Chin. Ann. of Math.
19B: 3(1998),273-280.

LOCALLY EKELAND'S VARIATIONAL PRINCIPLE AND SOME SURJECTIVE MAPPING THEOREMS**

Zhong Chengkui* Zhao Peihao*

Abstract

This paper shows that if a Gateaux differentiable functional f has a finite lower bound (although it need not attain it), then, for every $\varepsilon > 0$, there exists some point z_ε such that $\|f'(z_\varepsilon)\| \leq \frac{\varepsilon}{1+h(\|z_\varepsilon\|)}$, where $h : [0,\infty) \to [0,\infty)$ is a continuous function such that $\int_0^\infty \frac{1}{1+h(r)}dr = \infty$. Applications are given to extremum problem and some surjective mappings.

Keywords Variational principle, Extremum problem, Weak P.S. condition, Surjective mapping

1991 MR Subject Classification 47H09, 49J45, 58E05

Chinese Library Classification O177.3

§1. Introduction

Let (M, d) be a complete metric space, let $f : M \to R\bigcup\{+\infty\}$ be a lower semicontinuous function, not identically $+\infty$ and bounded from below. Then Ekeland's variational principle[4] states that, for every $\varepsilon > 0$, every $y \in M$ such that $f(y) < \inf_M f + \varepsilon$ and every $\lambda > 0$, there exists some point $z \in M$ such that

$$f(z) \leq f(y), \quad d(z, y) \leq \lambda,$$
$$f(x) \geq f(z) - \varepsilon d(x, z), \quad \forall x \in M.$$

It is well known that Ekeland's variational principle has many applications to optimization, optimal control, differential equations, fixed points, critical point theory and variants (see [4, 2, 5, 7]).

In Section 2 of this paper we prove the following general result:

Theorem 1.1. *Let $h : [0,\infty) \to [0,\infty)$ be a continuous function such that $\int_0^\infty \frac{1}{1+h(r)}dr = \infty$. Let (M, d) be a complete metric space, $x_0 \in M$ fixed. Suppose $f : M \to R\bigcup\{+\infty\}$ is a* l.s.c. *functional, $\not\equiv +\infty$, bounded from below. Then, for every $\varepsilon > 0$, every $y \in M$ such that*

$$f(y) < \inf_M f + \varepsilon, \tag{1.1}$$

and every $\lambda > 0$, there exist some point $z \in M$ and a neighborhood $B(z, \gamma) = \{x \in M | d(x, z) \leq \gamma\}$ such that

$$f(z) \leq f(y), \tag{1.2}$$

Manuscript received December 28, 1995. Revised April 8, 1996.

*Department of Mathematics, Lanzhou University, Lanzhou 730000, China.

**Project supported by the National Natural Science Foundation of China and the Chinese University Doctoral Foundation.

$$d(z, x_0) \leq r_0 + \overline{r}, \tag{1.3}$$

and

$$f(x) \geq f(z) - \frac{\varepsilon}{\lambda(1 + h(d(x_0, z)))} d(z, x), \quad \forall x \in B(z, \gamma), \tag{1.4}$$

where $r_0 = d(x_0, y)$ *and* $\overline{r}$ *is such that*

$$\int_{r_0}^{r_0+\overline{r}} \frac{1}{1 + h(r)} dr \geq 2\lambda. \tag{1.5}$$

Theorem 1.1 is called locally Ekeland's variational principle. The reason for it is that (1.4) holds locally.

As applications of Theorem 1.1, in Section 3 of this paper we derive the existence of minimal point for some functional with a weak "compactness" condition; in Section 4 we obtain some surjective mapping theorems, which generalize the similar results proved by W. O. Ray and A. M. Walker in [6] using an extension of Caristi's fixed point theorem.

