Chin. Ann. of Math.
19B: 3(1998),281-292.

CONVERGENCE ON RANDOMLY TRIMMED SUMS WITH A DEPENDENT SAMPLE**

LIN ZHENGYAN*

Abstract

Let $\{X_n\}$ be a sequence of random variables and $X_{n1} \le X_{n2} \le \cdots \le X_{nn}$ their order statistics. In this paper a central limit theorem and a strong law of large numbers for randomly trimmed sums $T_n = \sum\limits_{i=\alpha_n+1}^{\beta_n} X_{ni}$ are established in the case that α_n and β_n are positive integer-valued random variables such that α_n/n and β_n/n converge to random variables α and β respectively with $0 \le \alpha < \beta \le 1$ in certain sense, and $\{X_n\}$ is a φ-mixing sequence.

Keywords Randomly trimmed sums, φ-mixing, a.s.convergence, Asymptotic normality
1991 MR Subject Classification 60F05, 60G50
Chinese Library Classification O211.4, O211.5

§1. Introduction and Results

Let $\{X_n, n \ge 1\}$ be a sequence of random variables with a common distribution function $F(x)$ and let $X_{n1} \le X_{n2} \le \cdots \le X_{nn}$ be the order statistics of $X_1, X_2, \cdots, X_n$. Consider statistics of form

$$T_n = \sum_{i=\alpha_n+1}^{\beta_n} X_{ni},$$

where α_n and β_n are integers with $1 \le \alpha_n \le \beta_n \le n$. For an i.i.d. sequence $\{X_n\}$, many authors studied the asymptotic behavior of the trimmed sums T_n. In this paper, we try extending the research extent in two directions. First, we assume that $\{X_n\}$ is φ-mixing. Moreover, α_n and β_n may be positive integer-valued random variables such that the trimming fractions α_n and β_n converge to random variables α and β respectively with $0 \le \alpha < \beta \le 1$ in some sense.

For $0 \le x \le 1$, define the x-th quantile of F

$$F^-(x) = \inf\{t : F(t) \ge x\}, \tag{1.1}$$

which is left continuous. Let

$$m_{\alpha\beta} = \int_{(\alpha,\beta)} F^-(x)dx/(\beta - \alpha).$$

Manuscript received December 25, 1995. Revised November 25, 1996.
*Department of Mathematics, Hangzhou University, Hangzhou 310028, China.
**Project supported by the National Natural Science Foundation of China and the Natural Science Foundation of Zhejiang Province.

In the sequel, we always assume that $\{X_n\}$ is stationary and φ-mixing with coefficients $\varphi(n)$ satisfying

$$\sum_{n=1}^{\infty} \varphi^{\frac{1}{2}}(2^n) < \infty. \tag{1.2}$$

Theorem 1.1. *Suppose that $\alpha_n/n \to \alpha$, $\beta_n/n \to \beta$ a.s. as $n \to \infty$ and*

$$E|X_1 I(F^-(\alpha) < X_1 < F^-(\beta))| < \infty, \tag{1.3}$$

and suppose that both α and β are independent of $\{X_n\}$. Then

$$\frac{1}{n} T_n \to m_{\alpha\beta} \quad a.s. \text{ as } n \to \infty.$$

Put

$$\begin{aligned}\xi_i(\alpha,\beta) = {} & F^-(\alpha)I(X_i \le F(\alpha)) + F^-(\beta)I(X_i \ge F^-(\beta)) + X_i I(F^-(\alpha) < X_i < F^-(\beta)) \\ & - E\{[F^-(\alpha)I(X_i \le F^-(\alpha)) + F^-(\beta)I(X_i \ge F^-(\beta)) \\ & + X_i I(F^-(\alpha) < X_i < F^-(\beta))]|\alpha,\beta\},\end{aligned} \tag{1.4}$$

where $F^-(0)I(X_i \le F^-(0))$ and $F^-(1)I(X_i \ge F^-(1))$ are understood to be zero, and

$$\sigma^2_{\alpha\beta} = E\{\xi_1(\alpha,\beta)^2|\alpha,\beta\} + 2\sum_{i=2}^{\infty} E\{\xi_1(\alpha,\beta)\xi_i(\alpha,\beta)|\alpha,\beta\}.$$

Theorem 1.2. *Suppose that $\sqrt{n}(\alpha_n/n - \alpha) \xrightarrow{\text{P}} 0$, $\sqrt{n}(\beta_n/n - \beta) \xrightarrow{\text{P}} 0$ as $n \to \infty$ and suppose that both α and β are independent of $\{X_n\}$, and the series*

$$\sum_{i=2}^{\infty} E\{\xi_1(\alpha,\beta)\xi_i(\alpha,\beta)|\alpha,\beta\} \tag{1.5}$$

converges absolutely a.s. Then, in the case of $P(\alpha = 0) = P(\beta = 1) = 0$, if

$$\begin{aligned} & F^-(\alpha+\delta)I(\alpha > 0) \xrightarrow{\text{P}} F^-(\alpha)I(\alpha > 0), \\ & F^-(\beta+\delta)I(\beta < 1) \xrightarrow{\text{P}} F^-(\beta)I(\beta < 1) \quad as\ \delta \downarrow 0,\end{aligned} \tag{1.6}$$

we have

$$\frac{\sqrt{n}(\frac{1}{n}T_n - m_{\alpha\beta})}{\sigma_{\alpha\beta}} \xrightarrow{\text{D}} N(0,1) \quad as\ n \to \infty. \tag{1.7}$$

In the case of $P(\alpha = 0) > 0$ and/or $P(\beta = 1) > 0$, suppose that one of the following two sets of conditions is satisfied besides (1.6). *The first one is*

$$E\{X_1^2 I(F^-(\alpha) < X_1 < F^-(\beta))|\alpha,\beta\} < \infty \qquad a.s., \tag{1.8}$$

$\sqrt{x}|F^-(x)|$ and/or $\sqrt{x}F^-(1-x)$ are non-decreasing for x near 0 and

$$\alpha_n I(\alpha = 0) \vee (n - \beta_n)I(\beta = 1) = O_p(\log n), \tag{1.9}$$

condition (1.2) *is replaced by*

$$\sum_{n=1}^{\infty} \varphi^{\frac{1}{2+\gamma}}(2^n) < \infty, \quad for\ some\ \gamma > 0. \tag{1.10}$$

The second one is

$$E\{|X_1|^r I(F^-(\alpha) < X_1 < F^-(\beta))|\alpha,\beta\} < \infty \text{ a.s. for some } r > 2, \tag{1.8'}$$

$$\alpha_n I(\alpha=0) \vee (n-\beta_n) I(\beta=1) = O_p(n^{\frac{(r-2)}{2(r-1)}}). \tag{1.9'}$$

Then (1.7) *alse holds true.*

§2. Proofs

First of all, we show some lemmas which are useful for the proofs of our theorems.

