Chin. Ann. of Math.
19B: 3(1998),249-254.

A REMARK ON DUALITY FOR THE SUM OF CONVEX FUNCTIONS IN GENERAL BANACH SPACES**

ZHOU FENG*

Abstract

The author gives a new proof of Attouch-Brezis′ theorem concerned with the duality for the sum of convex functions in general Banach spaces, and gives also some sufficient conditions for the difference of two closed convex sets to be closed in reflexive Banach spaces.

Keywords Conjugate convex function, Asymptotic cone, Infimal convolution
1991 MR Subject Classification 46B10
Chinese Library Classification O177.2

§1. Introduction

Let E be a real Banach space, not necessarily reflexive. Let $\Phi,\ \Psi : E \to]-\infty,\infty]$ be two lower semi-continuous convex functions. We denote by $D(\Phi)$ (resp. $D(\Psi)$) the effective domain of Φ (resp. Ψ) and Φ^* (resp. Ψ^*) its conjugate convex function. The problem of computing the Fenchel conjugate of the sum $\Phi+\Psi$, namely

$$(\Phi+\Psi)^*(\varphi)=\sup_{x\in E}\{\langle\varphi,x\rangle-\Phi(x)-\Psi(x)\}$$

for $\varphi\in E^*$, is of great importance in this sense that many questions occuring in duality theory may be reduced to such a consideration. In [2], H. Attouch and H. Brezis have proved the following

Theorem 1.1.[2] *Assume that the above two convex functions* Φ *and* Ψ *satisfy*

$$\bigcup_{\lambda\geqslant 0}\lambda(D(\Phi)-D(\Psi))=H \quad \textit{is a closed linear subspace of } E. \tag{1.1}$$

Then we have

$$(\Phi+\Psi)^*=\Phi^*\Box\Psi^* \qquad \textit{on} \quad E^*. \tag{1.2}$$

Moreover, the infimal convolution in (1.2) *is exact.*

This result generalized a classical theorem of Fenchel (in the finite dimensional case) and an extension of Rockafellar[10] (see e.g. [5]) in which the assumption (1.1) is replaced by a much stronger and less geometrical qualification condition, more precisely, the following assumption:

$$\Phi \text{ (or } \Psi) \text{ is continuous at some point } x\in D(\Phi)\cap D(\Psi). \tag{1.3}$$

Manuscript received June 26, 1997. Revised October 8, 1997.
*Department of Mathematics, East China Normal University, Shanghai 200062, China.
**Project supported by the National Natural Science Foundation of China (19641001) and the Doctoral Foundation of the State Education Commission of China (19601017).

Our purpose of this paper is to give a new proof of the above theorem without using the famous Banach-Dieudonné-Krein-Smulian Theorem (See [6, Theorem V.5.7]) which was the key point of Attouch-Brezis′ proof. As an application of the theorem, we give also a closedness criterion for the difference of two closed convex sets to be closed in reflexive Banach spaces. More precisely, we obtain the following

Theorem 1.2. *Let E be a reflexive Banach space. Assume that the two closed convex sets A and B of E satisfy one of the following assumption:*

(1) *A (or B) is bounded,*

(2) *A (or B) is included in a finite dimensional linear subspace of E and* $\mathrm{ca}A \cap \mathrm{ca}B$ *is a linear subspace of E.*

Then $A - B$ is closed.

Here $\mathrm{ca}A = \bigcap_{\lambda>0} \lambda(A-a)$ denotes the asymptotic cone of A. It is a closed convex cone, and independent of the choice of $a \in A$. We note that this criterion is still valid for general Banach spaces, but the proof is quite different (see [4]).

The paper is organized as follows. In §2 we give some basic definitions and notations for the proof of theorems. In §3 and §4 we prove Theorem 1.1 and Theorem 1.2 respectively.