§2. Proof of Theorem 1.1

By the definition of Riemann integral, there is a partition

$$\Delta : r_0 < r_1 < \cdots < r_N < r_{N+1} = r_0 + \overline{r}$$

such that

$$\sum_{i=0}^{N} \frac{1}{1 + M_i}(r_{i+1} - r_i) \leq \int_{r_0}^{r_0+\overline{r}} \frac{1}{1 + h(r)} dr \leq \sum_{i=0}^{N} \frac{1}{1 + M_i}(r_{i+1} - r_i) + \frac{\lambda}{3}, \tag{2.1}$$

where $M_i = \max\limits_{r_i \leq r \leq r_{i+1}} h(r), i = 0, 1, \cdots, N$. Set $\delta = \min\limits_{0 \leq i \leq N}(r_{i+1} - r_i)$ and choose a natural number k_0 such that

$$\frac{3(N + 1)\delta}{\lambda} \leq k_0. \tag{2.2}$$

Now let us define inductively a sequence $\{x_n\} \subset M$ as follows:

Take $x_1 = y$ and suppose x_n is known. Then x_n is such that either

(1) $f(x) \geq f(x_n) - \frac{\varepsilon}{\lambda(1+h(d(x_0,x_n)))} d(x, x_n)$
whenever $x \in B(x_n, \frac{\delta}{2k_0}) = \{x \in M | d(x, x_n) \leq \frac{\delta}{2k_0}\}$, or

(2) $E_n = \{x \in B(x_n, \frac{\delta}{2k_0}) | f(x) < f(x_n) - \frac{\varepsilon}{\lambda(1+h(d(x_0,x_n)))} d(x, x_n)\} \neq \emptyset$.

If case (1) holds, we take $x_{n+1} = x_n$, and if case (2) holds, we choose $x_{n+1} \in E_n$ such that

$$f(x_{n+1}) < \inf_{E_n} f + \frac{1}{n + 1}. \tag{2.3}$$

Thus we obtain a sequence $\{x_n\}$ such that, for $n = 1, 2, \cdots$

$$d(x_{n+1}, x_n) \leq \frac{\delta}{2k_0} \tag{2.4}$$

and

$$f(x_{n+1}) \leq f(x_n) - \frac{\varepsilon}{\lambda(1 + h(d(x_0, x_n)))} d(x_n, x_{n+1}). \tag{2.5}$$

In the following, we prove in two steps that $\{x_n\}$ converges to some point z satisfying (1.2),(1.3) and (1.4).

Step 1. we prove that

$$d(x_0, x_n) \le r_0 + \overline{r}, \quad n = 1, 2, \cdots \tag{2.6}$$

First of all, $d(x_0, x_1) = d(x_0, y) = r_0 < r_0 + \overline{r}$. Now if we assume that there is some n_0 such that $d(x_0, x_n) \le r_0 + \overline{r}$ whenever $1 \le n < n_0$, but $d(x_0, x_{n_0}) > r_0 + \overline{r}$, we shall derive a contradiction as follows:

Set, for each $i, 1 \le i \le N$,

$$n_i^- = \max\{n | 1 \le n \le n_0, x_n \in B(x_0, r_i)\},$$
$$n_i^+ = \min\{n | n_i^- \le n \le n_0, x_n \notin B(x_0, r_{i+1})\}.$$

Since $x_{n_i^-} \in B(x_0, r_i)$ and $x_n \notin B(x_0, r_i)$ as $n > n_i^-$, it follows from (2.4) that

$$r_i < d(x_0, x_{n_i^-+1}) \le d(x_0, x_{n_i^-}) + d(x_{n_i^-}, x_{n_i^-+1}) \le r_i + \frac{\delta}{2k_0} \le r_{i+1}. \tag{2.7}$$

Hence $n_i^+ > n_i^-$ (in fact, $n_i^+ > n_i^- + 1$). By the definitions of n_i^- and n_i^+, we have

$$r_i < d(x_0, x_n) \le r_{i+1} \quad \text{as} \quad n_i^- < n < n_i^+. \tag{2.8}$$

Combining it with (2.4), we obtain

$$0 \le r_{i+1} - d(x_0, x_{n_i^+-1}) \le d(x_0, x_{n_i^+}) - d(x_0, x_{n_i^+-1}) \le d(x_{n_i^+}, x_{n_i^+-1}) \le \frac{\delta}{2k_0}. \tag{2.9}$$

