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MULTIPLE PARAMETERS IDENTIFICATION PROBLEMS IN RESISTIVITY WELL-LOGGING**

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Abstract

In petroleum exploitation, the main aim of resistivity well-logging is to determine the resistivity of the layers by measuring the potential on the electrodes. This mathematical problem can be described as an inverse problem for the elliptic equivalued surface boundary value problem. In this paper, the author gets the expression of the derivative functions of the potential on the electrodes with respect to the resistivity of the layers. This allows us to solve the identification problem of the resistivity of the layers.

Keywords Multiple parameters identification problem, Resistivity well-logging, Inverse problem, Equivalued surface boundary value problem

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§1. Introduction

Resistivity well-logging is one of the most common techniques in petroleum exploitation. As in the following figure, we suppose that the layers are symmetric about the well axis and the central plane. Here, Ω_1 is the wellbore filled with mud of resistivity R_1; Ω_2 is the surrounding rock of resistivity R_2; Ω_3 and Ω_4 are two parts of the objective layer, where Ω_3 is the area occupied by the log tool. Usually the objective layer is sandy rock which is porous material. The mud filter fluid penetrates into the porosity and changes the resistivity of the domain Ω_3. Therefore, Ω_3 is called invaded area and we denote the resistivity in this domain by R_3. Ω_4 is the part of the objective layer of resistivity R_4 which is not invaded by the conductive fluid. Thus the potential function $u = u(x, y, z)$ of the layers satisfies the following elliptic equivalued surface boundary value problem in the domain $\Omega = \bigcup\limits_{i=1}^{4} \Omega_i$:

$$\frac{\partial}{\partial x}\left(\frac{1}{R}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{R}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{R}\frac{\partial u}{\partial z}\right) = 0 \quad \text{in } \Omega_i \ (i = 1, 2, 3, 4), \tag{1.1}$$

$$u|_{\Gamma_1} = 0, \tag{1.2}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_2} = 0, \tag{1.3}$$

$$u|_{\Gamma_0^j} = c \text{ (unknown constant)}, \quad j = 1, 2, 3, 4, \tag{1.4}$$

$$\int_{\Gamma_0^j} \frac{1}{R}\frac{\partial u}{\partial n} ds = I_j \text{ (known constant)}, \quad j = 1, 2, 3, 4, \tag{1.5}$$

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$$u^+|_\gamma = u^-|_\gamma, \tag{1.6}$$

$$\left(\frac{1}{R}\frac{\partial u}{\partial n}\right)^+\Bigg|_\gamma = \left(\frac{1}{R}\frac{\partial u}{\partial n}\right)^-\Bigg|_\gamma, \tag{1.7}$$

where R is the resistivity of the layers. We suppose it is piecewise constant: $R|_{\Omega_i} = R_i$ $(1 \le i \le 4)$. Here γ is the interface between the two different domains. The superscripts "+" and "−" stand for the values on both sides of γ which have been prescribed in the figure above. We suppose that the unit normal vector $\overrightarrow{n}$ has the same direction on both sides of γ_j.

Γ_1

γ Ω_2

Γ_2 Ω_1 Γ_1

γ γ

Γ_0^4

γ Ω_3 γ Ω_4

$\vdots$

Γ_0^1

Γ_2

Figure

If the geometrical structure of the formation and the resistivity in each subdomain are all known, problem (1.1)–(1.7) has a unique H^1 solution[2]. Also it is equivalent to the following variational problem: for any given $\phi \in V$, there exists a unique solution $u \in V$, such that

$$\sum_{i=1}^{4}\int_{\Omega_i}\frac{1}{R_i}\nabla u\cdot\nabla\phi dx = \sum_{j=1}^{4} I_j\cdot\phi|_{\Gamma_0^j}, \tag{1.8}$$

where

$$V = \{v \mid v\in H^1(\Omega),\ v|_{\Gamma_1}=0,\ v|_{\Gamma_0} = \text{constant},\ j=1,2,3,4\}, \tag{1.9}$$

and $\Omega = \bigcup_{i=1}^{4}\Omega_i$.