Lemma 2.1. *Let $U_1, U_2, \cdots, U_n$ be φ-mixing $U(0,1)$ random variables satisfying condition* (1.2)*, let $U_{n1} \le U_{n2} \le \cdots \le U_{nn}$ be their order statistics. And let α_n be positive integer-valued random variables with $\alpha_n \le n$ and α a non-negative random variable independent of $U_1, U_2, \cdots, U_n$. Then for any $\delta > 0$,*

$$P\Big\{|U_{n,\alpha_n+1}-\alpha| > \delta, \Big|\frac{\alpha_n+1}{n}-\alpha\Big| < \frac{\delta}{2}\Big\} \le cn^{-2}\delta^{-4};$$

here and in the sequel, c stands for a positive constant, whose values are irrelevant.

Proof. We need a moment inequality for a ρ-mixing sequence[9]: Let $\{\eta_n, n \ge 1\}$ be a ρ-mixing sequence with $E\eta_n = 0$, $E|\eta_n|^q < \infty$ $(q \ge 2)$. Then for any $\epsilon > 0$ there exists a constant $K = K(q, \epsilon, \rho(\cdot))$ such that

$$\begin{aligned} E|S_k(n)|^q \le K\Big\{&\Big(n\exp\Big[(1+\epsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big]\max_{k<i\le k+n} E\eta_i^2\Big)^{q/2} \\ &+ n\exp\Big[K\sum_{i=0}^{[\log n]}\rho^{2/q}(2^i)\Big]\sup_{k<i\le k+n} E|\eta_i|^q\Big\}, \end{aligned}$$

where $S_k(n) = \sum\limits_{i=k+1}^{k+n}\eta_i$. Hence when $\{\eta_n\}$ is a φ-mixing sequence satisfying condition (1.2), by noting the fact $\rho(n) \le 2\varphi^{1/2}(n)$, we obtain under condition (1.2)

$$\begin{aligned} E|S_k(n)|^q &\le K\Big\{\Big(n\max_{k<i\le k+n}E\eta_i^2\Big)^{q/2} + n\exp\Big(K\sum_{i=0}^{[\log n]}\varphi^{1/q}(2^i)\Big)\max_{k<i\le k+n}E|\eta_i|^q\Big\} \\ &\le K\Big\{\Big(n\max_{k<i\le k+n}E\eta_i^2\Big)^{q/2} + n\exp(K(\log n)^{1-2/q})\max_{k<i\le k+n}E|\eta_i|^q\Big\}, \end{aligned} \tag{2.1}$$

since $\sum\limits_{i=0}^{[\log n]}\varphi^{1/q}(2^i) \le (1+\log n)^{1-2/q}\Big(\sum\limits_{i=0}^{\infty}\varphi^{1/2}(2^i)\Big)^{2/q}$.

Write

$$\begin{aligned} &P\Big\{|U_{n,\alpha_n+1}-\alpha| > \delta, \Big|\frac{\alpha_n+1}{n}-\alpha\Big| < \frac{\delta}{2}\Big\} \\ \le\, &P\Big\{U_{n,\alpha_n+1} > \alpha+\delta, \Big|\frac{\alpha_n+1}{n}-\alpha\Big| < \frac{\delta}{2}\Big\} \\ &+ P\Big\{U_{n,\alpha_n+1} < \alpha-\delta, \Big|\frac{\alpha_n+1}{n}-\alpha\Big| < \frac{\delta}{2}\Big\}, \\ &P\Big\{U_{n,\alpha_n+1} > \alpha+\delta, \Big|\frac{\alpha_n+1}{n}-\alpha\Big| < \frac{\delta}{2}\Big\} \\ \le\, &P\Big\{\sum_{i=1}^n I(U_i > \alpha+\delta) > n(1-\alpha-\frac{\delta}{2})\Big\} \\ =\, &E\Big(P\Big\{\sum_{i=1}^n (V_i - E(V_i \mid \alpha)) > \frac{n\delta}{2} \mid \alpha\Big\}\Big), \end{aligned} \tag{2.2}$$

where $V_i = I(U_i > \alpha + \delta)$. In view of independence of α and $\{U_i\}$, it is easy to verify that $\{V_i - E(V_i \mid \alpha), i = 1, \cdots, n\}$ is also φ-mixing when α is given. By (2.1) with $q = 4$, we have $P\big\{\sum\limits_{i=1}^{n}(V_i - E(V_i \mid \alpha)) > \frac{n\delta}{2} \mid \alpha\big\} \le cn^{-2}\delta^{-4}$. Hence

$$P\Big\{U_{n,\alpha_n+1} > \alpha + \delta, \Big|\frac{\alpha_n + 1}{n} - \alpha\Big| < \frac{\delta}{2}\Big\} \le cn^{-2}\delta^{-4}.$$

Similarly

$$P\Big\{U_{n,\alpha_n+1} < \alpha - \delta, \Big|\frac{\alpha_n + 1}{n} - \alpha\Big| < \frac{\delta}{2}\Big\} \le cn^{-2}\delta^{-4}.$$

The lemma is proved.

