Chin. Ann. of Math.
19B: 3(1998),255-264.

ASYMPTOTIC BEHAVIOUR OF SUPERSONIC FLOW PAST A CONVEX COMBINED WEDGE**

CHEN SHUXING*

Abstract

The supersonic flow past a convex combined wedge is discussed. Here the surface of the wedge is composed of two straight lines connected by a convex smooth curve. Under the assumptions that the shock is weak, the vertex of the wedge is less than a critical value and the difference of the slope of these two lines is small, the author proves the global existence of solution with shock front and obtains the asymptotic behaviour of the solution.

Keywords Supersonic flow, Convex combined wedge, Shock, Global solution, Asymptotic behaviour

1991 MR Subject Classification 35L60, 76J20

Chinese Library Classification O175.27, O354.5

§1. Introduction

In this paper we study uniform supersonic flow past a curved wedge with a small vertex angle. In this case an attached shock occurs. The location of the shock and the flow field behind the shock are unknown. For such a problem, the local existence of the solution near the vertex is known in the sixties. But the global existence is unknown until now. Furthermore, if the wedge keeps an constant starting from some point, could we predict the asymptotic behaviour of the flow? The answer to these problems seems to be natural from physical viewpoint, but it is interesting and necessary to give a rigorous proof to the answer. This is the purpose of our paper.

In the whole paper we only consider the upper half of the wedge without loss of generality. We also assume that the surface of the wedge has constant section, and its section is composed by two straight lines connecting by a smooth curve. The equation of the surface is

$$y = \tilde{f}(x) = \begin{cases} kx, & x \in (0,a), \\ f(x), & x \in (a,b), \\ f(b) + k_1(b-a), & x \in (b,\infty), \end{cases} \tag{1.1}$$

where $f(a) = ka,\ f'(a) = k,\ f'(b) = k_1,\ k > k_1 > 0$ and $f'(x) > 0, f''(x) < 0$ if $x \in (a,b)$.

As shown in Figure 1, $OH_1H_2H_3$ represents the surface of the wedge, where H_1H_2 is an arc corresponding to $x \in (a,b)$. Starting from H_1H_2 a rarefaction wave R_1 forms, then R_1

Manuscript received April 19, 1997. Revised January 26, 1998.

*Institute of Mathematics, Fudan University, Shanghai 200433, China.

**Project supported by the National Natural Science Foundation of China and the Doctoral Programme Foundation of Institution High Education of China.

is reflected by shock S at L_1L_2. After interaction of R_1 and its reflection a new rarefaction wave R_2 forms. Then R_2 propagates up to the wedge and is reflected by the wedge. Such an procedure will continue once and again, and it will be proved that under our assumptions the rarefaction wave will be reflected by shock S and the wedge infinite times, and no additional shock forms in between S and the wedge.

Take another wedge with constant slope k_1. We can also consider the problem of the same coming supersonic flow past the new wedge. In this case we have an oblique shock front $S_1 : y_1 = \sigma_1 x$, and a constant state in the domain $G_1 : k_1 x < y < \sigma_1 x$. Denote the flow parameters behind the corresponding shock in above two cases by $U(x, y)$ and $U_1(x, y)$ respectively. We confirm that S, U approach S_1, U_1 as $x \to \infty$. Namely, we have

Theorem 1.1. *If k is less than the critical value determined by the coming flow, $0 < k_1 < k$, and $b - a$ is small, then for the above-mentioned problem of supersonic flow past wedge the shock $S(x)$ and the flow field $U(x, y)$ in between the shock and the surface of the wedge $G : \tilde{f}(x) < y < S(x)$ can be globally determined. The solution is piecewise smooth. Moreover,*

$$\lim_{\alpha\to 0}\sup_{|x|\le K}\left|\alpha S\Big(\frac{x}{\alpha}\Big) - \sigma_1 x\right| = 0, \qquad \forall K > 0, \tag{1.2}$$

$$\lim_{\alpha\to 0}\sup_{|(x,y)|\le K}\left|U\Big(\frac{x}{\alpha}, \frac{y}{\alpha}\Big) - U_1\Big(\frac{x}{\alpha}, \frac{y}{\alpha}\Big)\right| = 0, \qquad \forall K > 0,\ (x, y) \in G \cap G_1. \tag{1.3}$$

Figure 1

The theorem will be proved by direct construction. In Section 2 we review some basic facts on the potential flow equation and hodograph transformation. In Section 3 and Section 4 we prove Theorem 1.1 by a series of lemmas. Particularly, Lemmas 3.1 and 3.2 in Section 3 describe the domain under hodograph transformation. Lemmas 3.3 and 3.4 give us the solution on (u, v) plane in the case when interaction happens. Then in Section 4 we prove the invertibility of the solution $x(u, v), y(u, v)$ in each subtriangle and then obtain the global existence on the physical space and the asymptotic behaviour of the solutions.