By (2.2),(2.7),(2.8) and (2.9), we get

$$\begin{aligned}
\sum_{i=0}^{N} \frac{1}{1+M_i}(r_{i+1} - r_i) &= \sum_{i=0}^{N} \frac{1}{1+M_i}\big[(r_{i+1} - d(x_0, x_{n_i^+-1})) \\
&\quad + (d(x_0, x_{n_i^+-1}) - d(x_0, x_{n_i^-+1})) + (d(x_0, x_{n_i^-+1}) - r_i)\big] \\
&\le \sum_{i=0}^{N} \frac{\delta}{k_0(1+M_i)} + \sum_{i=0}^{N} \frac{1}{1+M_i}(d(x_0, x_{n_i^+-1}) - d(x_0, x_{n_i^-+1})) \\
&\le \frac{(1+N)\delta}{k_0} + \sum_{i=0}^{N} \frac{1}{1+M_i} d(x_{n_i^+-1}, x_{n_i^-+1}) \\
&\le \frac{\lambda}{3} + \sum_{i=0}^{N} \sum_{n=n_i^-+1}^{n_i^+-2} \frac{1}{1+M_i} d(x_n, x_{n+1}) \\
&\le \frac{\lambda}{3} + \sum_{i=0}^{N} \sum_{n=n_i^-+1}^{n_i^+-2} \frac{1}{1+h(d(x_0, x_n))} d(x_n, x_{n+1}) \\
&\le \frac{\lambda}{3} + \sum_{n=1}^{n_0-1} \frac{1}{1+h(d(x_0, x_n))} d(x_n, x_{n+1}).
\end{aligned}$$

Using (1.1) and (2.5), we know that

$$\sum_{n=1}^{n_0-1} \frac{\varepsilon}{\lambda(1+h(d(x_0, x_n)))} d(x_n, x_{n+1}) \le f(x_1) - f(x_{n_0}) \le f(x_1) - \inf_M f < \varepsilon.$$

Therefore

$$\sum_{i=0}^{N} \frac{1}{1+M_i}(r_{i+1} - r_i) \le \frac{4}{3}\lambda.$$

Combining (1.5) and (2.1), we get

$$2\lambda \le \int_{r_0}^{r_0+\bar{r}} \frac{1}{1+h(r)}dr \le \sum_{i=0}^{N} \frac{1}{1+M_i}(r_{i+1}-r_i) + \frac{\lambda}{3} \le \frac{4}{3}\lambda + \frac{\lambda}{3} = \frac{5}{3}\lambda.$$

It is impossible. Hence (2.6) holds.

Step 2. We prove that $\{x_n\}$ converges to some point z in M for which (1.2)-(1.4) hold.

Using (1.1) and (2.5), we know that, for $n = 1, 2, \cdots$,

$$\sum_{k=1}^{n} \frac{\varepsilon}{\lambda(1+h(d(x_0,x_k)))} d(x_k, x_{k+1}) \le f(x_1) - f(x_{n+1}) \le f(x_1) - \inf_M f < \varepsilon.$$

Letting $n \to \infty$, we have

$$\sum_{k=1}^{\infty} \frac{1}{1+h(d(x_0,x_k))} d(x_k, x_{k+1}) \le \lambda. \tag{2.10}$$

Set

$$c_0 = \max_{0\le r\le r_0+\bar{r}} (1+h(r)). \tag{2.11}$$

It follows from step 1 that

$$\begin{aligned} &\frac{1}{c_0} d(x_{n+p}, x_n) \le \sum_{k=n}^{n+p-1} \frac{1}{c_0} d(x_k, x_{k+1}) \\ &\le \sum_{k=n}^{n+p-1} \frac{1}{1+h(d(x_0,x_k))} d(x_k, x_{k+1}) \le \sum_{k=n}^{\infty} \frac{1}{1+h(d(x_0,x_k))} d(x_k, x_{k+1}). \end{aligned}$$

Combined with (2.10), it implies that $\{x_n\}$ is a Cauchy sequence. Since M is complete, there exists some point z in M such that

$$\lim_{n\to\infty} x_n = z. \tag{2.12}$$

Take $\gamma = \frac{\delta}{4k_0}$. It remains to verify that (1.2),(1.3) and (1.4) hold.