§2. Preliminaries

Before the proof of Theorems, we review some basic definitions and notations in convex analysis theory. Let Φ be a convex function defined on E. The effective domain of Φ is defined by $D(\Phi) = \{x \in E; \Phi(x) < +\infty\}$. We denote by Φ^* its conjugate convex function which is defined by $\Phi^*(\varphi) = \sup_{x\in E}\{\langle\varphi, x\rangle_{E',E} - \Phi(x)\}$ for all $\varphi \in E'$. If A is a convex subset of E, Φ_A is the indicator function, i.e.

$$\Phi_A(x) = \begin{cases} 0, & \text{if} \quad x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

We know that Φ_A is lower semi-continuous if and only if A is closed. Now let Φ and Ψ be two convex functions defined on E. We define the inf-convolution of Φ and Ψ by

$$\Phi \Box \Psi(x) = \inf_{y\in E}\{\Phi(x-y) + \Psi(y)\} \quad \text{for all} \quad x \in E.$$

We say that the inf-convolution is exact if for any $x \in E$ there exists a y such that $\Phi\Box\Psi(x) = \Phi(x-y) + \Psi(y)$. Remark that we have $\Phi_A \Box \Phi_{-B} = \Phi_{A-B}$. One deduces that $A - B$ is closed in E if and only if the function $\Phi_A \Box \Phi_{-B}$ is lower semi-continuous.

§3. Proof of Theorem 1.1

We start with the following lemma due to B. Rodrigues and S. Simons[11], where the proof is given in Appendix.

Lemma 3.1. *Let X and Y be two metrisable topological vector spaces. Let $f : X \to Y$ be a linear application such that its graph, denoted by $G(f)$, is a complete linear subspace of $X \times Y$. Assume that $g : X \to]-\infty, +\infty]$ is a proper, lower semi-continuous convex function such that*

$$\bigcup_{\lambda\geqslant 0} \lambda f(D(g)) = Z \text{ is a complete linear subspace of } Y. \tag{3.1}$$

Let $\sigma(\alpha)=\{x\in X; g(x)\le\alpha\}$. *Then there exists* $\theta\geqslant 1$ *such that* $o\in int_Z f(\sigma(\theta))$.

Proof of Theorem 1.1. The proof of Theorem 1.1 is divided into two steps.

Step 1. We claim that assumption (1.1) is equivalent to

$$o\in \mathrm{int}_H(D(\Phi)-D(\Psi)). \tag{3.2}$$

Set $X=E\times E$ and $Y=E$. We define f, the continuous linear application from X to Y, by $f(x,y)=x-y$ for $(x,y)\in X$. By the continuity of f, its graph $G(f)$ is closed. We define also $g(x,y)=\Phi(x)+\Psi(y)$, where $(x,y)\in X$, the proper convex function defined from X to $]-\infty,+\infty]$. It is also lower semi-continuous and $\not\equiv+\infty$ such that $D(g)=D(\Phi)\times D(\Psi)$. Thus we have

$$Z=\bigcup_{\lambda\geqslant 0}\lambda f(D(g))=\bigcup_{\lambda\geqslant 0}\lambda(D(\Phi)-D(\Psi))=H,$$

which is a closed linear subspace of Y. In view of Lemma 3.1, there exists $\theta\ge 1$ such that $o\in\mathrm{int}_H f(\sigma(\theta))$ which is included in $\mathrm{int}_H f(D(g))$. Hence the conclusion (3.2) follows.

Step 2. Note that we have always $(\Phi\Box\Psi)^*=\Phi^*+\Psi^*$ on E^*. Moreover, in order to prove (1.2), it suffices to show that it is true at the origin, i.e.,

$$(\Phi+\Psi)^*(o)=\Phi^*\Box\Psi^*(o). \tag{3.3}$$

Indeed, suppose that we have proved (3.3). For any $\varphi\in E^*$, we set $\Phi_\varphi(x)=\Phi(x)+\langle-\varphi,x\rangle$. Then Φ_φ is also a lower semi-continuous convex and $\not\equiv+\infty$ such that $D(\Phi_\varphi)=D(\Phi)$. So we obtain $(\Phi_\varphi+\Psi)^*(o)=\Phi_\varphi^*\Box\Psi^*(o)$. On the other hand, for any $\psi\in E^*$, we see that

$$\Phi_\varphi^*(-\psi)=\sup_{x\in E}\{\langle-\psi,x\rangle+\langle\varphi,x\rangle-\Phi(x)\}=\Phi^*(\varphi-\psi).$$