In the real well-logging, the potential value c_j on the electrode Γ_0^j $(j = 1,2,3,4)$ can be measured by certain instruments. We hope to get the resistivity of each domain, and then to determine the amount of the petroleum in the earth. This is an inverse problem for the elliptic equivalued surface boundary value problem. The identification problem of destinating layer is solved in [3]. In this paper, we get the expression of the derivative functions of the potential on the electrodes with respect to the resistivity of the layers and use this to solve the identification problem of the resistivity of the layers.

§2. Continuous Differentiability of Measuring Potential Value to the Resistivity of the Layer

For convenience, we consider the more general case. Suppose the domain Ω is composed by m subdomains:

$$\Omega = \bigcup_{i=1}^{m} \Omega_i, \tag{2.1}$$

where $\Omega_i \cap \Omega_j = \emptyset$ $(i \neq j,\ 1 \leq i, j \leq m)$. Denote the conductance in each subdomain Ω_i (the inverse of the resistivity) by k_i $(1 \leq i \leq m)$, which is still a positive constant. Let the boundary of each subdomain Ω_i $(1 \leq i \leq m)$ be suitably smooth. Assume the boundary of Ω

$$\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2, \tag{2.2}$$

where

$$\Gamma_0 = \bigcup_{i=1}^{m} \Gamma_0^i. \tag{2.3}$$

Here

$$\Gamma_0 \cap \Gamma_1 = \emptyset, \quad \Gamma_0^i \cap \Gamma_0^j = \emptyset \ (i \neq j,\ 1 \leq i, j \leq m),$$

and Γ_0^i $(1 \leq i \leq m)$ are m electrodes put on m different positions. The intensity of the current discharged by Γ_0^i is I_i $(1 \leq i \leq m)$.

Denote

$$V = \{v \mid v \in H^1(\Omega), v|_{\Gamma_0} = 0, v|_{\Gamma_0^i} = \text{constant } (1 \leq i \leq m)\}, \tag{2.4}$$

and I_i $(1 \leq i \leq m)$ are all known. Thus the corresponding variational form of the resistivity well-logging problem is: find $u \in V$, such that

$$\sum_{i=1}^{m} \int_{\Omega_i} k_i \nabla u \cdot \nabla \phi dx = \sum_{i=1}^{m} I_i \cdot \phi|_{\Gamma_0^i}, \quad \forall \phi \in V. \tag{2.5}$$

Obviously, this problem admits a unique solution $u \in V$.

For any given i $(1 \leq i \leq m)$, assume that the intensity of the current of the i-th electrode Γ_0^i is $I_i = 1$, and those of the other electrodes are all null: $I_j = 0$ $(j \neq i,\ 1 \leq j \leq m)$. Denote by (P_i) the above problem, its solution $u_i \in V$. The potential value on electrode Γ_0^i is $c_i = u_i|_{\Gamma_0^i}$. In the case of fixing any other parameters, c_i is a function of parameters $k_1, \cdots, k_m$. By solving m equivalued surface boundary value problems, we get

$$c_i = c_i(k_1, \cdots, k_m), \quad 1 \leq i \leq m. \tag{2.6}$$

As mentioned above, the aim of the inverse problem is to determine the coefficients k_j $(1 \leq j \leq m)$ by measuring the potential values on the electrodes $c_i = \bar{c}_i$ $(1 \leq i \leq m)$. In fact, this is a problem finding the inverse functions. We hope to calculate the coefficients k_j $(1 \leq j \leq m)$ by the theorem of implicit function at least in the local area. If $(k_1, \cdots, k_m) = (k_1^0, \cdots, k_m^0)$, we get by (2.6)

$$(c_1, \cdots, c_m) = (c_1^0, \cdots, c_m^0).$$

We want to determine the unique coefficients $\bar{k} = (\bar{k}_1, \cdots, \bar{k}_m)$ in the neighborhood of $k^0 = (k_1^0, \cdots, k_m^0)$ by measuring value $\bar{c} = (\bar{c}_1, \cdots, \bar{c}_m)$ in the neighborhood of $c^0 = (c_1^0, \cdots, c_m^0)$.