Combining (2.5) and the lower semicontinuity of f, we have

$$f(z) \le \underline{\lim}_{n\to\infty} f(x_n) \le f(x_n) \le f(x_1) = f(y), \tag{2.13}$$

which implies (1.2) holds. Clearly (1.3) holds since $d(x_0, x_n) \le r_0 + \bar{r}$ and $\lim_{n\to\infty} x_n = z$. Finally we prove that z satisfies (1.4).

In fact, by the definition of $\{x_n\}$, we know that if there is some m such that x_m is defined as case (1), then all of $x_n, n \ge m$, are defined as case (1), that is, $x_n = x_m$ whenever $n \ge m$. Hence $z = x_m$, and (1.4) holds. So, without loss of generality, we assume that, for every n, x_n is defined as case (2). Now if (1.4) does not hold, then there exists some $z_1 \in B(z, \frac{\delta}{4k_0})$ such that

$$f(z_1) < f(z) - \frac{\varepsilon}{\lambda(1+h(d(x_0,z)))} d(z_1, z). \tag{2.14}$$

Using (2.13),(2.14) and the continuity of h, there exists some n_1 such that, for $n \ge n_1$, $d(x_n, z) < \frac{\delta}{4k_0}$ and

$$f(z_1) < f(x_n) - \frac{\varepsilon}{\lambda(1+h(d(x_0,x_n)))} d(z_1, x_n). \tag*{$(2.15)_n$}$$

This shows that $z_1 \in E_n$ whenever $n \ge n_1$.

Combining (2.3) and $(2.15)_{n+1}$, we obtain

$$f(z_1)+\frac{1}{n+1}\ge \inf_{E_n} f+\frac{1}{n+1}\ge f(x_{n+1})>f(z_1)+\frac{\varepsilon}{\lambda(1+h(d(x_0,x_{n+1})))}d(x_{n+1},z_1),$$

which implies that

$$d(x_{n+1},z_1)\le \frac{\lambda}{\varepsilon}(1+h(d(x_0,x_{n+1})))\frac{1}{n+1}\le \frac{c_0\lambda}{\varepsilon}\frac{1}{n+1},$$

where c_0 is given by (2.11). Letting $n\to\infty$, we get $d(z,z_1)=0$. Therefore $z_1=z$. But it contradicts (2.14) and completes the proof.

§3. The Weak P. S. Condition and the Existence of Minimal Point

Throughout this section X will denote a Banach space. Recall that a function $f: X\to R$ is said to admit Gateaux derivative at x_0 if there exists a continuous linear functional $f'(x_0)$ such that, for every $y\in X$,

$$\lim_{t\to 0}\frac{f(x_0+ty)-f(x_0)}{t}=\langle f'(x_0),y\rangle.$$

Theorem 3.1. *Let $h:[0,\infty)\to[0,\infty)$ be a continuous function satisfying $\int_0^\infty \frac{1}{1+h(r)}dr=\infty$. Let X be a Banach space, $x_0\in X$ fixed. Suppose that $f: X\to R$ is a l.s.c. function, having Gateaux derivative at every point x in X and bounded from below. Then, for every $\varepsilon>0$, every $y\in X$ such that*

$$f(y)<\inf_X f+\varepsilon, \tag{3.1}$$

and every $\lambda>0$, there exists $z\in X$ such that

$$f(z)\le f(y), \tag{3.2}$$

$$\|z-x_0\|\le r_0+\overline{r}, \tag{3.3}$$

$$\|f'(z)\|\le \frac{\varepsilon}{\lambda(1+h(\|z-x_0\|))}, \tag{3.4}$$

where $r_0=\|x_0-y\|$ and $\overline{r}$ is such that

$$\int_{r_0}^{r_0+\overline{r}}\frac{1}{1+h(r)}dr\ge 2\lambda. \tag{3.5}$$

We remark that if we take $h(r)\equiv 0$ and $x_0=y$, then (3.3) and (3.4) become respectively

$$\|z-y\|\le 2\lambda, \tag{3.6}$$

$$\|f'(z)\|\le \frac{\varepsilon}{\lambda}. \tag{3.7}$$

This is almost the same as Theorem 2.1 of [4].