Therefore we have

$$\begin{aligned}(\Phi+\Psi)^*(\varphi)&=\sup_{x\in X}\{-\Phi_\varphi(x)-\Psi(x)\}=(\Phi_\varphi+\Psi)^*(o)\\&=\Phi_\varphi^*\Box\Psi^*(o)=\inf_{\psi\in E^*}\{\Phi_\varphi^*(-\psi)+\Psi^*(\psi)\}.\end{aligned}$$

Thus we prove that $(\Phi+\Psi)^*(\varphi)=\Phi^*\Box\Psi^*(\varphi)\quad$ for all $\quad\varphi\in E^*$.

In order to prove (3.3), we remark at first that

$$(\Phi+\Psi)^*(o)\le\Phi^*\Box\Psi^*(o). \tag{3.4}$$

Without loss of generality, we can assume that $D(\Phi)$ and $D(\Psi)$ are included in H.

Let $\Phi_0: H\to]-\infty,+\infty]$ be the convex function defined by

$$\Phi_0(x)=\inf_{z\in H}\{\Phi(x+z)+\Psi(z)\}.$$

Clearly $D(\Phi_0)=D(\Phi)-D(\Psi)$. By (3.2), we get $o\in\mathrm{int}_H D(\Phi_0)$. We conclude that Φ_0 is continuous at o and $\partial\Phi_0(o)\ne\emptyset$. Let $\varphi_0\in\partial\Phi_0(o)$ and $\varphi_0\in H^*$, i.e., $\Phi_0(x)\ge\Phi_0(o)+\langle\varphi_0,x\rangle$ for all $x\in H$. Hence we define $\Phi_1(x)=\inf_{z\in E}\{\Phi(x+z)+\Psi(z)\}$ for $x\in E$. We see that $\Phi_1(x)=\Phi_0(x)$ if $x\in H$ and $\Phi_1(x)=+\infty$ otherwise. That means Φ_1 is an extension of Φ_0 on E, and in particular, we have

$$\Phi_1(o)=\inf_{z\in E}\{\Phi(z)+\Psi(z)\}=-(\Phi+\Psi)^*(o). \tag{3.5}$$

According to Hahn-Banach Theorem, there exists $\varphi_1\in E^*$, the extension of φ_0, such that $\varphi_1|_H=\varphi_0$. Then we obtain $\Phi_1(x)\ge\Phi_1(o)+\langle\varphi_1,x\rangle$ for all $x\in E$ which is equivalent to $\varphi_1\in\partial\Phi_1(o)$ and $\Phi_1(o)+\Phi_1^*(\varphi_1)=0$. We have also $\Phi_1(x)=(\Phi\Box\Psi')(x)$, where $\Psi'(x)=$

$\Psi(-x)$. This yields $\Phi_1^* = (\Phi\Box\Psi')^* = \Phi^* + \Psi'^*$. But we have also $\Psi'^*(\varphi_1) = \Psi^*(-\varphi_1)$. Therefore one deduces that

$$\Phi_1(o) + \Phi^*(\varphi_1) + \Psi^*(-\varphi_1) = 0. \tag{3.6}$$

On the other hand, we have

$$\Phi^*\Box\Psi^*(o) \le \Phi^*(\varphi_1) + \Psi^*(-\varphi_1). \tag{3.7}$$

Together with (3.4), (3.5), and (3.6), we see that (3.3) holds and the infimal convolution is exact, which completes the proof of Theorem 1.1.

Remark 3.1. We can also give an alternative proof of Theorem 1.1 by applying a theorem of Rockafellar[9] with Lagrangian functions and duality spaces. But it seems that we should first prove Lemma 3.1. So we have the same kind of difficulty to overcome.