Obviously, to solve the multiple parameters identification problem, we should have the property mentioned above. This is an important step to solve the whole identification problem.

To make use of the theorem of implicit function, we consider the C^1 continuity of function (2.6), and get the expression of Jacobian determinant

$$|J| = \left|\frac{\partial c}{\partial k}\right|_{k=k^0}.$$

Given some suitable conditions, such that $|J| \neq 0$, we can determine the coefficients k_i $(1 \le i \le m)$ near $k = k^0$.

First we prove

Theorem 2.1. *The functions* $c_i = c_i(k_1, \cdots, k_m)$ $(1 \le i \le m)$ *are continuously differentiable, and*

$$\frac{\partial c_i}{\partial k_j} = -\int_{\Omega_j} |\nabla u_i|^2 dx, \quad 1 \le i, j \le m \tag{2.7}$$

hold, where u_i *is the solution to problem* (P_i).

Proof. For any fixed i $(1 \le i \le m)$, we consider problem (P_i). Let $u_i \in V$ and $u_i^n \in V$ be the solutions to problem (P_i) with parameters $(k_1, \cdots, k_j, \cdots, k_m)$ and $(k_1, \cdots, k_{j-1}, k_j^n, k_{j+1}, \cdots, k_m)$ respectively. And also let $k_j^n \to k_j > 0$ $(n \to \infty)$.

By (2.5), u_i, $u_i^n \in V$, and for any $\phi \in V$,

$$\sum_{s \neq j} \int_{\Omega_s} k_s \nabla u_i \cdot \nabla \phi dx + \int_{\Omega_j} k_j \nabla u_i \cdot \nabla \phi dx = \phi|_{\Gamma_0^i} \tag{2.8}$$

and

$$\sum_{s \neq j} \int_{\Omega_s} k_s \nabla u_i^n \cdot \nabla \phi dx + \int_{\Omega_j} k_j^n \nabla u_i^n \cdot \nabla \phi dx = \phi|_{\Gamma_0^i} \tag{2.9}$$

hold.

Substracting (2.8) from (2.9), we get

$$\begin{aligned}&\sum_{s \neq j} \int_{\Omega_s} k_s (\nabla u_i^n - \nabla u_i) \cdot \nabla \phi dx + \int_{\Omega_j} k_j^n (\nabla u_i^n - \nabla u_i) \cdot \nabla \phi dx \\ &+ \int_{\Omega_j} (k_j^n - k_j) \nabla u_i \cdot \nabla \phi dx = 0, \quad \forall \phi \in V.\end{aligned} \tag{2.10}$$

Since u_i, $u_i^n \in V$, we choose $\phi = u_i^n - u_i \in V$ in (2.10), and have

$$\begin{aligned}&\sum_{s \neq j} \int_{\Omega_s} k_s |\nabla u_i^n - \nabla u_i|^2 dx + \int_{\Omega_j} k_j^n |\nabla u_i^n - \nabla u_i|^2 dx \\ &+ \int_{\Omega_j} (k_j^n - k_j) \nabla u_i \cdot (\nabla u_i^n - \nabla u_i) dx = 0.\end{aligned} \tag{2.11}$$