Proof of Theorem 3.1. Using Theorem 1.1 directly, we see that there exist $z\in X$ and a neighborhood $B(z,\gamma)$ such that (3.2), (3.3) hold and

$$f(x)\ge f(z)-\frac{\varepsilon}{\lambda(1+h(\|z-x_0\|))}\|x-z\|,\quad \forall x\in B(z,\gamma). \tag{3.8}$$

For every $v\in X$ and every $t>0$ small enough, $x=z+tv\in B(z,\gamma)$. Hence (3.8) gives

$$\frac{f(z+tv)-f(z)}{t}\ge \frac{-\varepsilon}{\lambda(1+h(\|x_0-z\|))}\|v\|.$$

Letting $t \to 0$, we obtain

$$\langle f'(z), v\rangle \geq \frac{-\varepsilon}{\lambda(1+h(\|x_0-z\|))}\|v\|. \tag{3.9}$$

The inequality (3.9), holding for every $v \in X$, means that

$$\|f'(z)\| \leq \frac{\varepsilon}{\lambda(1+h(\|x_0-z\|))}.$$

The proof is completed.

Corollary 3.1. *Under the hypotheses of Theorem* 3.1*, for every $\varepsilon > 0$, there exists some point z_ε such that*

$$f(z_\varepsilon) < \inf_X f + \varepsilon^2, \tag{3.10}$$

$$\|f'(z_\varepsilon)\| \leq \frac{\varepsilon}{1+h(\|z_\varepsilon\|)}. \tag{3.11}$$

Proof. Just take ε^2 instead of ε, ε instead of λ and 0 instead of x_0 in the preceding theorem.

Corollary 3.2. *Under the hypotheses of Theorem* 3.1, *there exists a minimizing sequence $\{z_n\}$ of f such that*

$$\|f'(z_n)\|(1+h(\|z_n\|)) \to 0. \tag{3.12}$$

Proof. Take $\varepsilon = \frac{1}{n}, n = 1, 2, \cdots$ in the preceding corollary.

Definition 3.1. *Let X be a Banach space, $f : X \to R$ a function having Gateaux derivative at every point x in X. We say that f satisfies the weak P.S. condition if the existence of a sequence $\{x_n\}$ in X such that $\{f(x_n)\}$ is bounded and $\|f'(x_n)\|(1+h(\|x_n\|)) \to 0$ implies that $\{x_n\}$ has a convergent subsequence, where $h : [0,\infty) \to [0,\infty)$ is a continuous function such that $\int_0^\infty \frac{1}{1+h(r)}dr = \infty$.*

Remark 3.1. If we take $h(r) \equiv 0$ and $h(r) \equiv r$ respectively, then the weak P.S. condition is just the famous P.S. condition and (C) condition respectively (see [1, 3]). In critical point theory, many results still hold if we take the weak P.S. condition instead of P.S. condition.

Theorem 3.2. *Under the hypotheses of Theorem* 3.1, *suppose that f satisfies the weak* P.S. *condition. Then f has a minimal point.*

Proof. By Corollary 3.2, there is a sequence $\{x_n\}$ in X such that $f(x_n) \to \inf_X f$ and $\|f'(x_n)\|(1+h(\|x_n\|)) \to 0$. The weak P.S. condition implies that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ convergent to some point x^*. Since f is a l.s.c. function, we obtain

$$\inf_X f \leq f(x^*) \leq \underline{\lim}_{k\to\infty} f(x_{n_k}) \leq \inf_X f.$$

Therefore, $f(x^*) = \inf\limits_X f$. The proof is completed.

§4. Some Surjective Mapping Theorems

In this section we apply Theorem 1.1 to obtain two surjective mapping theorems which generalize Theorem 3.1 and Theorem 3.3 of [6] respectively.

Let X and Y be Banach spaces and F a mapping from X to Y; F is said to be Gateaux differentiable if for each $x \in X$ there is a function $dF_x : X \to Y$ satisfying

$$\lim_{t\to 0} \frac{F(x+ty)+F(x)}{t} = dF_x(y) \quad (y \in X) \quad .$$

Note we do not require that dF_x be linear; it follows from the definition, however, that dF_x is homogeneous, i.e., $dF_x(\lambda y) = \lambda dF_x(y)$ for all λ.