Lemma 2.2. *Let $G_n(x)$ be the empirical distribution function of $U_1, \cdots, U_n$ in Lemma* 2.1. *Define the empirical process $Y_n(t) = \sqrt{n}(G_n(t) - t)$, $0 \le t \le 1$. Then we have for any* $0 \le s \le s + a \le 1, \lambda > 1$

$$P\Big\{\sup_{s \le t \le s+a} |Y_n(t) - Y_n(s)| \ge \lambda\Big\} \le ca^{3/2}/\lambda^4. \tag{2.3}$$

Proof. Obviously, for any $s \le t$,

$$E((I(U_1 \le t) - t) - (I(U_1 \le s) - s))^i \le t - s, \quad i = 2, 4.$$

Therefore, we have by (2.1)

$$E(Y_n(t) - Y_n(s))^4 \le c((t-s)^2 + n^{-1}\exp(K(\log n)^{\frac{1}{2}})(t-s)) \le c(t-s)^{\frac{3}{2}} \tag{2.4}$$

if $t - s \ge 1/n$. Consider the random variables $Y_n(s + ip) - Y_n(s + (i-1)p), i = 1, \cdots, m$, where m is a positive integer. Using Theorem 12.2 in [1], we obtain

$$P\Big\{\max_{1 \le i \le m} |Y_n(s + ip) - Y_n(s)| \ge \lambda\Big\} \le \frac{c}{\lambda^4}(mp)^{3/2}.$$

Note that for $0 \le t \le p$, $I(U_i \le t) - t \le (I(U_i \le p) - p) + p$. We have

$$|Y_n(t)| \le |Y_n(p)| + p\sqrt{n}, \quad 0 \le t \le p. \tag{2.5}$$

Similarly $|Y_n(t) - Y_n(s)| \le |Y_n(s + p) - Y_n(s)| + p\sqrt{n}$, $s \le t \le s + p$. Hence

$$\sup_{s \le t \le s+mp} |Y_n(t) - Y_n(s)| \le 3 \max_{1 \le i \le m} |Y_n(s + ip) - Y_n(s)| + p\sqrt{n}.$$

Then with $p = 1/n, m = [an] + 1$,

$$P\Big\{\sup_{s \le t \le s+mp} |Y_n(t) - Y_n(s)| \ge \lambda\Big\} \le P\Big\{\max_{1 \le i \le m} |Y_n(s + ip) - Y_n(s)| \ge \frac{\lambda}{4}\Big\} \le \frac{c}{\lambda^4}(mp)^{\frac{3}{2}},$$

which implies (2.3).

Remark 2.1. If condition (1.10) is satisfied, we have

$$E|Y_n(t) - Y_n(s)|^{2+\gamma} \le c(t-s)^{1+\gamma/2}$$

instead of (2.4) (using the first inequality in (2.1) instead of the second one). Then (2.3) can be rewritten as

$$P\Big\{\sup_{s \le t \le s+a} |Y_n(t) - Y_n(s)| \ge \lambda\Big\} \le ca^{1+\frac{\gamma}{2}}/\lambda^{2+\gamma}. \tag{2.3'}$$

Lemma 2.3. *With the notations in Lemma* 2.2*, suppose that condition* (1.10) *is satisfied. Then*

$$P\Big\{\sup_{1/n \le t \le a} |Y_n(t)/\sqrt{t}| \ge \lambda(\log n)^{\frac{1}{2}}\Big\} \le c/\lambda^{2+\gamma}, \tag{2.6}$$

$$P\Big\{\sup_{1/n\le t\le a}\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\frac{I(\frac{1}{n}<U_i\le t)}{\sqrt{U_i}}-\frac{EI(\frac{1}{n}<U_i\le t)}{\sqrt{U_i}}\Big)\Big|\ge\lambda(\log n)^{\frac12}\Big\}\le\frac{c}{\lambda^{2+\gamma}}. \tag{2.7}$$

Proof. We have for $1\le i<j\le m$ (recalling $p=1/n, m=[an]+1$),

$$E\Big\{\frac{I(U_1\le jp)-jp}{\sqrt{jp}}-\frac{I(U_1\le ip)-ip}{\sqrt{ip}}\Big\}^2$$
$$=2-(i+j)p-2\sqrt{\frac{i}{j}}(1-jp)\le 2\Big(1-\sqrt{\frac{i}{j}}\ \Big)\le 2\frac{j-i}{j}$$

and

$$E\Big|\frac{I(U_1\le jp)-jp}{\sqrt{jp}}-\frac{I(U_1\le ip)-ip}{\sqrt{ip}}\Big|^{2+\gamma}$$
$$\le cE\Big(I(ip<U_1\le jp)\frac{1}{\sqrt{jp}}\Big)^{2+\gamma}+cE\Big\{I(U_1\le ip)\Big(\frac{1}{\sqrt{ip}}-\frac{1}{\sqrt{jp}}\Big)\Big\}^{2+\gamma}$$
$$=c(jp)^{-\frac{\gamma}{2}}\frac{j-i}{j}+c(ip)^{-\frac{\gamma}{2}}\Big(\frac{\sqrt{j}-\sqrt{i}}{\sqrt{j}}\Big)^{2+\gamma}\le cp^{-\frac{\gamma}{2}}\Big(\frac{j-i}{j}\Big)^{1+\frac{\gamma}{2}}.$$

Therefore from the first inequality of (2.1) we obtain

$$E\Big|\frac{Y_n(jp)}{\sqrt{jp}}-\frac{Y_n(ip)}{\sqrt{ip}}\Big|^{2+\gamma}\le c\Big\{\Big(\frac{j-i}{j}\Big)^{1+\frac{\gamma}{2}}+n^{-\frac{\gamma}{2}}p^{-\frac{\gamma}{2}}\Big(\frac{j-i}{j}\Big)^{1+\frac{\gamma}{2}}\Big\}\le c\Big(\sum_{k=i+1}^{j}\frac{1}{k}\Big)^{1+\frac{\gamma}{2}}.$$

Then using Theorem 12.2 in [1], we have

$$P\Big\{\max_{1\le i\le m}\Big|\frac{Y_n(ip)}{\sqrt{ip}}-\frac{Y_n(p)}{\sqrt{p}}\Big|\ge\lambda\Big\}\le\frac{c(\sum_{k=2}^{m}\frac{1}{k})^{1+\gamma/2}}{\lambda^{2+\gamma}}\le\frac{c(\log m)^{1+\gamma/2}}{\lambda^{2+\gamma}}$$

and further

$$P\Big\{\max_{1\le i\le m}\frac{|Y_n(ip)|}{\sqrt{ip}}\ge\lambda\Big\}\le\frac{c(\log m)^{1+\gamma/2}}{\lambda^{2+\gamma}}+P\Big\{\frac{|Y_n(p)|}{\sqrt{p}}\ge\frac{\lambda}{2}\Big\}\le\frac{c(\log m)^{1+\gamma/2}}{\lambda^{2+\gamma}}.$$