Noticing that $k_j^n \to k_j > 0$ as $n \to \infty$, we see that there exists $\delta > 0$, such that $k_j^n \ge \delta > 0$

for sufficiently large n. By (2.11),

$$\begin{aligned}
&\|\nabla u_i^n - \nabla u_i\|^2_{L^2(\Omega)} \\
&= \sum_{s\neq j}\int_{\Omega_s} |\nabla u_i^n - \nabla u_i|^2 dx + \int_{\Omega_j} |\nabla u_i^n - \nabla u_i|^2 dx \\
&\le C\left(\sum_{s\neq j}\int_{\Omega_s} k_s|\nabla u_i^n - \nabla u_i|^2 dx + \int_{\Omega_j} k_j^n|\nabla u_i^n - \nabla u_i|^2 dx\right) \\
&\le C|k_j^n - k_j|\left(\int_{\Omega_j} |\nabla u_i|^2 dx\right)^{1/2}\left(\int_{\Omega_j} |\nabla u_i^n - \nabla u_i|^2 dx\right)^{1/2},
\end{aligned}$$

where C is a positive constant independent of n. Noticing that $\left(\int_{\Omega_j} |\nabla u|^2 dx\right)^{1/2}$ is independent of n, we have

$$\|u_i^n - u_i\|_{H^1(\Omega)} \le M|k_j^n - k_j|, \tag{2.12}$$

where M is a positive constant independent of n. So, when $k_j^n \to k_j$,

$$u_i^n \to u_i \quad (n \to \infty) \quad \text{strongly convergent in } H^1(\Omega). \tag{2.13}$$

So

$$\int_{\Omega_j} |\nabla u_i^n|^2 dx \to \int_{\Omega_j} |\nabla u_i|^2 dx \tag{2.14}$$

and

$$\begin{aligned}
&c_i(k_1,\cdots,k_{j-1},k_j^n,k_{j+1},\cdots,k_m) = u_i^n|_{\Gamma_0^i} \\
&\to c_i(k_1,\cdots,k_j,\cdots,k_m) = u_i|_{\Gamma_0^i}.
\end{aligned} \tag{2.15}$$

By (2.8) and the trace theorem, $\|\nabla u_i\|_\Omega$ is uniformly bounded in any given neighborhood of $(k_1,\cdots,k_j,\cdots,k_m)$. So M in (2.12) can be chosen as a common constant in this neighborhood of the parameters. Thus, the convergence of (2.15) is uniform. Hence, we prove the continuity of the function $c_i = c_i(k_1,\cdots,k_m)$.

On the other hand, by (2.12),

$$\left\|\frac{u_i^n - u_i}{k_j^n - k_j}\right\|_{H^1(\Omega)} \le M \tag{2.16}$$

holds. So by weak compactness, there exists a subsequence $\{n_l\}$, such that

$$\frac{u_i^{n_l} - u_i}{k_j^{n_l} - k_j} \rightharpoonup \overline{u} \quad \text{weakly convergent in } H^1(\Omega). \tag{2.17}$$

It is easy to show $\overline{u} \in V$.

Choosing $n = n_l$ in (2.9), and substracting (2.8) from (2.9), we get

$$\begin{aligned}
&\sum_{s\neq j}\int_{\Omega_s} k_s\frac{\nabla u_i^{n_l} - \nabla u_i}{k_j^{n_l} - k_j}\cdot\nabla\phi dx + \int_{\Omega_j} k_j\frac{\nabla u_i^{n_l} - \nabla u_i}{k_j^{n_l} - k_j}\cdot\nabla\phi dx \\
&+ \int_{\Omega_j} \nabla u_i^{n_l}\cdot\nabla\phi dx = 0, \quad \forall\phi \in V.
\end{aligned} \tag{2.18}$$

Letting $n_l \to \infty$, and noticing (2.17) and (2.13), we have

$$\sum_{s\neq j}\int_{\Omega_s} k_s \nabla\overline{u}\cdot\nabla\phi dx + \int_{\Omega_j} k_j\nabla\overline{u}\cdot\nabla\phi dx + \int_{\Omega_j}\nabla u_i\cdot\nabla\phi dx = 0, \quad \forall\phi\in V. \tag{2.19}$$

The existence and uniqueness of the solution $\overline{u}\in V$ can also be proved by Lax-Milgram theorem. By the uniqueness of $\overline{u}$, when $k_j^n \to k_j$, the whole sequence

$$\frac{u_i^n - u_i}{k_j^n - k_j} \rightharpoonup \overline{u} \quad \text{weakly convergent in } H^1(\Omega). \tag{2.20}$$