Theorem 4.1. *Let X and Y be Banach spaces and F be a continuous and Gateaux differentiable mapping from X to Y. Let $h : [0, \infty) \to [0, \infty)$ be a continuous function satisfying $\int_0^\infty \frac{1}{1+h(r)} dr = \infty$ and suppose, for each $x \in X$, that*

$$dF_x(B(0, 1 + h(\|x\|))) \supset B(0, 1). \tag{4.1}$$

Then F is surjective.

We remark that Theorem 4.1 generalizes Theorem 3.1 of [6] where it is assumed that h is nondecreasing.

Proof of Theorem 4.1. For every fixed $w \in Y$, set $f(x) = \|F(x) - w\|$. Then f is a continuous mapping from $(X, \|.\|)$ to $[0, \infty)$.

Applying Theorem 1.1, we see that for $\varepsilon < 1$, there exist $z_\varepsilon \in X$ and a neighborhood $B(z_\varepsilon, \gamma)$ such that

$$\|F(x) - w\| \geq \|F(z_\varepsilon) - w\| - \frac{\varepsilon}{1 + h(\|z_\varepsilon\|)} \|x - z_\varepsilon\|, \quad \forall x \in B(z_\varepsilon, \gamma). \tag{4.2}$$

Hence, for each fixed $v \in X$ and $t > 0$ small enough,

$$\|F(z_\varepsilon + tv) - w\| - \|F(z_\varepsilon) - w\| \geq -\frac{\varepsilon}{1 + h(\|z_\varepsilon\|)} t\|v\|. \tag{4.3}$$

Choose $y_t^* \in Y^*$ such that $\|y_t^*\| = 1$ and

$$\langle y_t^*, F(z_\varepsilon + tv) - w\rangle = \|F(z_\varepsilon + tv) - w\|. \tag{4.4}$$

Combining (4.3) and (4.4), we get

$$\langle y_t^*, F(z_\varepsilon + tv) - F(z_\varepsilon)\rangle \geq \|F(z_\varepsilon + tv) - w\| - \|F(z_\varepsilon) - w\| \geq \frac{-\varepsilon t\|v\|}{1 + h(\|z_\varepsilon\|)}. \tag{4.5}$$

It is well known that if a Banach space is separable then the unit ball of its dual space is weak sequentially compact. Therefore there exists some $y_0^* \in Y^*$ such that, for $y \in Y_1 = \overline{\text{span}\{F(z_\varepsilon + tv)\}}$, the closed linear hull of $\{F(z_\varepsilon + tv)\}$, $\langle y_{t_n}^*, y\rangle \to \langle y_0^*, y\rangle$ as $t_n \to 0$. Using (4.4) and (4.5) respectively, we have

$$\langle y_0^*, F(z_\varepsilon) - w\rangle = \|F(z_\varepsilon) - w\|, \tag{4.6}$$

$$\langle y_0^*, dF_{z_\varepsilon}(v)\rangle \geq -\frac{\varepsilon\|v\|}{1 + h(\|z_\varepsilon\|)}. \tag{4.7}$$

By virtue of (4.1), there exists some $v \in X$ such that

$$\|v\| \leq (1 + h(\|z_\varepsilon\|))\|F(z_\varepsilon) - w\|, \quad dF_{z_\varepsilon}(v) = -(F(z_\varepsilon) - w).$$

Combining (4.6) and (4.7), we obtain

$$\|F(z_\varepsilon) - w\| = \langle y_0^*, F(z_\varepsilon) - w\rangle = \langle y_0^*, -dF_{z_\varepsilon}(v)\rangle \leq \frac{\varepsilon\|v\|}{1 + h(\|z_\varepsilon\|)} \leq \varepsilon\|F(z_\varepsilon) - w\|,$$

which implies that $F(z_\varepsilon) = w$ and completes the proof.

Theorem 4.2. *Let X and Y be Banach spaces, F be an open and continuous mapping from X to Y. Let $h : [0, \infty) \to [0, \infty)$ be a continuous function for which $\int_0^\infty \frac{1}{1+h(r)} dr = \infty$. Suppose for each $x \in X$ there is a $\delta(x) > 0$ such that if $\|x - \overline{x}\| < \delta(x)$, then*

$$\frac{1}{1 + h(\|x\|)} \|x - \overline{x}\| \leq \|F(x) - F(\overline{x})\|. \tag{4.8}$$

Then $F(X) = Y$.