Recalling (2.5), we have

$$\sup_{1/n\le t\le a}\frac{|Y_n(t)|}{\sqrt{t}}\le\max_{1\le i\le m}\Big\{\frac{|Y_n((i+1)p)|}{\sqrt{(i+1)p}}\sqrt{\frac{i+1}{i}}+\sqrt{\frac{pn}{i}}\Big\}.$$

Hence, it follows that

$$P\Big\{\sup_{1/n\le t\le a}\frac{|Y_n(t)|}{\sqrt{t}}\ge\lambda(\log n)^{\frac12}\Big\} \tag{2.8}$$
$$\le P\Big\{\max_{1\le i\le m}\frac{|Y_n((i+1)p)|}{\sqrt{(i+1)p}}\ge\frac12\lambda(\log n)^{\frac12}\Big\}\le\frac{c(\log m)^{1+\gamma/2}}{\lambda^{2+\gamma}(\log n)^{1+\gamma/2}}\le\frac{c}{\lambda^{2+\gamma}}.$$

(2.6) is proved.

(2.17) can be showed in the same way. We omit it.

Lemma 2.4. *Let $r>2$. With the notations of Lemma 2.2, but putting $p=n^{\frac{-r}{2(r-1)}}$, we have*

$$P\Big\{\sup_{p\le t\le a}\frac{|Y_n(t)|}{t^{1/r}}\ge\lambda\Big\}\le\frac{c}{\lambda^r}, \tag{2.9}$$

$$P\Big\{\sup_{p\le t\le a}\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\frac{I(p<U_i\le t)}{U_i^{1/r}}-E\frac{I(p<U_i\le t)}{U_i^{1/r}}\Big)\Big|\ge\lambda\Big\}\le\frac{c}{\lambda^r}. \tag{2.10}$$

Proof. The lines of the proof is similar to that of Lemma 2.3 except that the second inequality of (2.1) is used instead of the first one. The details are omitted.

We appoint that $\int_{(b,a)} = -\int_{(a,b)}$, $I(b<x<a) = I(a<x<b)$ for any $a<b$.

Proof of Theorem 1.1. Define U_i by $X_i = F^-(U_i)$. Then $\{U_n, n\ge 1\}$ is a sequence of $U(0,1)$ random variables with the same mixing property. The corresponding order statistics are denoted by $U_{ni}, i=1,\cdots,n$. Consequently, $X_{ni} = F^-(U_{ni}), i=1,\cdots,n$. Let $F_n(x)$ and $G_n(x)$ be the empirical distribution functions of $X_1,\cdots,X_n$ and $U_1,\cdots,U_n$ respectively. Noting the well-known fact that $F^-(t)\le x$ if and only if $t\le F(x)$, we have $F_n(x) = G_n(F(x))$.

Write

$$\begin{aligned}\frac{1}{n}T_n &= \int_{[U_{n,\alpha_n+1},U_{n\beta_n}]} F^-(x)dG_n(x)\\ &= F^-(U_{n\beta_n}+)G_n(U_{n\beta_n}) - F^-(U_{n,\alpha_n+1})G_n(U_{n,\alpha_n+1}-)\\ &\quad - \int_{[U_{n,\alpha_n+1},U_{n\beta_n}]} G_n(x)dF^-(x) =: t_1+t_2+t_3 \end{aligned}\tag{2.11}$$

and

$$\begin{aligned} m_{\alpha\beta} &= \int_{(\alpha,\beta)} F^-(x)dx\\ &= F^-(U_{n\beta_n}+)U_{n\beta_n} - F^-(U_{n,\alpha_n+1})U_{n,\alpha_n+1} - \int_{[U_{n,\alpha_n+1},U_{n\beta_n}]} xdF^-(x)\\ &\quad + \int_{(\alpha,U_{n,\alpha_n+1})} F^-(x)dx + \int_{(U_{n\beta_n},\beta)} F^-(x)dx\\ &=: m_1+m_2+m_3+m_4+m_5. \end{aligned}\tag{2.12}$$

Now we show that as $n\to\infty$

$$U_{n,\alpha_n+1}\to\alpha \ \text{ and } \ U_{n\beta_n}\to\beta \text{ a.s.}\tag{2.13}$$

By Lemma 2.1 and the Borel-Cantelli Lemma for any $\delta>0$ we have

$$P\left(\{|U_{n,\alpha_n+1}-\alpha|>\delta\}\bigcap\Big\{\Big|\frac{\alpha_n+1}{n}-\alpha\Big|<\frac{\delta}{2}\Big\},\ \text{i.o.}\right)=0,$$

i.e. there is $\Omega_0\subset\Omega$ with $P(\Omega_0)=1$ such that for any $\omega\in\Omega_0$ either $|U_{n,\alpha_n+1}-\alpha|\le\delta$ or $\big|\frac{\alpha_n+1}{n}-\alpha\big|\ge\frac{\delta}{2}$ for each $n\ge$ some $n_0(\omega)$. But the condition $\alpha_n/n\to\alpha$ *a.s.* implies that there is $\Omega_1\subset\Omega$ with $P(\Omega_1)=1$ such that for any $\omega\in\Omega_1$, $\big|\frac{\alpha_n+1}{n}-\alpha\big|<\frac{\delta}{2}$ for each $n\ge$ some $n_1(\omega)$. Hence for any $\omega\in\Omega_0\bigcap\Omega_1$ and each $n\ge n_0(\omega)\vee n_1(\omega)$,

$$|U_{n,\alpha_n+1}-\alpha|\le\delta.$$

This proves the first limit of (2.13). Similarly we can show another limit.