§2. Equation and Hodograph Transformation

Throughout this paper we will use the potential flow equation to describe the supersonic flow.

$$\frac{\partial}{\partial x}(\Phi_x H) + \frac{\partial}{\partial y}(\Phi_y H) = 0 \tag{2.1}$$

where Φ is potential satisfying $\nabla\Phi = \vec{v}$, H represents the density, which is determined by

Bernoulli's relation

$$\frac{1}{2}(\Phi_x^2+\Phi_y^2)+\frac{\gamma H^{\gamma-1}}{\gamma-1}=\frac{1}{2}\tilde{q}^2 \tag{2.2}$$

with γ being the adiabatic exponent satisfying $1\le\gamma\le 3$.

Denote by (u_0, v_0, ρ_0) the state of coming flow, and by $[\ \cdot\ ]$ the jump of corresponding quantity, then the flow parameters on the both sides of a shock should satisfy

$$[u]+\sigma[v]=0,\quad \sigma[\rho u]-[\rho v]=0, \tag{2.3}$$

where $\sigma=\frac{dy}{dx}$ is the slope of the shock. (2.3) implies

$$\cos(\theta-\theta_0)=\frac{\rho q^2+\rho_0 q_0^2}{(\rho+\rho_0)q_0 q}, \tag{2.4}$$

where $q=\sqrt{u^2+v^2}, \theta=\arctan v/u$. The equation (2.4) is called the equation of shock polar. The graph of (2.4) is a curve with a double point at $A(u_0, v_0)$. In the sequel the shock polar with double point (u_0, v_0) will be denoted by Pol_A, and the branch with positive (resp. negative) slope near A is denoted by Pol_A^+ (resp. Pol_A^-).

Turn to the rarefaction wave, we rewrite (2.1) as

$$\begin{cases} (u^2-a^2)u_x+uv(u_y+v_x)+(v^2-a^2)v_y=0, \\ u_y=v_x, \end{cases} \tag{2.5}$$

where a represents the sonic speed. Introduce the hodograph transformation

$$T: u=u(x,y), v=v(x,y), \tag{2.6}$$

which is invertible. If $J=\frac{\partial(u,v)}{\partial(x,y)}\neq 0$, then the system can be rewritten as

$$\begin{cases} (u^2-a^2)y_v-uv(y_u+x_v)+(v^2-a^2)x_u=0, \\ x_v=y_u. \end{cases} \tag{2.7}$$

The characteristic directions for (2.5) and (2.7) are

$$\begin{aligned} \Big(\frac{dy}{dx}\Big)_\pm &= \lambda_\pm=\frac{uv\pm a\sqrt{u^2+v^2-a^2}}{u^2-a^2}, \\ \Big(\frac{dv}{du}\Big)_\pm &= \mu_\pm=\frac{-uv\mp a\sqrt{u^2+v^2-a^2}}{v^2-a^2} \quad (=-(\lambda_\mp)^{-1}). \end{aligned} \tag{2.8}$$

The characteristics through P on (u,v) plane corresponding to $\mu_\pm$ is denoted by $\mathrm{Char}_P^\pm$.

Replace u, v by $q=(u^2+v^2)^{1/2}, \theta=\arctan(v/u)$. Then the following differential relation holds on any characteristic line:

$$\frac{d\theta}{dq}=\frac{(udv-vdu)/q^2}{(udu+vdv)/q}=\frac{1}{q}\frac{u+\lambda_\pm v}{v-\lambda_\pm u}=\mp\frac{\sqrt{q^2-a^2}}{aq}. \tag{2.9}$$

On the (u,v) plane the integration of (2.8) is epicycloid:

$$\begin{cases} u=a_*(\cos\mu(\omega-\omega_*)\cos\omega+\mu^{-1}\sin\mu(\omega-\omega_*)\sin\omega), \\ v=a_*(\cos\mu(\omega-\omega_*)\sin\omega-\mu^{-1}\sin\mu(\omega-\omega_*)\cos\omega), \end{cases} \tag{2.10}$$

where $\mu=(\frac{\gamma-1}{\gamma+1})^{\frac{1}{2}}$, a_* is the critical sonic speed. (2.10) can also be the condition for the state which can be connected with the state (u_0, v_0) by a rarefaction wave.

For fixed ω_*, the part $\omega>\omega_*$ (resp. $\omega<\omega_*$) of the arc determined by (2.10) corresponds to μ_+ (resp. μ_-) characteristics.

$$(\omega-\omega_*)=\pm\mu^{-1}\Big(\frac{\pi}{2}-\arcsin\frac{a}{a_*}\Big). \tag{2.11}$$

Through any point (u, v) satisfying $a_*^2 < u^2 + v^2 < \mu^{-2} a_*^2$ there are two epicycloids corresponding to positive and negative signs in (2.11). When $\omega - \omega_* > 0$ (resp. < 0), the corresponding curve is μ_+ (resp. μ_-) characteristics. Hence ω, ω_* are denoted by ω_+, ω_{*+} (resp. ω_-, ω_{*-}) respectively.

