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ALGEBRAIC GROUPS ASSOCIATED WITH GRADED LIE ALGEBRAS OF CARTAN TYPE***

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Abstract

For a graded simple Lie algebra of Cartan type $L = X(m:\mathbf{n})^{(2)}$, $X \in \{W, S, H, K\}$, over a field F of odd characteristic p, the group generated by one-parameter subgroups of the form $\exp(t\mathrm{ad}y)$ is described, where $y \in L + Fu$ satisfying $y^p = 0$, $t \in F$ and u is some fixed element of the p-envelope of L in $\mathrm{Der}\mathfrak{A}(m:\mathbf{n})$.

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§0. Introduction

It is well known that over algebraically closed fields of characteristic 0, there exists a good correspondence between the connected semisimple algebraic groups and the semisimple Lie algebras. But when the ground field F is of characteristic $p > 0$, the situation is rather different. Simple algebraic groups correspond only to classical Lie algebras which may not be simple (for instance $\mathrm{SL}(n)$ when p divides n). By the classification theorem (see [14]), if $p > 7$, a simple Lie algebra is either classical or of Cartan type. It might be interesting to investigate the possible relation between the simple algebraic groups and the Lie algebras of Cartan type. When $p > 3$, the (restricted) Lie algebra $\mathfrak{g}$ of a semisimple algebraic group G is generated by $Y(\mathfrak{g}) = \{y \in \mathfrak{g} | y^p = 0\}$, and G can be described as the group generated by the one-dimensional subgroups $\exp t\rho(y)$, $t \in F$, $y \in Y(\mathfrak{g})$, where ρ is a nontrivial irreducible representation of $\mathfrak{g}$. Similarly, a restricted Lie algebra L of Cartan type is generated by $Y(L)$. Let ρ be a nontrivial irreducible p-representation of L in the module V and $G(Y(L), \rho)$ the group generated by $\exp t\rho(y)$, $t \in F$, $y \in Y(L)$, A. I. Kostrikin has raised the problem of the classification of the groups $G(Y(L), \rho)$ for restricted Cartan type Lie algebras L. Later, A. A. Premet showed in [7] that $G(Y(L), \rho)$ coincides with one of the groups $\mathrm{SL}(V)$, $\mathrm{SO}(V)$ or $\mathrm{Sp}(V)$ except when $L = W(1,1)$ and $p = 7$, in which case the group is G_2.

In this paper, Premet's results in [7] are generalized. Especially, when L is a graded Lie algebra of Cartan type, $Y(L)$ in general does not generate L. However, if adding a suitable element u of a p-envelope $\mathfrak{L}$ in $Y(L)$ (in some cases, u may be taken to be 0), then the algebra $\langle\langle u, Y(L)\rangle\rangle$ generated by u and $Y(L)$ is $\langle\langle u, L\rangle\rangle(= Fu + L)$ which contains L as its

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unique minimal ideal. Let ad:$\langle\langle u, L\rangle\rangle \to L$ be the adjoint representation of L extended to $\langle\langle u, L\rangle\rangle$. Then the group $G(Y(L+Fu)$,ad) is simple and the main results of [7] is also valid for $G(Y(L+Fu)$,ad).

In principle, the group $G(Y(L+Fu),\rho)$ can be considered similarly for any irreducible representation ρ of L that can be extented to $\mathfrak{L}$. Those representations were determined in [11, 12] and [5, 6].

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§1. Generalized Premet's Results

In the present section, we will generalize the main results for restricted simple Lie algebras in [7] to nonrestricted cases. In the following, F always denotes a field of characteristic $p>2$. All Lie algebras in the paper are over F. If Y is a subset of a Lie algebra, $\langle\langle Y\rangle\rangle$ (resp. $\langle Y\rangle$) will stand for the Lie subalgebra generated (resp. the subspace F-spanned) by Y.