Remark 3.2. We have a simpler proof without using Lemma 3.1 if E is a reflexive Banach space. In fact, in the case where $H = E$, we set for $R > 0$

$$D_R = \{x \in X; \Phi(x) \le R \quad \text{and} \quad \|x\| \le R\} - \{x \in X; \Psi(x) \le R \quad \text{and} \quad \|x\| \le R\}.$$

The reflexivity of E shows that D_R is then closed. On the other hand $E = \bigcup_{n\ge 1} nD_n$. By Baire$'$ theorem, there is $k \in \mathbb{N}$ such that $\mathrm{int}D_k \neq \emptyset$. We can therefore prove that $o \in \mathrm{int}(D(\Phi) - D(\Psi))$ and the conclusion will be derived.

§4. An Application

Notice that we have also proved the following

Theorem 1.1$'$. *Let E be a reflexive Banach space. Assume Φ and Ψ to be two convex, lower semi-continuous functions, $\not\equiv +\infty$ and satisfying*

$$\bigcup_{\lambda\geqslant 0} \lambda(D(\Phi^*) - D(\Psi^*)) = H \quad \textit{is a closed linear subspace of } E^*. \tag{4.1}$$

Then we have

$$(\Phi^* + \Psi^*)^* = \Phi\Box\Psi \qquad \textit{on} \quad E. \tag{4.2}$$

Moreover, the infimal convolution is exact.

However, it was shown in [2] that the conclusion falses in the case of a non reflexive Banach space. In this section, we will use the above theorem to prove Theorem 1.2.

Proof of Theorem 1.2. We claim that under the conditions (1) or (2), we have

$$D_1 + D_2 \quad \text{is a closed linear subspace of } E^*, \tag{4.3}$$

where D_1 (resp. D_2) is the effective domain of Φ_A^* (resp. Φ_B^*). We postpone the proof of the claim and present the proof of the theorem. By the definition, we have $D(\phi_{-B}^*) = \{\varphi \in E^*; \sup_{x\in -B} \langle\varphi, x\rangle < +\infty\} = -D_2$. Since D_i, $i = 1, 2$, are convex cones, we obtain

$$\bigcup_{\lambda\geqslant 0} \lambda(D(\phi_A^*) - D(\phi_{-B}^*)) = D_1 + D_2.$$

By claim (4.3) and Theorem 1.1$'$, we deduce that

$$\phi_{A-B}(x) = \phi_A\Box\phi_{-B}(x) = (\phi_A^* + \phi_{-B}^*)(x).$$

In particular, it is a lower semi-continuous function on E and this implies that $A-B$ is closed.

We now turn to the proof of claim (4.3). Clearly in the case of (1), i.e. if A (or B) is bounded, then D_1 (or D_2) is E^*, (4.3) is obvious.

In the case of (2), we assume that A is included in a finite dimensional linear subspace G of E and set $G_1 = \mathrm{ca}A \cap \mathrm{ca}B$ which is a linear subspace. Without loss of generality, we suppose that $o \in A \cap B$. We will prove $D_1 + D_2 = G_1^{\perp}$.

First, for any $d_i \in D_i$ $(i=1,2)$ and any $x \in G_1$, we observe that

$$\begin{aligned}\sup_{\lambda\geqslant 0}\{\langle d_1+d_2,\lambda x\rangle\} &\leqslant \sup_{\lambda\geqslant 0}\{\langle d_1,\lambda x\rangle\}+\sup_{\lambda\geqslant 0}\{\langle d_2,\lambda x\rangle\}\\ &\leqslant \sup_{y\in A}\{\langle d_1,y\rangle\}+\sup_{y\in B}\{\langle d_2,y\rangle\}=\phi_A^*(d_1)+\phi_B^*(d_2)<+\infty.\end{aligned}$$

This yields $\langle d_1+d_2,x\rangle = 0$ for all $x \in G_1$, which proves $D_1+D_2 \subset G_1^{\perp}$.