By (2.13) and condition (1.3), we have

$$m_4=\int_{(\alpha,U_{n,\alpha_n+1})}F^-(x)dx\to 0,\ m_5=\int_{(U_{n\beta_n},\beta)}F^-(x)dx\to 0 \text{ a.s. as } n\to\infty.\tag{2.14}$$

At first we consider the case of $\alpha>0$. In this case $F^-(U_{n,\alpha_n+1})$ are bounded a.s. for all large n by (2.13). Hence we have

$$t_2-m_2 = F^-(U_{n,\alpha_n+1})\Big(U_{n,\alpha_n+1}-\frac{\alpha_n}{n}\Big)\to 0 \text{ a.s.}$$
$$\text{on } \{\alpha>0\}\ as\ n\to\infty.\tag{2.15}$$

Under condition (1.2), the sequence $\{U_n\}$ obeys the strong law of large numbers (a consequence of Corollary 3.4 in [8]), consequently, which leads to the Glivenko-Cantelli theorem for $\{U_n\}$:

$$\sup_{-\infty<x<\infty} |G_n(x) - x| \to 0 \quad \text{a.s. as } n \to \infty. \tag{2.16}$$

Noting that $F^-(x)$ is bounded on $[U_{n,\alpha_n+1}, (U_{n,\alpha_n+1} + U_{n,\beta_n})/2]$ for all large n, by (2.16) we obtain

$$\int_{[U_{n,\alpha_n+1},(U_{n,\alpha_n+1}+U_{n,\beta_n})/2]} (x - G_n(x))dF^-(x) \to 0 \tag{2.17}$$

a.s. on $\{\alpha > 0\}$ as $n \to \infty$. Similarly we have

$$\begin{aligned} t_1 - m_1 =& F^-(U_{n,\beta_n}+)(G_n(U_{n,\beta_n}) - U_{n,\beta_n}) \to 0, \\ \int_{[(U_{n,\alpha_n+1}+U_{n,\beta_n})/2,U_{n,\beta_n}]} &(G_n(x) - x)dF^-(x) \to 0 \quad \text{a.s. on } \{\beta < 1\} \text{ as } n \to \infty. \end{aligned} \tag{2.18}$$

Now consider the case of $\alpha = 0$ and/or $\beta = 1$. We deal with only the limit on $\mathcal{A} = \{\alpha = 0, \beta = 1)$. By (2.13), for any $\delta > 0$ and almost all $\omega \in \mathcal{A}$, there exists $n_0(\omega)$ such that $U_{n,\alpha_n+1} \le \delta$ for $n \ge n_0$. Then, using (1.2) and (1.3), we can employ the SLLN (also from Corollary 3.4 in [8]) and obtain

$$\begin{aligned} \left| \int_{(0,U_{n,\alpha_n+1}]} F^-(x)dG_n(x) \right| &\le \int_{(0,\delta]} \left|F^-(x)\right| dG_n(x) \\ = \frac{1}{n}\sum_{i=1}^{n} \left|F^-(U_i)I(0 < U_i \le \delta)\right| &\to \int_{(0,\delta]} \left|F^-(x)\right| dx < \epsilon \end{aligned} \tag{2.19}$$

provided $\delta > 0$ is small enough. Note that $F^-(x)$ is non-decreasing. Moreover, since the case that $F^-(x)$ is bounded from below can be treated as in the case of $\alpha > 0$, we assume $F^-(x) < 0$ for $0 < x \le \delta$. Hence

$$\left|F^-(t)G_n(t)\right| \le \int_{(0,t]} \left|F^-(x)\right| dG_n(x) < \epsilon \tag{2.20}$$

for $0 < t \le \delta$ and $n \ge n_0$. And, by (2.13)

$$|t_2| = \left|F^-(U_{n,\alpha_n+1})G_n(U_{n,\alpha_n+1}-)\right| \le \left|F^-(U_{n,\alpha_n+1})G_n(U_{n,\alpha_n+1})\right| < \epsilon \tag{2.21}$$

a.s. on $\mathcal{A}$ for $n \ge n_0$. Noting the last inequality in (2.19), we have $|F^-(t)t| < \epsilon$ for $0 < t \le \delta$ and $n \ge n_0$, and hence,

$$|m_2| = \left|F^-(U_{n,\alpha_n+1})U_{n,\alpha_n+1}\right| < \epsilon \text{ a.s. on } \mathcal{A} \text{ for } n \ge n_0. \tag{2.22}$$

Similarly we have

$$t_1 = F^-(U_{n,\beta_n}+)G_n(U_{n,\beta_n}) \to 0 \text{ and } m_1 = F^-(U_{n,\beta_n}+)U_{n,\beta_n} \to 0 \tag{2.23}$$

a.s. on $\mathcal{A}$ as $n \to \infty$. (2.14) still holds on $\mathcal{A}$ obviously. Now we consider

$$\begin{aligned} t_3 - m_3 = &\int_{(0,U_{n,\alpha_n+1})} (G_n(x) - x)dF^-(x) \\ &+ \int_{(U_{n,\beta_n},1)} (G_n(x) - x)dF^-(x) + \int_{(0,1)} (x - G_n(x))dF^-(x). \end{aligned} \tag{2.24}$$

It is easy to see from the proofs of (2.21)–(2.23) that

$$\int_{(0,1)} (x - G_n(x))dF^-(x) = \int_{(0,1)} F^-(x)d(x - G_n(x)) =: \frac{1}{n}\sum_{i=1}^{n}(\xi_i - E\xi_i),$$

where $\xi_i = -F^-(U_i)$. By the SLLN (cf. (2.19)), we obtain

$$\int_{(0,1)} (x - G_n(x))dF^-(x) \to 0 \text{ a.s. on } \mathcal{A} \text{ as } n \to \infty.$$

Moreover, recalling (2.21), (2.22) and (2.19), (2.20) we have

$$\begin{aligned}&\lim_{n\to\infty}\int_{(0,U_{n,\alpha_n+1})} (G_n(x) - x)dF^-(x)\\ &= \lim_{n\to\infty}\int_{(0,U_{n,\alpha_n+1})} F^-(x)d(G_n(x) - x) = 0 \text{ a.s. on } \mathcal{A}\,. \end{aligned} \tag{2.25}$$

Similarly

$$\int_{(U_{n,\beta_n},1)} (G_n(x) - x)dF^-(x) \to 0 \text{ a.s. on } \mathcal{A} \text{ as } n \to \infty. \tag{2.26}$$

Thus, the theorem is proved by combining these results.