Proof. For every fixed $w \in Y$, set $f(x) = \|F(x) - w\|$. Applying Theorem 1.1, for $\varepsilon < 1$, there exist some point $z_\varepsilon \in X$ and a neighborhood $B(z_\varepsilon, \gamma)$ such that

$$\|F(x) - w\| \geq \|F(z_\varepsilon) - w\| - \frac{\varepsilon}{1 + h(\|z_\varepsilon\|)}\|x - z_\varepsilon\|, \quad \forall x \in B(z_\varepsilon, \gamma). \tag{4.9}$$

We proceed by contradiction and suppose that $F(z_\varepsilon) \neq w$. Take $\delta_1 = \min\{\delta(z_\varepsilon), \gamma\}$. Since F is an open mapping,

$$F(B(z_\varepsilon, \delta_1)) \cap \{tF(z_\varepsilon) + (1-t)w | 0 < t < 1\} \neq \emptyset.$$

Thus there exist some point $v \in B(z_\varepsilon, \delta_1)$ and $0 < t_0 < 1$ such that

$$F(v) = t_0 F(z_\varepsilon) + (1 - t_0)w. \tag{4.10}$$

From (4.9)

$$\|F(z_\varepsilon) - w\| - \|F(v) - w\| \leq \frac{\varepsilon}{1 + h(\|z_\varepsilon\|)}\|v - z_\varepsilon\|. \tag{4.11}$$

Choose $y^* \in Y^*$ such that $\|y^*\| = 1$ and

$$y^*(F(z_\varepsilon) - w) = \|F(z_\varepsilon) - w\|. \tag{4.12}$$

Using (4.10), we have $F(v) - w = t_0(F(z_\varepsilon) - w)$, and $F(z_\varepsilon) - F(v) = (1 - t_0)(F(z_\varepsilon) - w)$. Hence

$$y^*(F(v) - w) = \|F(v) - w\|, \tag{4.13}$$

$$y^*(F(z_\varepsilon) - F(v)) = \|F(z_\varepsilon) - F(v)\|. \tag{4.14}$$

By (4.8),(4.11)–(4.14), we obtain

$$\begin{aligned}\frac{1}{1 + h(\|z_\varepsilon\|)}\|z_\varepsilon - v\| &\leq \|F(z_\varepsilon) - F(v)\| = y^*(F(z_\varepsilon) - F(v)) \\ &= y^*(F(z_\varepsilon) - w) - y^*(F(v) - w) \\ &= \|F(z_\varepsilon) - w\| - \|F(v) - w\| \\ &\leq \frac{\varepsilon}{1 + h(\|z_\varepsilon\|)}\|z_\varepsilon - v\|.\end{aligned}$$

It is impossible and completes the proof.

References

[1] Bartolo, P., Benci, V. & Fortunato, D., Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity, *Nonlinear Anal. Theory, Methods & Appl.*, **7**:9(1983), 981–1012.

[2] Brezis, H. & Nirenberg, L., Remarks on finding critical points, *Comm. Pure Appl. Math.*, **64** (1991), 939–963.

[3] Cerami, G., Un criterio di esistenza per i punti critici su varieta illimitate, *Rc. Ist. Lomb. Sci. Lett.*, **112** (1978), 332–336.

[4] Ekeland, I., On the variational principle, *J. Math. Anal. Applic.*, **47**(1974), 324–357.

[5] Ray, W. O. & Walker, A. M., Mapping theorems for Gateaux differentiable and accretive operators, *Nonlinear Anal. Theory, Mathods & Appl.*, **6**:5 (1982), 423–433.

[6] Mawhin, J. & Willem, M., Critical point theory and Hamiltonian systems, Springer, Verlay, Berlin 1989.

[7] Shi, S. Z., Ekeland's variational principle and the mountain pass lemma, *Acta Math. Sinca* (*NS*), **1**(1985), 348–358.