Thus

$$\frac{c_i(k_1,\cdots,k_{j-1},k_j^n,k_{j+1},\cdots,k_m) - c_i(k_1,\cdots,k_j,\cdots,k_m)}{k_j^n - k_j} = \left.\frac{u_i^n - u_i}{k_j^n - k_j}\right|_{\Gamma_0^i}$$
$$\to \overline{u}|_{\Gamma_0^i} \quad \text{strongly convergent in } L^2(\Gamma_0^i). \tag{2.21}$$

This means $\dfrac{\partial c_i}{\partial k_j}$ exists, and

$$\frac{\partial c_i(k_1,\cdots,k_m)}{\partial k_j} = \overline{u}|_{\Gamma_0^i}. \tag{2.22}$$

Choosing $\phi = u_i$ and $\phi = u_i^n$ in (2.8) and (2.9) respectively, we get

$$\sum_{s\neq j}\int_{\Omega_s} k_s|\nabla u_i|^2 dx + \int_{\Omega_j} k_j|\nabla u_i|^2 dx = u_i|_{\Gamma_0^i} = c_i(k_1,\cdots,k_j,\cdots,k_m) \tag{2.23}$$

and

$$\sum_{s\neq j}\int_{\Omega_s} k_s|\nabla u_i^n|^2 dx + \int_{\Omega_j} k_j^n|\nabla u_i^n|^2 dx = u_i^n|_{\Gamma_0^i} = c_i(k_1,\cdots,k_{j-1},k_j^n,k_{j+1},\cdots,k_m). \tag{2.24}$$

Substract (2.23) from (2.24), and divide the result by $(k_j^n - k_j)$. Then we get

$$\begin{aligned}&\frac{c_i(k_1,\cdots,k_{j-1},k_j^n,k_{j+1},\cdots,k_m) - c_i(k_1,\cdots,k_{j-1},k_j,k_{j+1},\cdots,k_m)}{k_j^n - k_j}\\ &= \sum_{s\neq j}\int_{\Omega_s} k_s\frac{(\nabla u_i^n + \nabla u_i)\cdot(\nabla u_i^n - \nabla u_i)}{k_j^n - k_j}dx\\ &\quad + \int_{\Omega_j} k_j\frac{(\nabla u_i^n + \nabla u_i)\cdot(\nabla u_i^n - \nabla u_i)}{k_j^n - k_j}dx + \int_{\Omega_j}|\nabla u_i^n|^2 dx.\end{aligned} \tag{2.25}$$

Letting $k_j^n \to k_j$, and noticing (2.20), (2.13)–(2.14), (2.8) and (2.22), we have

$$\begin{aligned}\frac{\partial c_i}{\partial k_j} &= 2\sum_{s\neq j}\int_{\Omega_s} k_s\nabla u_i\cdot\nabla\overline{u}dx + 2\int_{\Omega_j} k_j\nabla u_i\cdot\nabla\overline{u}dx + \int_{\Omega_j}|\nabla u_i|^2 dx\\ &= 2\overline{u}|_{\Gamma_0^i} + \int_{\Omega_j}|\nabla u_i|^2 dx\\ &= 2\frac{\partial c_i}{\partial k_j} + \int_{\Omega_j}|\nabla u_i|^2 dx.\end{aligned} \tag{2.26}$$

This has proved (2.7).

Using (2.14) again, in the same way as in the discussion of the continuity of $c_i(k_1, \cdots, k_m)$, we can prove the continuity of $\dfrac{\partial c_i}{\partial k_j}(k_1, \cdots, k_m)$ by (2.7).