Proof of Theorem 1.2. At first, we note that by the condition $\sqrt{n}(\alpha_n/n - \alpha) \xrightarrow{P} 0$ and $\sqrt{n}(\beta_n/n - \beta) \xrightarrow{P} 0$ as $n \to \infty$ and Lemma 2.1, for any $\epsilon > 0$ there exist $K > 0$ and n_0 such that

$$P(\sqrt{n}\,|U_{n,\alpha_n+1} - \alpha| > K) < \epsilon, \quad P(\sqrt{n}\,|U_{n,\beta_n} - \beta| > K) < \epsilon \text{ for } n \geq n_0. \tag{2.27}$$

On the set $\{\alpha > 0\}$, write

$$m_2 + m_4 = \int_{(\alpha,U_{n,\alpha_n+1})} (F^-(x) - F^-(U_{n,\alpha_n+1}))dx - F^-(U_{n,\alpha_n+1})\alpha$$

and hence

$$t_2 - m_2 - m_4 = \int_{(\alpha,U_{n,\alpha_n+1})} (F^-(U_{n,\alpha_n+1}) - F^-(x))dx - F^-(U_{n,\alpha_n+1})\Big(\frac{\alpha_n}{n} - \alpha\Big). \tag{2.28}$$

For any $\epsilon > 0$ there is $0 < \gamma < 1/2$, such that $P(\alpha > 0) - P(\gamma < \alpha < 1 - \gamma) < \epsilon$. So we can consider $\{\gamma < \alpha < 1 - \gamma\}$ instead of $\{\alpha > 0\}$. From (2.27) we have

$$P\{\big|F^-(U_{n,\alpha_n+1})\big| > M, \gamma < \alpha < 1 - \gamma\} < \epsilon$$

for some $M > 0$ and large n. By condition (1.6) we have

$$P\{\big|F^-(U_{n,\alpha_n+1}) - F^-(\alpha)\big| > \epsilon^2, |U_{n,\alpha_n+1} - \alpha| < \delta, \gamma < \alpha < 1 - \gamma\} < \epsilon$$

provided $\delta > 0$ is small enough and n large enough. Therefore, from (2.28)

$$\begin{aligned}&P\{\sqrt{n}\,|t_2 - m_2 - m_4| > \epsilon, \gamma < \alpha < 1 - \gamma\}\\ &\leq P\Big\{M\sqrt{n}\,\Big|\frac{\alpha_n}{n} - \alpha\Big| > \frac{\epsilon}{2}, \gamma < \alpha < 1 - \gamma\Big\} + \epsilon\\ &\quad + P\{\epsilon^2\sqrt{n}\,|U_{n,\alpha_n+1} - \alpha| > \frac{\epsilon}{2}, |U_{n,\alpha_n+1} - \alpha| < \delta, \gamma < \alpha < 1 - \gamma\} + \epsilon\\ &\quad + P\{|U_{n,\alpha_n+1} - \alpha| > \delta\} \leq 5\epsilon \end{aligned} \tag{2.29}$$

for all large n. Similarly

$$P\{\sqrt{n}\,|t_1 - m_1 - m_5| > \epsilon, \gamma < \delta < 1 - \gamma\} \leq 5\epsilon. \tag{2.30}$$

Now write

$$t_3 - m_3 = \int_{(\alpha,\beta)} (x - G_n(x))dF^-(x) + \int_{(\alpha,U_{n,\alpha_n+1})} (G_n(x) - x)dF^-(x)$$
$$+ \int_{(U_{n,\beta_n},\beta)} (G_n(x) - x)dF^-(x) \tag{2.31}$$

and

$$\sqrt{n}\int_{(\alpha,U_{n,\alpha_n+1})} (G_n(x) - x)dF^-(x)$$
$$= \sqrt{n}(G_n(U_{n,\alpha_n+1}-) - U_{n,\alpha_n+1})(F^-(U_{n,\alpha+1}) - F^-(\alpha))$$
$$- \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(I(\alpha < U_i < U_{n,\alpha_n+1}) - (U_{n,\alpha_n+1} - \alpha))I(\alpha < U_i < U_{n,\alpha_n+1})$$
$$\cdot (F^-(U_i) - F^-(\alpha))$$
$$- \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(U_{n,\alpha_n+1} - \alpha)I(\alpha < U_i < U_{n,\alpha_n+1})\ (F^-(U_i) - F^-(\alpha))$$
$$+ \sqrt{n}\int_{(\alpha,U_{n,\alpha_n+1})} (F^-(x) - F^-(\alpha))dx =: \sum_{i=1}^{4} D_{ni}. \tag{2.32}$$

By Lemma 2.2, putting

$$A_1 = \Big\{ \sup_{0\le t\le a} \big|\sqrt{n}(G_n(\alpha+t) - (\alpha+t))\big| \ge \lambda, \gamma < \alpha < 1-\gamma \Big\},$$

we have for $0 < a < \gamma$

$$P(A_1) \le c/\lambda^4, \quad n \ge n_0. \tag{2.33}$$

Moreover, for given $\lambda \ge 1/\epsilon$ large enough, taking $a > 0$ to be small enough and putting

$$A_2 = \Big\{ \sup_{0\le t\le a} \big|F^-(\alpha+t) - F^-(\alpha)\big| \ge \lambda^{-2}, \gamma < \alpha < 1-\gamma \Big\},$$

we have

$$P(A_2) < \epsilon. \tag{2.34}$$

When $\alpha + t$ is replaced by $\alpha - t$ in (2.33) and (2.34), we have the same estimators. Thus

$$P\{|D_{n1}| > \epsilon, \gamma < \alpha < 1-\gamma\} \le P\{|U_{n,\alpha_n+1} - \alpha| > a\} + P(A_1) + P(A_2)$$
$$\le 2\epsilon + c\epsilon^4 \tag{2.35}$$

for all large n. For D_{n2},

$$0 \le D_{n2} \le (F^-(U_{n,\alpha_n+1}) - F^-(\alpha))\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(I(\alpha < U_i < U_{n,\alpha_n+1}) - (U_{n,\alpha_n+1} - \alpha))\Big|.$$

Then we have the result similar to D_{n1}.