Proposition 1.1. *Let L be a simple Lie algebra, $\mathfrak{L}$ a restricted Lie algebra containing L and a subalgebra $\mathfrak{L}_1$ with $\mathfrak{L}_1$ and $\langle\langle Y(\mathfrak{L}_1)\rangle\rangle \supset L$. Suppose (ρ, V) to be a p-representation of $\mathfrak{L}$ which satisfies the following conditions:* (i) *there exists $y\in L$ such that $\rho(y)\neq 0$ and $\rho(y)^2=0$;* (ii) *$(\rho|_L, V)$ is nontrivial and irreducible. Then $G(Y(\mathfrak{L}_1),\rho|_{\mathfrak{L}_1})$ has the following properties:*

(1) $\rho(L)\subset \mathrm{Lie}(G(Y(\mathfrak{L}_1),\rho|_{\mathfrak{L}_1}))$.

(2) *$G(Y(\mathfrak{L}_1),\rho|_{\mathfrak{L}_1})$ is a semisimple irreducible group in $SL(V)$.*

(3) *$G(Y(\mathfrak{L}_1),\rho|_{\mathfrak{L}_1})$ is a simple algebraic group.*

Proof. (1) and (2) follow directly from the proof of [7, Lemma 8].

(3) Set $\mathfrak{g}=\mathrm{Lie}(G(Y(\mathfrak{L}_1),\rho|_{\mathfrak{L}_1}))$. $\mathfrak{g}$ is a restricted Lie algebra. Assume that the semisimple algebraic group $G(Y(\mathfrak{L}_1),\rho|_{\mathfrak{L}_1})$ is not simple. Then $\mathfrak{g}$ would be decomposed into a direct sum of nonablian p-ideals $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2$ by [3, §6]. By the same method as in the proof of [7, Lemma 9], $\rho(y)\in\mathfrak{g}_1$ or $\rho(y)\in\mathfrak{g}_2$. Suppose $\rho(y)\in\mathfrak{g}_1$. Then $\rho|_L^{-1}(\mathfrak{g}_1\cap\rho(L))$ is a nonzero ideal of L. Thus $\mathfrak{g}_1\supset\rho(L)$. But V is an irreducible L-module, the centralizer of $\rho(L)$ in $\mathfrak{gl}(V)$ consists of scalar matrices. Thus $\mathfrak{g}_2$ consists of scalar matrices, contradicting the non-commutativity of $\mathfrak{g}_2$.

The following lemma follows directly from [8, Theorem 2].

Lemma 1.1. *Let T be an irreducible rational representation of a simple algebraic group G, V its associated G-module and*

$$Q(G,T)=\{x\in G \mid T(x)\neq 1 \quad \text{and} \quad (T(x)-1)^2=0\}.$$

Suppose that not all elements $x\in Q(G,T)$ with $\dim(T(x)-1)V<\frac{1}{2}\dim V$ are pairwise conjugate in G. Then $T(G)$ coincides with one of the following groups: $\mathrm{SL}(V)$, $\mathrm{SO}(V)$ *or* $\mathrm{Sp}(V)$.

By a direct deduction we have

Proposition 1.2. *Let $\mathfrak{L}$, $\mathfrak{L}_1$, L and ρ be the same as in Proposition* 1.1. *Suppose that $\rho|_L$ satisfies the so-called Premet condition: there exist $y_i\in L$, $i=1,2$, $\rho(y_i)\neq 0$ and $\rho(y_i)^2=0$ such that* $\dim y_iV<\frac{1}{2}\dim V$ *and* $\dim y_1V\neq\dim y_2V$. *Then $G(Y(\mathfrak{L}_1),\rho|_{\mathfrak{L}_1})$ coincides with one of* $\mathrm{SL}(V)$, $\mathrm{SO}(V)$ *or* $\mathrm{Sp}(V)$.

Especially, suppose $\mathfrak{L}$ to be a p-envelope of L. We have an analog of Lemma 12 of [7].