Under the hodograph transformation the whole domain in between the wedge and the shock maps a curved triangle enclosed by shock polar A_1B, characteristics BC_1 and the straight line C_1A_1 (see Figure 2). Here the point B (resp. A_1) represents the intersection of Pol_A^- with the ray $v = ku$ (resp. $v = k_1u$), and C_1 represents the intersection of Char_B^- with $v = k_1u$.

As mentioned in Section 1, the rarefaction wave R_1 is mapped by T on the characteristics BC_1. Near shock S the rarefaction wave R_1 is reflected, and interaction between R_1 and its reflective wave R_2 takes place. Through the intersection L_1 of S with the tail of R_1 draw a λ_- characteristics. If $b - a$ is small, the λ_- characteristics intersects the head of R_1 at M_1, then the curved triangle $\triangle L_1M_1L_2$ is mapped on the triangle BC_1D_1 on (u, v) plane, where C_1D_1 is a μ_+ characteristic curve. The curve C_1D_1 is also the image of whole reflected rarefaction wave R_2 with its head L_2N_1 and tail M_1H_3. Then R_2 is reflected by the wedge, and another interaction takes place. The new interaction area on (x, y) plane is the curved triangle $\triangle H_3N_1H_4$, whose image on (u, v) plane is the triangle $\triangle C_1D_1C_2$. The reflection of R_2 is another rarefaction wave R_3 starting from N_1H_4 with image D_1C_2. Such a reflective procedure will continue infinitely, and $\triangle A_1BC_1$ will be filled up by all sub-triangles $\triangle C_iD_iC_{i+1}$ and $\triangle D_iC_{i+1}D_{i+1} (i = 1, 2, \cdots)$. We may solve (2.7) in each sub-triangle, and then obtain the solution of (2.6) by means of T (or its inverse). Here the two key points are: 1) The invertibity of map T on each sub-triangle. 2) All simple waves produced by successively reflection are expanding.

Figure 2

§3. The Solution on Phase Plane

Lemma 3.1. *For any point P outside the sonic circle,* Pol_P *and* Char_P *are tangential to each other at P.*

Proof. For our convenience we use the polar coordinates (ρ, θ), and simply take P as A without lose of generality. Denote $M_0 = (u_0^2 + v_0^2)^{1/2}/a_0, t = \rho/\rho_0$. The equation of shock polar can be written as

$$\cos(\theta-\theta_0)=\frac{M_0^2-\frac{2}{\gamma-1}\frac{t}{t+1}(t^{\gamma-1}-1)}{M_0\sqrt{M_0^2-\frac{2}{\gamma-1}(t^{\gamma-1}-1)}}. \tag{3.1}$$

Therefore

$$\begin{aligned}\frac{d\theta}{dt}&=(-\sin(\theta-\theta_0))^{-1}\frac{d}{dt}\left(\frac{M_0^2-\frac{2}{\gamma-1}\frac{t}{t+1}(t^{\gamma-1}-1)}{M_0\sqrt{M_0^2-\frac{2}{\gamma-1}(t^{\gamma-1}-1)}}\right)\\&=\pm\frac{(t-1)[(\frac{\gamma-1}{2}t^{\gamma-2}+\frac{t^{\gamma-1}-1}{t^2-1})M_0^2-\frac{2}{\gamma-1}\frac{t^{\gamma-1}-1}{t^2-1}(\frac{\gamma-1}{2}t^{\gamma}+\frac{\gamma+1}{2}t^{\gamma-1}-1)]}{(M_0^2-\frac{2}{\gamma-1}(t^{\gamma-1}-1))\sqrt{(t^2-1)(t^{\gamma-1}-1)(\frac{\gamma-1}{2}M_0^2-\frac{t^{\gamma-1}-1}{t^2-1}t^2)}}.\end{aligned} \tag{3.2}$$

When $t\to1$, we have $\frac{t^{\gamma-1}-1}{t^2-1}\to\frac{\gamma-1}{2}$, then

$$\frac{d\theta}{dt}\to\pm\frac{\sqrt{M_0^2-1}}{M_0^2}. \tag{3.3}$$

On the other hand, (2.11) indicates that on the characteristics we have

$$\frac{d\theta}{d\rho}=\frac{d\theta}{dq}\frac{dq}{d\rho}=\frac{d\theta}{dq}\Big(-\frac{a^2}{\rho q}\Big)=\pm\frac{t^{\frac{\gamma-3}{2}}\sqrt{M_0^2-\frac{2}{\gamma-1}(t^{\gamma-1}-1)-t^{\gamma-1}}}{\rho_0(M_0^2-\frac{2}{\gamma-1}(t^{\gamma-1}-1))}, \tag{3.4}$$

which also implies (3.3). Hence Pol_A is tangential to Char_A at A.