As for D_{n3} and D_{n4}, a combination of (2.27) and condition (1.6) yields the required estimators. Consequently we obtain

$$P\Big\{\Big|\sqrt{n}\int_{(\alpha,U_{n,\alpha_n+1})} (G_n(x) - x)dF^-(x)\Big| > \epsilon, \gamma < \alpha < 1-\gamma\Big\} \le c\epsilon \tag{2.36}$$

provided n is large enough. Similarly

$$P\Big\{\Big|\sqrt{n}\int_{(U_{n,\beta_n},\beta)} (G_n(x) - x)dF^-(x)\Big| > \epsilon, \gamma < \beta < 1-\gamma\Big\} \le c\epsilon. \tag{2.37}$$

Now we turn to the case of $\alpha = 0$ and/or $\beta = 1$. As an instance we consider the case of $\alpha = 0$. First, suppose that conditions (1.8)–(1.10) are satisfied. Define

$$f(x) = \sqrt{x \log x^{-1}}\, \big|F^-(x)\big|. \tag{2.38}$$

Conditions (1.8) and (1.9) imply that $f(x) \to 0$ as $x \to 0$ (see [6]). Hence, with $q = n^{-1}\log n$

$$\Big|\sqrt{n}\int_0^q F^-(x)dx\Big| \le \sqrt{n}\int_0^q (x\log x^{-1})^{-1/2} f(x)dx = o(1) \text{ as } n \to \infty. \tag{2.39}$$

Therefore $E(\sqrt{n}\int_0^q F^-(x)dG_n(x))^2$, equivalently

$$E\Big\{\frac{1}{\sqrt{n}}\sum_{i=1}^n [F^-(U_i)I(0<U_i<q) - E(F^-(U_i)I(0<U_i<q))]\Big\}^2$$
$$\le c\int_0^q F^-(x)^2 dx \to 0 \text{ as } n \to \infty$$

by (2.1) and (1.8). This implies

$$\Big|\sqrt{n}\int_0^q F^-(x)dG_n(x)\Big| \xrightarrow{P} 0 \quad \text{as } n \to \infty. \tag{2.40}$$

Then similarly to (2.20) we obtain

$$\sqrt{n}\sup_{0<x\le q}\big|F^-(x)x\big| \to 0 \;\text{ and }\; \sqrt{n}\sup_{0<x\le q}\big|F^-(x)G_n(x)\big| \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{2.41}$$

Then putting $g = Kn^{-1/2}$, where K is defined by (2.27), we have

$$\begin{aligned} &P\{\sqrt{n}\,|t_2 - m_2| \ge \epsilon, \alpha = 0\} \le P\Big\{\sqrt{n}\sup_{0<x\le q}\big|(G_n(x)-x)F^-(x)\big| \ge \epsilon\Big\} \\ &+ P\Big\{\sqrt{n}\sup_{q<x\le g}\big|(G_n(x)-x)F^-(x)\big| \ge \epsilon\Big\} + P\{\sqrt{n}U_{n,\alpha_n+1} > K, \alpha = 0\} \\ &=: w_1 + w_2 + w_3, \end{aligned} \tag{2.42}$$

where $w_1 \le \epsilon$ and $w_3 \le \epsilon$ by (2.41) and (2.27) for large n respectively,

$$w_2 \le P\Big\{\sqrt{n}\sup_{q<x\le g}\big|(G_n(x)-x)/\sqrt{x}\big| \ge \epsilon \inf_{q<x\le g}(\log x^{-1})^{1/2}/f(x)\Big\} \le \epsilon$$

by (2.6) of Lemma 2.3 provided n is large enough.

Consider m_4. By noting condition (1.9) and recalling the proof of (2.39), we have

$$P\Big\{\sqrt{n}\Big|\int_0^{(\alpha_n+1)/n} F^-(x)dx\Big| \ge \epsilon, \alpha = 0\Big\} \to 0 \text{ as } n \to \infty.$$

Therefore, we may treat

$$\sqrt{n}\int_{(\alpha_n+1)/n}^{U_{n,\alpha_n+1}} F^-(x)dx I\Big(U_{n,\alpha_n+1} > \frac{\alpha_n+1}{n}\Big)$$

instead of $\sqrt{n}m_4$. From (1.9), for $\epsilon > 0$ there exists $M > 0$ such that $P(\alpha_n \ge M\log n) < \epsilon$ for large n. Put

$$\epsilon_n = \epsilon(\log n)^{1/2}/(nf(1/n)).$$

Then

$$\epsilon_n \le \epsilon/\Big(\sqrt{n}\,\Big|F^-\Big(\frac{\alpha_n+1}{n}\Big)\Big|\Big)$$

by (2.38). Hence

$$
\begin{aligned}
&P\Big\{\sqrt{n}\int_{(\alpha_n+1)/n}^{U_{n,\alpha_n+1}} F^-(x)dx \ge \epsilon, \alpha=0, U_{n,\alpha_n+1} > \frac{\alpha_n+1}{n}\Big\}\\
&\le P\Big\{\Big|\sqrt{n}\Big(U_{n,\alpha_n+1}-\frac{\alpha_n+1}{n}\Big)F^-\Big(\frac{\alpha_n+1}{n}\Big)\Big| \ge \epsilon, \alpha=0, U_{n,\alpha_n+1} > \frac{\alpha_n+1}{n}\Big\}\\
&\le P\Big\{\alpha_n+1 \ge \sum_{i=1}^n I\Big(U_i \le \frac{\alpha_n+1}{n}+\epsilon_n\Big), \alpha=0\Big\}\\
&\le P\Big\{\sum_{i=1}^n \Big[I\Big(U_i > \frac{\alpha_n+1}{n}+\epsilon_n\Big) - \Big(1-\frac{\alpha_n+1}{n}-\epsilon_n\Big)\Big] \ge n\epsilon_n, \alpha=0, \alpha_n \le M\log n\Big\}\\
&\quad + P(\alpha_n > M\log n, \alpha=0)\\
&\le P\Big\{\sup_{0\le t\le (2M\log n)/n} \big|Y_n(t)\big| \ge \sqrt{n}\epsilon_n\Big\} + \epsilon \le c(M/\epsilon^2)^{1+\gamma/2}(f(1/n))^{2+\gamma} + \epsilon,
\end{aligned}
$$

where (2.13)' in Remark 2.1 is used. Therefore we obtain

$$
P\{\sqrt{n}\big|m_4\big| \ge \epsilon, \alpha=0\} \to 0 \quad \text{as } n\to\infty. \tag{2.43}
$$