By Theorem 2.1 and the theorem of implicit function, we can get immediately

Theorem 2.2. *Denote*

$$J = \left(\frac{\partial c_i}{\partial k_j}\right)_{1\le i,j\le m} = \begin{pmatrix} \dfrac{\partial c_1}{\partial k_1} & \cdots & \dfrac{\partial c_1}{\partial k_m} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial c_m}{\partial k_1} & \cdots & \dfrac{\partial c_m}{\partial k_m} \end{pmatrix}. \tag{2.27}$$

If

$$|J|_{k=k^0} = \det J|_{k=k^0} \neq 0, \tag{2.28}$$

then we can get

$$k_i = k_i(c_1, \cdots, c_m), \quad 1 \le i \le m, \tag{2.29}$$

near $k = k^0$. *Thus we can determine the parameters* $\overline{k} = k(\overline{c})$ *near* k^0 *by measuring*

$$c^0 = c(k^0).$$

Remark 2.1. To identify m parameters, we put m electrodes on m different positions. Because of linearity, only m types of the discharged current are independent. We choose the intensity of the current of i-th electrode $I_i = 1$, and those of the other electrodes $I_j = 0$ ($j \neq i$, $1 \le j \le m$). Also we can use another method using only one electrode. Put the electrode on m different positions, the intensity of the current are always 1. In this case, the potential function $u \in V$ also satisfies the variational problem (2.3). Similar to Theorems 2.1 and 2.2, we can get (2.7), the derivative function of c_i with respect to k_j, and (2.28), the condition of the inverse problem.

§3. Identification Problem of Single Resistivity of the Layer

To determine a single resistivity of the layer, we only need one measuring electrode. Let k_s ($1 \le s \le m$, $s \neq j$) be all known constants, k_j be an inverse coefficient. Denote $k = k_j$. Then the function $c = c(k)$ is a one variable function of k. By Theorem 2.1 we can get the following theorem immediately.

Theorem 3.1. *Function* $c = c(k)$ *is strictly decreasing about* k. *Thus there exists a unique continuously differentiable inverse function.*

Proof. Obviously, $\dfrac{dc}{dk} \le 0$. So it suffices to prove $\dfrac{dc}{dk} \neq 0$. Assume the contrary; if $\dfrac{dc}{dk} = 0$, by (2.7)

$$\int_{\Omega_j} |\nabla u|^2 dx = 0. \tag{3.1}$$

So u is a constant in Ω_j. By adjointing condition, without loss of generality, suppose Ω_s ($1 \le s \le m$, $s \neq j$) adjoints Ω_j. Then

$$u \equiv \text{constant}, \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega_s \cap \partial\Omega_j.$$

By Holmogren Theorem (see [5]), u is a constant in Ω_s. Since Ω is a connected domain, u is a constant in the whole domain Ω. By (1.2), $u = 0$ on Γ_1, hence $u \equiv 0$ in the whole domain Ω. This contradicts (1.5). Thus this proves

$$\frac{dc}{dk} < 0.$$

By the theorem of inverse function, there exists a unique continuously differentiable inverse function

$$k = k(c).$$

By Theorem 3.1, to single resistivity of the layer, the potential function on the measuring electrode $c = c(k)$ is invertible in the whole domain of the parameter. Thus we can determine the resistivity of any layer by measuring the potential value c on the electrode Γ_0.

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References

[1] Li Ta-tsien et al., Applications of FEM in electric logs, Petroleum Industry Press, 1980.
[2] Li Ta-tsien et al., On boundary value problem of the self-adjoint elliptic equation with equal value boundary conditions (I), *Fudan Journal* (*Natural Science*), 1 (1976), 61–71.
[3] Tan Yongji, An inverse problem for nonlocal elliptic BVP and resistivity identification, Partial Differential Equations (Proceedings, Tianjin 1986, edited by S. S. Chern), Lecture Notes in Mathematics 1306, Springer-Verlag, 1988, 149–159.
[4] Necas, Les methods directes en theorie des equations elliptiques, Masson, Paris, 1967.
[5] Treves, F., Basic linear partial differential equations, Academic Press, Inc., 1975.