Lemma 3.2. *The angle between the curves BC_1 and BA_1 is not zero, and is a quantity of order $O(s^2)$, where s is the length of the arc BA.*

Proof. Assume that B is a point on the curve Pol_A other than A. Then the angle between Pol_A and Pol_B at point B is not zero. In fact, denoting the coordinates of point B by (u_1,v_1), the shock polar Pol_B^- by $\phi(\tilde t)$, and denoting $M_1=(u_1^2+v_1^2)^{1/2}/a_1,\theta_1=\arctan(v_1/u_1),t_1=\rho_1/\rho_0,\tilde t=\rho/\rho_1$, we have

$$\cos(\phi-\theta_1)=\frac{M_1^2-\frac{2}{\gamma-1}\frac{\tilde t}{\tilde t+1}(\tilde t^{\gamma-1}-1)}{M_1\sqrt{M_1^2-\frac{2}{\gamma-1}(\tilde t^{\gamma-1}-1)}} \tag{3.5}$$

and $\frac{d\phi}{d\tilde t}|_{\tilde t=1}=-\frac{\sqrt{M_1^2-1}}{M_1^2}$. Because of $M_1^2=\frac{q_1^2}{a_1^2}=(M_0^2-\frac{2}{\gamma-1}(t_1^{\gamma-1}-1))/t_1^{\gamma-1}$, we have

$$\frac{d\phi}{dt}\Big|_{t=t_1}=\frac{\rho_0}{\rho_1}\frac{d\phi}{d\tilde t}\Big|_{\tilde t=1}=-\frac{t_1^{\frac{\gamma-3}{2}}\sqrt{M_0^2-\frac{\gamma+1}{\gamma-1}t_1^{\gamma-1}+\frac{2}{\gamma-1}}}{M_0^2-\frac{2}{\gamma-1}(t_1^{\gamma-1}-1)}. \tag{3.6}$$

Let $s=t_1-1$ be a small quantity. Then by Taylor expansion $t_1^{\gamma-1}=1+(\gamma-1)s+\frac12(\gamma-1)(\gamma-2)s^2+\cdots$, and

$$\begin{aligned}\frac{t_1^{\gamma-1}-1}{t_1^2-1}&=\frac{\gamma-1}{2}\Big(1+\frac{\gamma-3}{2}s+\frac{(\gamma-2)(\gamma-3)}{6}s^2\Big)+O(s^3),\\M_0^2-\frac{2}{\gamma-1}(t_1^{\gamma-1}-1)&=M_0^2\Big(1-\frac{2}{M_0^2}s-\frac{\gamma-2}{M_0^2}s^2\Big)+O(s^3),\\M_0^2-\frac{\gamma+1}{\gamma-1}t_1^{\gamma-1}-\frac{2}{\gamma-1}&=N\Big(1-\frac{\gamma+1}{N}s-\frac{(\gamma+1)(\gamma-2)}{2N}s^2\Big)+O(s^3),\end{aligned}$$

where $N = M_0^2 - 1$. Substituting them into the expression of $\frac{d\phi}{dt}$, we have

$$\frac{d\phi}{dt} = \frac{N^{\frac{1}{2}}}{M_0^2}\Big[1 + \Big(\frac{\gamma-3}{2} - \frac{\gamma+1}{2N} + \frac{2}{M_0^2}\Big)s + \Big(\frac{(\gamma-3)(\gamma-5)}{8} - \frac{(\gamma-1)(\gamma-2)}{4N} - \frac{(\gamma+1)^2}{8N^2} + \frac{\gamma-2}{M_0^2} + \frac{4}{M_0^4} - \frac{(\gamma-3)(\gamma+1)}{4N} + \frac{\gamma-3}{M_0^2} - \frac{\gamma+1}{NM_0^2}\Big)s^2\Big] + O(s^3). \tag{3.7}$$

On the other hand, (3.2) implies at $t = t_1$

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{N^{\frac{1}{2}}}{M_0^2}\Big(1 + \Big(\frac{3\gamma-7}{4} - \frac{3\gamma+3}{4N}\Big)s + \Big(\frac{(\gamma-3)(8\gamma-19)}{24} - \frac{20\gamma^2-25\gamma-45}{24N}\Big)s^2\Big)\\
&\quad\cdot\Big(1 - \frac{2}{M_0^2}s - \frac{\gamma-2}{M_0^2}s^2\Big)^{-1}\Big(1 + \frac{\gamma-1}{2}s + \frac{(\gamma-2)(2\gamma-3)}{12}s^2\Big)^{-\frac{1}{2}}\\
&\quad\cdot\Big(1 - \frac{\gamma+1}{2N}s - \frac{2\gamma^2-\gamma-3}{12N}s^2\Big)^{-\frac{1}{2}} + O(s^3)\\
&= \frac{N^{\frac{1}{2}}}{M_0^2}\Big[1 + \Big(\frac{3\gamma-7}{4} - \frac{3\gamma+3}{4N} + \frac{2}{M_0^2} - \frac{\gamma-1}{4} + \frac{\gamma+1}{4N}\Big)s\\
&\quad + \Big(\frac{(\gamma-3)(8\gamma-19)}{24} + \frac{-20\gamma^2+25\gamma+45}{24N} + \frac{\gamma-2}{M_0^2} + \frac{4}{M_0^4} - \frac{(\gamma-2)(2\gamma-3)}{24}\\
&\quad + \frac{3(\gamma-1)^2}{32} + \frac{2\gamma^2-\gamma-3}{24N} + \frac{3(\gamma+1)^2}{32N^2} + \frac{2}{M_0^2}\Big(-\frac{\gamma-1}{4} + \frac{\gamma+1}{4N}\Big)\\
&\quad + \Big(\frac{3\gamma-7}{4} - \frac{3\gamma+3}{4N}\Big)\Big(\frac{2}{M_0^2} - \frac{\gamma-1}{4} + \frac{\gamma+1}{4N}\Big) - \frac{\gamma^2-1}{16N}\Big)s^2\Big] + O(s^3).\end{aligned} \tag{3.8}$$