Consider $t_3 - m_3$. Remember (2.31). We investigate

$$
\sqrt{n}\int_{(0,U_{n,\alpha_n+1})} (G_n(x)-x)dF^-(x)I(\alpha=0),
$$

equivalently

$$
\sqrt{n}\int_{(0,U_{n,\alpha_n+1})} F^-(x)d(G_n(x)-x)I(\alpha=0),
$$

by (2.42) (and the estimators for w_1, w_2 and w_3), and further, it suffices to consider

$$
\sqrt{n}\int_{(1/n,U_{n,\alpha_n+1})} F^-(x)d(G_n(x)-x)I\Big(\alpha=0, \frac{1}{n} < U_{n,\alpha_n+1} \le K/\sqrt{n}\Big)
$$

by (2.39), (2.40) and (2.27). Noting $F^-(x)<0$ for x near zero enough, we have

$$
\begin{aligned}
&\Big|\sqrt{n}\int_{(1/n,U_{n,\alpha_n+1})} (-F^-(x))d(G_n(x)-x)I\Big(\alpha=0, \frac{1}{n} < U_{n,\alpha_n+1} \le K/\sqrt{n}\Big)\Big|\\
&= \Big|\sqrt{n}\int_{(1/n,U_{n,\alpha_n+1})} (x\log x^{-1})^{-\frac{1}{2}} f(x)d(G_n(x)-x)I\Big(\alpha=0, \frac{1}{n} < U_{n,\alpha_n+1} \le \frac{K}{\sqrt{n}}\Big)\Big|,
\end{aligned}
$$

which, as $n\to\infty$, is equivalent to

$$
\begin{aligned}
&\frac{\sqrt{n}\delta_n}{\sqrt{\log n}}\left|\int_{(1/n,U_{n,\alpha_n+1})} x^{-1/2}d(G_n(x)-x)I\Big(\alpha=0, \frac{1}{n} < U_{n,\alpha_n+1} \le K/\sqrt{n}\Big)\right|\\
&\le \frac{\delta_n}{\sqrt{\log n}} \sup_{1/n<t\le K/\sqrt{n}} \left|\frac{1}{\sqrt{n}}\sum_{i=1}^n \Big(\frac{1}{\sqrt{U_i}} I\Big(\frac{1}{n} < U_i \le t\Big) - E\frac{1}{\sqrt{U_i}} I\Big(\frac{1}{n} < U_i \le t\Big)\Big)\right|
\end{aligned} \tag{2.44}
$$

for some $\delta_n \to 0$ as $n\to\infty$. Now (2.17) of Lemma 2.3 implies that the right hand side of (2.44) tends to zero in probability as $n\to\infty$. Hence

$$
\sqrt{n}\int_{(0,U_{n,\alpha_n+1})} (G_n(x)-x)dF^-(x)I(\alpha=0) \xrightarrow{P} 0 \quad \text{as } n\to\infty. \tag{2.45}
$$

For

$$
\sqrt{n}\int_{(U_{n,\beta_n},1)} (G_n(x)-x)dF^-(x)I(\beta=1),
$$

we have the same convergence.

Suppose that conditions (1.8)' and (1.9)' are satisfied. Instead of (2.38), we can write

$$F^{-}(x) = x^{-1/r} f(x),$$

where $f(x) \to 0$ as $x \to 0$. (Note $t\,|F^{-}(x)|^{r} \le \int_0^t |F^{-}(x)|^{r}\,dx \to 0$). Then the above conclusions are also true by using Lemma 2.4 instead of Lemma 2.3.

Finally we consider $\sqrt{n}\int_{(\alpha,\beta)}(x - G_n(x))dF^{-}(x)$ in (2.31). For $\alpha < a < b < \beta$, write

$$\begin{aligned}
&\int_{(a,b)} (x - I(U_i \le x))dF^{-}(x) \\
&= F^{-}(a+)[I(U_i \le a) - a] - F^{-}(b)[I(U_i < b) - 1) - (b-1)] \\
&\quad + F^{-}(U_i)I(a < U_i < b) - \int_{(a,b)} F^{-}(x)dx.
\end{aligned}$$

Therefore in the case of $\alpha > 0$ and $\beta < 1$, when $a \downarrow \alpha$ and $b \uparrow \beta$, by recalling (1.4) and noting condition (1.6), it follows that

$$\sqrt{n}\int_{(a,b)} (x - G_n(x))dF^{-}(x) \xrightarrow{P} \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \xi_i(\alpha,\beta). \tag{2.46}$$

If $\alpha = 0$, $\sqrt{n}F^{-}(a+)[G_n(a) - a)] \xrightarrow{P} 0$ as $a \downarrow 0$ since for fixed n

$$P\{|F^{-}(a+)(I(U_i \le a) - a)| > \epsilon\} \le 2a|F^{-}(a+)|/\epsilon \to 0 \text{ as } a \downarrow 0$$

by (2.19). When $\beta = 1$, we have the similar conclusion. So, if we understand $F^{-}(0+)[G_n(0) - 0]$ to mean zero, (2.46) holds for any cases. Now, by the central limit theorem for a φ-mixing sequence (e.g., cf. [7]), we obtain (1.7) of Theorem 1.2.

References

[1] Billingsley, P., Convergence of probability measure, J. Wiley, New York, 1968.

[2] Cheng, S. H., A complete solution for weak convergence of heavy trimmed sums, *Science in China,* **35A** (1992), 641–656.

[3] Csörgő, S., Haeusler, E. & Mason, D. M., The asymptotic distribution of trimmed sums, *Ann. Probab.,* **16** (1988), 672–699.

[4] Csörgő, S., Haeusler,E. & Mason, D. M., The asymptotic distribution of extreme sums, *Ann. Probab.,* **19** (1991), 783–811.

[5] Griffn, P. S. & Pruitt ,W. E., Asymptotic normality and subsequential limits of trimmed sums, *Ann. Probab.,* **17** (1989), 1186–1219.

[6] James, B. R., A functional law of the iterated logarithm for weighted empirical distributions, *Ann. Probab.,* **3** (1975), 762–772.

[7] Peligrad, M., An invariance principle for ϕ-mixing sequences, *Ann. Probab.,* **13** (1983), 1304–1313.

[8] Shao, Q. M., A moment inequality and its applications, *Acta Math. Sinica,* **31** (1988), 736–747.

[9] Shao, Q. M., On complete convergence for a ρ–mixing sequence, *Acta Math. Sinica,* **32** (1989), 377–393.

[10] Stigler, S. M., The asymptotic distribution of the trimmed mean, *Ann. Statist.,* **1** (1973), 472–477.