On the other hand, let L be the topological complement of G. Then we have $E^* = L^{\perp}+G^{\perp}$ and $\dim L^{\perp} = \dim G$. Moreover, if G decomposes into $G = G_1 + G_2$, where G_2 is also a linear subspace, we get also $E = G+L = G_1+G_2+L$ and G_1+L (resp. G_2+L) can be considered as a topological complement of G_2 (resp. G_1) (see e.g. [5]).

We should prove $G_1^{\perp} \subset D_1+D_2$. It is true if $G_2 = \{o\}$, since we have $G = G_1$ and A is included in $\mathrm{ca}B$. Assume that $(G_1+L)^{\perp} \neq \{o\}$. We see that $G^{\perp} = (G_1+G_2)^{\perp}$ is included in D_1. We claim that $P_{(G_1+L)^{\perp}}(D_1+D_2) = (G_1+L)^{\perp}$, which is a finite dimensional subspace, where P_F denotes the canonical projection on F. Otherwise, since it is a convex cone, Hahn-Banach theorem implies that there is some $u \neq o$, $u \in E$ such that

$$P_{(G_1+L)^{\perp}}(D_1+D_2) \subset \{\varphi \in (G_1+L)^{\perp}; \langle \varphi, u\rangle \leqslant 0\}, \tag{4.4}$$

i.e., there is $v = P_{G_2}(u) \in G_2$, $v \neq o$ and $\langle \varphi, u\rangle = \langle \varphi, v\rangle$. Indeed, v belongs to G_1. If it is not true, there exists some $\lambda > 0$ such that λv does not belong to one of A or B. For example, suppose that $\lambda v \notin A$. Again by Hahn-Banach theorem, there exists $\varphi \in E^*$ such that $\sup_{y\in A}\langle \varphi, y\rangle < \alpha < \langle \varphi, \lambda v\rangle$. Consequently, if we set $\xi = P_{(G_1+L)^{\perp}}(\varphi)$, this yields $\varphi \in D_1$ and $\langle \xi, v\rangle > 0$. We deduce that $\xi \in P_{(G_1+L)^{\perp}}(D_1+D_2)$, which gives a contradiction with (4.4). Therefore for any $\varphi \in (G_1+L)^{\perp}$, there are $d_i \in D_i$ $(i=1,2)$ such that

$$\varphi = P_{(G_1+L)^{\perp}}(d_1+d_2) = P_{(G_1+L)^{\perp}}(\varphi_1+\varphi_2) = \varphi_1$$

with $\varphi_1 \in (G_1+L)^{\perp}$ and $\varphi_2 \in G^{\perp}$. This implies that $\varphi = d_1+d_2-\varphi_2 \in D_1+D_2$, which proves claim (4.3) and completes the proof of Theorem 1.2.

Appendix

Proof of Lemma 3.1. Let U be a neighborhood of o in X and T a ball centered at o in X such that $T+T \subset U$. For any $m \geqslant 1$, we set $D_m = f(mT \cap \sigma(m))$. Since T is an absorbing set, we get $Z = \bigcup_{i,m\geqslant 1} iD_m$. According to Baire's theorem, there exist $i,m \geq 1$ such that $\mathrm{int}_Z\, \overline{iD_m}$ is not empty. Let $y_0 \in \mathrm{int}_Z\, \overline{iD_m}$. There are $j,n \geq 1$ such that $-y_0 \in jD_n$. Set $k = \max(m,n)$. By the convexity of g, we obtain

$$iD_m - y_0 \subset iD_m + jD_n \subset iD_k + jD_k \subset (i+j)f(kU \cap \sigma(k)),$$

which implies that

$$o = y_0 - y_0 \in \mathrm{int}_Z \overline{iD_m} - y_0 = \mathrm{int}_Z \overline{iD_m - y_0} \subset (i+j)\mathrm{int}_Z \overline{f(kU \cap \sigma(k))}.$$

Thus we have proved that for any neighborhood U of $o \in X$, there exists $k \geq 1$ such that $o \in \mathrm{int}_Z \overline{f(kU \cap \sigma(k))}$. Let $\{U_n\}_{n\geq 1}$ be a sequence of balls centered at o which is a basis of neighborhood of o in X and satisfies $U_{n+1} + U_{n+1} \subset U_n$. Let $\{V_n\}_{n\geq 1}$ be an arbitrary basis of neighborhood of o in Y.