Therefore, we have

$$\frac{d\theta}{dt} - \frac{d\phi}{dt} = Ks^2 + O(s^3), \tag{3.9}$$

where

$$K = \frac{\gamma+1}{32}N^{-\frac{3}{2}}(4 + (\gamma-3)M_0^2).$$

This means $\frac{d\theta}{dt} - \frac{d\phi}{dt} \geq \delta_0 > 0$, provided s is small and $s \geq s_0 > 0$.

Remark. Notice that all BC_1, C_1D_1, D_1C_2 are epicycloid described by equation (2.9) (with different parameter $\omega_{*\pm}$). Therefore, for given k, k_1 the angle $\angle BC_1D_1$, $\angle D_1C_1C_2$, $\angle C_1D_1C_2$ and $\angle C_2D_1A_1$ are all bounded away from 0, i.e. each sub-triangle on Figure 2 is a triangle with vertex angles being away from zero.

Next we solve the system (2.7) in $\triangle BC_1D_1$ and $\triangle C_1D_1C_2$. On boundary BC_1 the functions x, y are known. In fact, u and v are constant on each λ_+ characteristic line in R_1 on (x, y) plane. Hence the direction of the velocity is $(1, f'(x))$, where x is the coordinate of the end of characteristics on the wedge. From (2.10) we have

$$f'(x) = \frac{\cos\mu(\omega-\omega_*)\sin\omega - \mu^{-1}\sin\mu(\omega-\omega_*)\cos\omega}{\cos\mu(\omega-\omega_*)\cos\omega + \mu^{-1}\sin\mu(\omega-\omega_*)\sin\omega}. \tag{3.10}$$

Denote the left side of (3.10) by $\ell(\omega)$, its derivative is positive. In view of the monotonicity of $f'(x)$, (3.10) determines a smooth function $x = x(\omega)$ with $x'(\omega) > 0$. Correspondingly we have

$$y = f(x(\omega)) = y(\omega).$$

Therefore, the functions x, y are given on BC_1 :

$$x = x_0, \quad y = y_0. \tag{3.11}$$

The boundary BD_1 is a part of the shock polar, on which the shock relation (2.3) is satisfied. Therefore,

$$\frac{dy}{dx} = \frac{q_0 - u}{v} \tag{3.12}$$

in view of $\sigma = \frac{dy}{dx}$ and $(u_0, v_0) = (q_0, 0)$.

Remark. Taking $\alpha = \omega_{*+}, \beta = \omega_{*-}$ as new variables instead of u, v, we can introduce a curved coordinate system with two characteristics as its coordinate curve. In this coordinate system BC_1 becomes $\beta = 0$ and BD_1 becomes $\beta = h(\alpha)$. In view of Lemma 3.1, we have $h'(\alpha) \geq c_0 > 0$, and $\alpha \mapsto \beta = h(\alpha)$ is a one-to-one map.

Lemma 3.3. *The boundary value problem* (2.7), (3.11), (3.12) *has a unique differentiable solution.*

Proof. Introducing

$$X = (v^2 - a^2)(x + \mu_+ y), Y = (v^2 - a^2)(x + \mu_- y), \tag{3.13}$$

we have

$$x = \frac{\mu_- X - \mu_+ Y}{(v^2 - a^2)(\mu_- - \mu_+)}, y = \frac{X - Y}{(v^2 - a^2)(\mu_- - \mu_+)}, \tag{3.14}$$

and the system (3.7) can be reduced to a diagonal form

$$\begin{cases} \frac{\partial X}{\partial u} + \mu_- \frac{\partial X}{\partial v} + A_+ X + B_+ Y = 0, \\ \frac{\partial Y}{\partial u} + \mu_+ \frac{\partial Y}{\partial v} + A_- X + B_- Y = 0, \end{cases} \tag{3.15}$$

where

$$A_\pm = \frac{(\mu_\pm (v^2 - a^2))_u + \mu_\mp (\mu_\pm (v^2 - a^2))_v + \mu_- ((v^2 - a^2)_u + (\mu_\mp (v^2 - a^2))_v)}{(v^2 - a^2)(\mu_+ - \mu_-)},$$

$$B_\pm = \frac{(\mu_\pm (v^2 - a^2))_u + \mu_\mp (\mu_\pm (v^2 - a^2))_v + \mu_+ ((v^2 - a^2)_u + (\mu_\mp (v^2 - a^2))_v)}{(v^2 - a^2)(\mu_- - \mu_+)}.$$

