma C_2{C_1}^{-\alpha}x^{\gamma}(1-x^{-\alpha})$. From Theorem 5.1, we obtain (5.5).

§6. Comparison Between $\widehat{P}_{n,d}$ and $\widehat{H}_n$

We consider the minima of asymptotic mean squared errors of $\widehat{H}_n$ and $\widehat{P}_{n,d}$ for the ditributions in the class (5.4). As in Corollary 5.1, (2.1) holds for $b(t)=t^{-\alpha}$. Hence,

$$\begin{aligned}f(t)=(-bb')^{\to}(t)&=\sup\{x:-x^{-\alpha}(-\alpha x^{-\alpha-1})>t\}\\&=\sup\Big\{x:x^{-(2\alpha+1)}>\frac{t}{\alpha}\Big\}=\alpha^{1/(2\alpha+1)}t^{-1/(2\alpha+1)}.\end{aligned}$$

From Theorem 4.1 and Theorem 5.1, we have

$$\begin{aligned}\inf_{k_1}E(\widehat{P}_{n,d}-\gamma)^2&\sim\inf_{k_1}m_{1,n}\Big(\frac{n+1}{dk_1}\Big)=m_{1,n}\Big(\frac{n+1}{dk_{1,0}}\Big)\\&=m_{1,n}\Big(f\Big(\frac{\gamma^2(d-1)\alpha^2}{2C^2(1-d^{-\alpha})^2(n+1)}\Big)\Big)\\&=(n+1)^{-\frac{2\alpha}{2\alpha+1}}(\gamma^2(d-1))^{\frac{2\alpha}{2\alpha+1}}(C^2)^{\frac{1}{2\alpha+1}}\\&\quad(1-d^{-\alpha})^{\frac{2}{2\alpha+1}}(1+2\alpha)2^{-\frac{2\alpha}{2\alpha+1}}\alpha^{-\frac{2\alpha+2}{2\alpha+1}}(\log d)^{-2},\end{aligned}$$

where $C=-\gamma C_2{C_1}^{-\alpha}\alpha$, and

$$\begin{aligned}\inf_{k}E(\widehat{H}_n-\gamma)^2&\sim\inf_{k}m_{2,n}\Big(\frac{n+1}{k}\Big)=m_{2,n}\Big(\frac{n+1}{k_0}\Big)\\&=m_{2,n}\Big(f\Big(\frac{(\alpha+1)^2\gamma^2}{2(n+1)C^2}\Big)\Big)\\&=m_{2,n}\Big(\alpha^{\frac{1}{2\alpha+1}}\Big(\frac{2(n+1)C^2}{(\alpha+1)^2\gamma^2}\Big)^{\frac{1}{2\alpha+1}}\Big)\\&=(n+1)^{-\frac{2\alpha}{2\alpha+1}}(\gamma^2)^{\frac{2\alpha}{2\alpha+1}}(2\alpha)^{\frac{1}{2\alpha+1}}\\&\quad(C^2)^{\frac{1}{2\alpha+1}}(\alpha+1)^{-\frac{2}{2\alpha+1}}\Big(1+\frac{(\alpha+1)^2}{2\alpha C^2}\Big).\end{aligned}$$

Hence

$$\frac{\inf_{k_1} E(\widehat{P}_{n,d}-\gamma)^2}{\inf_k E(\widehat{H}_n-\gamma)^2} \sim R(d,\gamma,\alpha) =: \frac{(\alpha+1)^{\frac{2}{2\alpha+1}}(d-1)^{\frac{2\alpha}{2\alpha+1}}(1-d^{-\alpha})^{\frac{2}{2\alpha+1}}(1+2\alpha)}{2\alpha^{\frac{2\alpha+3}{2\alpha+1}}(\log d)^2(1+\frac{(\alpha+1)^2}{2\alpha C^2})}.$$

When $\alpha = 1$,

$$R(d,\gamma,1) = \frac{3(d-1)^{4/3}}{2^{1/3}d^{2/3}(\log d)^2(1+\frac{2C_1{}^2}{\gamma^2 C_2{}^2})}.$$

When d and γ are such that $R(d,\gamma,\alpha) < 1$ (or $R(d,\gamma,\alpha) > 1$), $\widehat{P}_{n,d}$ (or $\widehat{H}_n$) is better than $\widehat{H}_n$ (or $\widehat{P}_{n,d}$).

Specially, when $d = 2$, $R(2,\gamma,1) < 1$ is equivalent to

$$0 < \gamma < \Big|\frac{C_1}{C_2}\Big| \frac{2\log 2}{\sqrt{3-2(\log 2)^2}}. \tag{6.1}$$

This shows that when γ satisfies (6.1), the estimator

$$\widehat{P_n} = \frac{1}{\log 2}(\log X_{n-k+1,n} - \log X_{n-2k+1,n})$$

is better than Hill's estimator for the distribution F in the class:

$$1 - F(x) = C_1 x^{-1/\gamma}\{1 + C_2 x^{-1/\gamma}(1+o(1))\}, x \to \infty.$$

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