Set $U = 2^n U_n, n \geq 1$. Then $\exists k_n \geq 1$ such that $o \in \mathrm{int}_Z \overline{f(U_n \cap r_n \sigma(k_n))}$, where $r_n = (2^n k_n)^{-1}$. We set $\theta = \big(\sum\limits_{n\geq 1} r_n\big)^{-1} \geq 1$. Then we have

$$\theta\ \overline{f(U_1 \cap r_1 \sigma(k_1))} \subset f(\sigma(\theta)). \tag{$*$}$$

In fact, letting $y \in \overline{f(U_1 \cap r_1\sigma(k_1))}$, we can find a sequence $\{x_n\}_{n\geq 1}$ such that $\forall n \geq 1$, $x_n \in U_n \cap r_n \sigma(k_n)$ and

$$y - \sum_{m=1}^{n} f(x_m) \in \overline{f(U_{n+1} \cap r_{n+1}\sigma(k_{n+1}))} \cap V_n.$$

The sequence $\left\{\big(\sum\limits_{m=1}^{n} x_m, \sum\limits_{m=1}^{n} f(x_m)\big)\right\}_{n\geq 1}$ is a Cauchy sequence in $G(f)$ which is complete. Then there exists $x \in X$ such such that $(x, f(x))$ is the limit as $n \to +\infty$ and we have $y = f(x)$.

On the other hand, if we set $v_n = \big(\sum\limits_{m=1}^{n} x_m\big)\big(\sum\limits_{m=1}^{n} r_m\big)^{-1}$, since g is convex, we get, $\forall m \geq 1$,

$$g(2^m k_m x_m) \leq k_m \quad \text{and} \quad \forall n \geq 1, \quad g(v_n) \leq \Big(\sum_{m=1}^{n} 2^{-m}\Big) \Big/ \Big(\sum_{m=1}^{n} r_m\Big).$$

The lower semicontinuity of g implies that $g(\theta x) \leq \theta$ and $\theta y = \theta(f(x)) = f(\theta x)$ with $\theta x \in \sigma(\theta)$. We have proved ($*$), which achieves the proof of Lemma 3.1.

Acknowledgement. The author would like to thank the referee for helpful comments and suggestions.

References

[1] Attouch, H., On the maximality of the sum of two maximal operators, *Nonlinear Anal.,* **5** (1981), 143–147.

[2] Attouch, H. & Brezis, H., Duality of the sum of convex functions in general Banach spaces, Aspects of Math. and its applications, J. Barroso Ed., North Holland, 1986, 125–133.

[3] Baiocchi, C., Gastaldi, F. & Tomarelli, F., Some existence results on non coercive variational inequalities, *Pavia,* **428** (1984).

[4] Beaulieu, A. & Zhou, F., A closedness criterion for the difference of two closed convex sets in general Banach spaces, Preprint.

[5] Brezis, H., Analyse fonctionnelle, théorie et applications, Masson, 1983.

[6] Dunford, N. & Schwartz, F., Linear operators Part I, Interscience, 1964.

[7] Fenchel, W., Convex cones, sets and functions, Lecture Notes, Princeton, 1953.

[8] Rockafellar, T., Conjugate duality and optimization, Regional Conferences Series in Applied Math., SIAM Coll., Philadelphia, 1976.

[9] Rockafellar, T., Convex analysis, Princeton, 1970.

[10] Rockafellar, T., Extension of Fenchel's duality theorem for convex functions, *Duke Math. J.,* **33** (1976), 81–90.

[11] Rodrigues, B. & Simons, S., Conjugate functions and subdifferentials in non normed situation for operators with complete graphs, *Nonlinear Anal.,* **12** (1988), 1069–1078.