Take α, β as new variables. Then (3.15) becomes

$$\begin{cases} \frac{\partial X}{\partial \alpha} + A_+ X + B_+ Y = 0, \\ \frac{\partial Y}{\partial \beta} + A_- X + B_- Y = 0. \end{cases} \tag{3.16}$$

The boundary conditions are

$$Y = Y_0(\alpha) \qquad \text{on } \beta = 0, \tag{3.17}$$

$$\frac{dy}{dx} = g(\alpha) \qquad \text{on } \beta = h(\alpha). \tag{3.18}$$

Now we take an iterative scheme as follows. First, $(X^{(0)}, Y^{(0)})$ is chosen to satisfy (3.17), (3.18). When $(X^{(n)}, Y^{(n)})$ are given, we take

$$Y^{(n+1)}(\alpha, \beta) = \int_0^\beta (A_- X^{(n)} + B_- Y^{(n)}) d\beta + Y_0(\alpha), \tag{3.19}$$

which also gives the value of $Y^{(n+1)}$ on BD_1. Denote $Y^{(n+1)}(\alpha, h(\alpha))$ by $d^{(n+1)}(\alpha)$. Then (3.14) implies

$$y^{(n+1)}(\alpha, h(\alpha)) = -\mu_-^{-1} x^{(n+1)}(\alpha, h(\alpha)) + \mu_-^{-1} d^{(n+1)}(\alpha)/(v^2 - a^2). \tag{3.20}$$

By differentiating we have

$$(g(\alpha)+\mu_-^{-1})\frac{d}{d\alpha}x^{(n+1)}(\alpha,h(\alpha))=-(\mu_-^{-1})'x^{(n+1)}(\alpha,h(\alpha))+\Big(\frac{d^{(n+1)}(\alpha)}{\mu_-(v^2-a^2)}\Big)'. \tag{3.21}$$

Because of $g(\alpha)+\mu_-^{-1}\neq 0$, (3.21) can be solved explicitly. Finally, the first equation in (3.16) gives

$$X^{(n+1)}(\alpha,\beta)=\int_{h^{-1}(\beta)}^{\alpha}(A_+X^{(n)}+B_+Y^{(n)})d\alpha+X^{(n+1)}(h^{-1}(\beta),\beta). \tag{3.22}$$

Similarly, for the difference

$$\tilde{X}^{(n+1)}=X^{(n+1)}-X^{(n)},\tilde{Y}^{(n+1)}=Y^{(n+1)}-Y^{(n)},$$

we can establish the estimate

$$|\tilde{X}^{(n)}|,|\tilde{Y}^{(n)}|\le K_1C^n\frac{(\alpha+\beta)^n}{n!}, \tag{3.23}$$

which implies the convergence of the sequence $\{X^{(n)},Y^{(n)}\}$ and the existence of the solution immediately.

Besides, from the system (3.16) itself we know $\frac{\partial X}{\partial\alpha},\frac{\partial Y}{\partial\beta}$ are continuous. By differentiating (3.19)-(3.22) we can obtain the convergence of $\{\frac{\partial Y^{(n+1)}}{\partial\alpha}\}$ and $\{\frac{\partial X^{(n+1)}}{\partial\beta}\}$, which leads to the differentiability of $X(\alpha,\beta)$ and $Y(\alpha,\beta)$.

By using similar method we can establish

Lemma 3.4. *There is a unique differentiable solution of the equation* (2.7) *with the boundary conditions*

$$x,y\ \text{are}\ C^1\ \text{on}\ C_1D_1\ \text{ and satisfy the characteristic relation}, \tag{3.24}$$

$$y=k_1x\ \text{on}\ C_1C_2. \tag{3.25}$$

§4. The Solution on Physical Plane

Let us turn to the solution of (2.5) now. Obviously, if the determinant $|\frac{\partial(x,y)}{\partial(u,v)}|$ is not zero, then the solution of (2.7) will also give us a solution $(u(x,y),v(x,y))$ of (2.5). Indeed, we have

Lemma 4.1. *The determinant* $|\frac{\partial(x,y)}{\partial(u,v)}|$ *is not zero in* $\triangle BC_1D_1$.

Proof. As did in the proof of Lemma 3.3, we take α,β as two characteristic variables. Since $\frac{\partial(\alpha,\beta)}{\partial(u,v)}$ is an invertible matrix, we only need to indicate $J_1=\frac{\partial x}{\partial\alpha}\frac{\partial y}{\partial\beta}-\frac{\partial x}{\partial\beta}\frac{\partial y}{\partial\alpha}\neq 0$ instead.

Noticing that

$$\frac{\partial y}{\partial\alpha}=\lambda_-\frac{\partial x}{\partial\alpha},\frac{\partial y}{\partial\beta}=\lambda_+\frac{\partial x}{\partial\beta}, \tag{4.1}$$

we have

$$J_1=(\lambda_+-\lambda_-)\frac{\partial x}{\partial\alpha}\frac{\partial x}{\partial\beta}. \tag{4.2}$$

To indicate $\frac{\partial x}{\partial\alpha}\neq 0$ and $\frac{\partial x}{\partial\beta}\neq 0$, we differentiate (4.1)

$$\frac{\partial^2 y}{\partial\alpha\partial\beta}=\frac{\partial\lambda_-}{\partial\beta}\frac{\partial x}{\partial\alpha}+\lambda_-\frac{\partial^2 x}{\partial\alpha\partial\beta}\frac{\partial^2 y}{\partial\alpha\partial\beta}=\frac{\partial\lambda_+}{\partial\alpha}\frac{\partial x}{\partial\beta}+\lambda_+\frac{\partial^2 x}{\partial\alpha\partial\beta}.$$

Then

$$(\lambda_+-\lambda_-)\frac{\partial^2 x}{\partial\alpha\partial\beta}=-\frac{\partial\lambda_+}{\partial\alpha}\frac{\partial x}{\partial\beta}+\frac{\partial\lambda_-}{\partial\beta}\frac{\partial x}{\partial\alpha}. \tag{4.3}$$

By using $\lambda_\pm = -\mu_\mp^{-1}$, we have

$$(\lambda_+ - \lambda_-)\frac{\partial^2 x}{\partial\alpha\partial\beta} = -\frac{1}{\mu_-^2}\frac{\partial\mu_-}{\partial\alpha}\frac{\partial x}{\partial\beta} + \frac{1}{\mu_+^2}\frac{\partial\mu_+}{\partial\beta}\frac{\partial x}{\partial\alpha}, \tag{4.4}$$

where $\frac{\partial\mu_+}{\partial\beta} > 0, \frac{\partial\mu_-}{\partial\alpha} < 0$ according to the meaning of α, β.

The differentiation of (3.10) yields $\frac{\partial x}{\partial\omega_{*+}} > 0$, which implies $\frac{\partial x}{\partial\alpha} > 0$ on $\beta = 0$. Consider the value $\frac{\partial x}{\partial\beta}$ at B. Since $\frac{dy}{dx} = \sigma$ on $\beta = h(\alpha)$, we have

$$\frac{\partial y}{\partial\alpha} + \frac{\partial y}{\partial\beta}h'(\alpha) = \sigma\Big(\frac{\partial x}{\partial\alpha} + \frac{\partial x}{\partial\beta}h'(\alpha)\Big). \tag{4.5}$$

Then

$$(\lambda_- - \sigma)\frac{\partial x}{\partial\alpha} + (\lambda_+ - \sigma)\frac{\partial x}{\partial\beta}h'(\alpha) = 0,$$

$$\frac{\partial x}{\partial\beta} = \frac{1}{h'(\alpha)}\frac{\sigma - \lambda_-}{\lambda_+ - \sigma}\frac{\partial x}{\partial\alpha}. \tag{4.6}$$

In view of $h'(\alpha) > 0$ we have $\frac{\partial x}{\partial\beta} > 0$ at B.

Now we can prove $\frac{\partial x}{\partial\alpha} > 0, \frac{\partial x}{\partial\beta} > 0$ in $\triangle BC_1D_1$ by reduction to absurdity. Suppose that α_1 is the maximun of α_0, such that $\frac{\partial x}{\partial\alpha} > 0, \frac{\partial x}{\partial\beta} > 0$ hold for $\alpha < \alpha_0$. Then there is (α_1, β_1) such that $\frac{\partial x}{\partial\alpha}\frac{\partial x}{\partial\beta} = 0$. Without loss of generality we assume $\frac{\partial x}{\partial\alpha} = 0$. However, from (4.4) we have $\frac{\partial^2 x}{\partial\alpha\partial\beta} > 0$, which yields

$$\frac{\partial x}{\partial\alpha}(\alpha_1, \beta_1) = \int_0^{\beta_1}\frac{\partial^2 x}{\partial\alpha\partial\beta}d\beta + \frac{\partial x}{\partial\alpha}(\alpha_1, 0) > 0. \tag{4.7}$$

The contradiction implies $\frac{\partial x}{\partial\alpha}$ and $\frac{\partial x}{\partial\beta}$ must be positive in the whole triangle. This proves $J_1 > 0$.

According to Lemma 4.1 we can obtain the solution $u(x,y), v(x,y)$ in the triangle $\triangle L_1M_1L_2$ from $x(u,v), y(u,v)$ in $\triangle BC_1D_1$. In order to construct the solution on physical plane following the procedure mentioned in the end of Section 2, we need the following proposition.

Lemma 4.2. *The λ_- characteristics is divergent along M_1L_2.*

Proof. Because the λ_- characteristics on (x,y) plane is perpendicular to the corresponding μ_+ characteristics on (u,v) plane, the conclusion of this lemma can be derived from the fact that the slope $\ell(\omega)$ of μ_+ characteristics is monotonically increasing. And this is just the conclusion of (3.10).

Remark. Lemma 4.2 also indicates that in the condition (3.24) $X_0(\beta)$ satisfies $\frac{\partial X_0}{\partial\beta} > 0$.

Lemma 4.3. $|\frac{\partial(x,y)}{\partial(u,v)}| \neq 0$ *in* $\triangle C_1D_1C_2$.

Proof. From (3.25) we can derive

$$\ell'(\beta)\frac{\partial y}{\partial\alpha} + \frac{\partial y}{\partial\beta} = h_1\Big(\ell'(\beta)\frac{\partial x}{\partial\alpha} + \frac{\partial x}{\partial\beta}\Big), \tag{4.9}$$

therefore,

$$x_\beta = \ell'\frac{h_1 - \lambda_-}{\lambda_+ - h_1}x_\alpha. \tag{4.10}$$

The next part of the proof is similar to Lemma 4.1 with (4.6) replaced by (4.10).

By alternatively using the argument in Lemmas 4.1–4.3 the solution $u(x, y)$ and $v(x, y)$ in each domain of interaction on (x, y) plane can be obtained from the solution $x(u, v), y(u, v)$ in corresponding curved triangle on (u, v) plane. As mentioned in Section 2 we may connect these solutions by simple wave and constant to obtain a global solution in the whole domain between the shock and the surface of the wedge.

Finally, let us prove the second part of Theorem 1.1. Obviously, following the procedure of our construction (u, v) tends to the point A_1. That means

$$\lim u(x, y) \to u_{A_1}, \quad \lim v(x, y) \to v_{A_1}. \tag{4.11}$$

From Bernoulli's relation we have $\rho(x, y) \to \rho_{A_1}$ and the slope of characteristics on (x, y) plane tends to a constant. Particularly, we have

$$\sigma = \frac{\rho v - \rho_0 v_0}{\rho u - \rho_0 u_0} \to \sigma_1 \Big(= \frac{\rho_{A_1} v_{A_1} - \rho_0 v_0}{\rho_{A_1} u_{A_1} - \rho_0 u_0} \Big). \tag{4.12}$$

Then for any $\epsilon > 0$ there is $d > 0$ such that $|\sigma - \sigma_1| < \epsilon$ if $x > d$. Therefore

$$|s(x) - s(d) - \sigma_1(x - d)| = \Big| \int_d^x (\sigma(\xi) - \sigma_1) d\xi \Big| < \epsilon(x - d). \tag{4.13}$$

In view of $s_1(x) = \sigma_1 x$, by replacing x by $\frac{x}{\alpha}$ we have

$$\alpha \Big| s\Big(\frac{x}{\alpha}\Big) - s_1\Big(\frac{x}{\alpha}\Big) - s(d) + \sigma_1 d \Big| < \epsilon x - \alpha\epsilon d. \tag{4.14}$$

Hence for any given K, we have

$$\lim_{\alpha \to 0} \sup_{|x| \le K} \alpha \Big| s\Big(\frac{x}{\alpha}\Big) - s_1\Big(\frac{x}{\alpha}\Big) \Big| < \epsilon K. \tag{4.15}$$

This implies (1.2), because ϵ can be arbitrarily small. Besides, (1.3) is the direct conclusion of (4.11).

Remark. The conclusion in our theorem still holds in the case $a = b$, when the surface of wedge becomes a broken line on (x, y) plane, and the rarefaction wave becomes a centre wave issuing from the point (a, ka).

References

[1] Chen, S. X., Linear approximation of shock reflection at a wedge with large angles, *Communication in PDEs,* **21:** 7& 8 (1996), 1103–1118.

[2] Courant, R. & Friedrichs, K. O., Supersonic flow and shock waves, Springer-Verlag, New York, 1976.

[3] Majda, A. J., Compressible fluid flow and systems of conservation laws in several space variables, Appl. Math. Sciences 53, Springer-Verlag, New York, 1984.

[4] Majda, A. J., One perspective on open problems in multi-dimensional conservation laws, *IMA,* (1990), 217–237.

[5] Morawetz, C. S., Potential theory for regular and Mach reflection of a shock at a wedge, *Comm. Pure Appl. Math.,* **47** (1994), 593-624.

[6] Smoller, J., Shock waves and reaction-diffusion equation, Springer-Verlag, New York, Heidelberg, Berlin, 1982.