Proposition 1.3. *Let $\mathfrak{L}$, $\mathfrak{L}_1$, L and ρ be the same as in Proposition* 1.1, *and $\mathfrak{L}$ a p-envelope of L. Suppose $\rho|_L$ satisfies the Premet condition. Then $G(Y(\mathfrak{L}_1), \rho|_{\mathfrak{L}_1})$ is a subgroup of* $\mathrm{SO}(V)$ (*resp.* $\mathrm{Sp}(V)$) *if and only if V has a nondegenerate L-invariant symmetric* (*resp. skew-symmetric*) *bilinear form.*

Proof. If f is a nonzero L-invariant form of V, then f is also an $\mathfrak{L}_1$-invariant form because $\mathfrak{L}_1$ is contained in a p-envelope of L. Let $y \in Y(\mathfrak{L}_1)$ and z, z' be arbitrary elements in V. For $q < p$,

$$\sum_{i=0}^{q} \frac{1}{i!(q-i)!} f(\rho(y)^i z, \rho(y)^{q-i} z') = \frac{f(z, \rho(y)^q z')}{m!} \sum_{i=0}^{q} (-1)^i \begin{pmatrix} q \\ i \end{pmatrix} = 0.$$

For $q > p$,

$$f(\rho(y)^i z, \rho(y)^{q-i} z') = (-1)^i f(z, \rho(y)^q z') = 0.$$

Thus

$$f(\exp \rho(y) z, \exp \rho(y) z') = \sum_{q=0}^{\infty} \sum_{i=0}^{q} \frac{1}{i!(q-i)!} f(\rho(y)^i z, \rho(y)^{q-i} z') = f(z, z').$$

We have

$$G(Y(\mathfrak{L}_1), \rho|_{\mathfrak{L}_1}) \subset \mathrm{Aut} f \cap \mathrm{SL}(V).$$

Conversely, if $G(Y(\mathfrak{L}_1), \rho|_{\mathfrak{L}_1}) \subset \mathrm{SO}(V)$ or $\mathrm{Sp}(V)$, then $\mathfrak{g}$ is contained in $\mathfrak{so}(V)$ or $\mathfrak{sp}(V)$. Note that

$$\rho(L) \subset \rho(\mathfrak{L}_1) \subset \mathfrak{g},$$

which implies that V has a nondegenerate L-invariant bilinear form.

Furthermore, if there exists another subalgebra $\mathfrak{L}_2$ such that $\langle\langle Y(\mathfrak{L}_2)\rangle\rangle \supset L$, then $G(Y(\mathfrak{L}_2),\ \rho|_{\mathfrak{L}_2})$ is also one of $\mathrm{SL}(V)$, $\mathrm{SO}(V)$ or $\mathrm{Sp}(V)$ and the following interesting result follows directly from Proposition 1.3.

Corollary 1.1. *$G(Y(\mathfrak{L}_1), \rho|_{\mathfrak{L}_1}) = G(Y(\mathfrak{L}_2), \rho|_{\mathfrak{L}_2})$ if $\langle\langle Y(\mathfrak{L}_i)\rangle\rangle \supset L$, $i = 1, 2$.*

§2. Groups Associated with Non-Restricted Graded Simple Lie Algebras of Cartan Type

In the rest of the paper, we focus our attention to the graded Lie algebras of Cartan type. Let L be any one of these algebras, i.e. $L = X(m : \mathbf{n})^{(2)}$, $X = W, S, H$ or K (for their definitions see [10] and [15]), where $m \in \mathbb{N}$ and $m \geqslant 3$ if $X = S$, m is even if $X = H$ and m is odd and $\geqslant 3$ if $X = K$, and $\mathbf{n} = (n_1, n_2, \cdots, n_m) \in \mathbb{Z}_+^m$. For the notations of Lie algebras of Cartan type we will follow [10] and [15]. Set $Y(L) = \{y \in L | (\mathrm{ad} y)^p = 0\}$ and denote by $\langle\langle Y(L)\rangle\rangle$ the Lie algebra generated by $Y(L)$. Set $L_i = L_{\mathbf{1}+\delta_{X,K}\epsilon_m, i}$, i.e. $\{L_i\}$ is the noncontractible filtration of L (see [15]). By the results in [13], a description of $\langle\langle Y(L)\rangle\rangle$ can be given as follows:

Proposition 2.1. *For $L = X(m : \mathbf{n})^{(2)}$, $X = W, S, H$ or K.*

(1) *If $\mathbf{n} \not> \mathbf{1}$, then $\langle\langle Y(L)\rangle\rangle = L$.*

(2) *If $\mathbf{n} > \mathbf{1}$, then $\langle\langle Y(L)\rangle\rangle = L_0$ for $X = S$ or H, $\langle\langle Y(L)\rangle\rangle = W(m : \mathbf{n})_0^{(1)}$ for $X = W$, and $\langle\langle Y(L)\rangle\rangle \supset \{\mathcal{D}_K(x^\alpha) \mid \|\alpha\| \geqslant 0$ and $\alpha \neq p^k \epsilon_m\}$ for $X = K$.*

From now on, let $L = X(m:\mathbf{n})^{(2)}$ be nonrestricted, i.e. $\mathbf{n} \neq \mathbf{1}$, $X \in \{W, S, H, K\}$. Fix $u = D_i^{p^{n_i-1}}$, $i \in \{1, 2, \cdots, m\}$ for $X = W, S$ or H and $u = \mathcal{D}_K(x_{\bar{i}})^{p^{n_i-1}}$, $i \in \{1, 2, \cdots, m-1\}$ or $u = \mathcal{D}_K(1)^{p^{n_m-1}}$ for $X = K$. Let $\mathfrak{L}_1$ be the Lie algebra generated by L and u. As u is contained in a p-envelope of L, $\mathfrak{L}_1 = L + Fu$. By computation, we see $(\mathrm{ad}u)^p = 0$. In fact, as $u \in \mathrm{Der}(\mathfrak{A}(m:\mathbf{n}))$, a Lie p-algebra, in order to prove $(\mathrm{ad}u)^p = 0$ it is sufficient to verify $u^p = 0$, i.e. $D_i^{p^{n_i}}(x^{p^s\epsilon_j}) = 0$ for $X = W, S$ or H, $\mathcal{D}_K(x_{\bar{i}})^{p^{n_i}}(x^{p^s\epsilon_j}) = 0$ or $\mathcal{D}_K(1)^{p^{n_m}}(x^{p^s\epsilon_j}) = 0$, $j = 1, 2, \cdots, m$, $0 \leqslant s < n_j$ for $X = K$, where $\epsilon_j = (\delta_{j1}, \cdots, \delta_{jm})$. Thus

$$Y(\mathfrak{L}_1) = Y(L) \cup \{Fu\}.$$

According to Proposition 2.1, it is easily verified that $\langle\langle Y(\mathfrak{L}_1)\rangle\rangle = \mathfrak{L}_1$. A nontrivial ideal of $\mathfrak{L}_1$ can be considered as an $\mathfrak{L}_1$-module and the corresponding representation will be still denoted by ad. Let $G(Y(\mathfrak{L}_1),\mathrm{ad})$ be the group generated by $\exp(t\mathrm{ad}y)$, $t \in F$, $y \in Y(\mathfrak{L}_1)$. $G(Y(\mathfrak{L}_1),\mathrm{ad}) \in \mathrm{GL}(L)$ is a connected algebraic group according to [4, 7.5]. By Proposition 1.1 we have

Lemma 2.1. *$G(Y(\mathfrak{L}_1)$,ad) is a simple algebraic group.*

For the graded Lie algebra $L = \bigoplus_{i=-2}^{s} L_{[i]}$, $L_{[s]} \neq 0$ where s is called the length of the gradation. Since $\mathbf{n} \neq \mathbf{1}$, $s \geqslant 7$.

Lemma 2.2. *For any $y \in L_{[s-1]} + L_{[s]}$, $\dim \mathrm{ad}\, y(L) < \frac{1}{2}\dim L$.*

Proof. We have $\mathrm{ad}y(L) = \mathrm{ad}y(L_{[-2]} + L_{[-1]} + L_{[0]} + L_{[1]})$ and

$$\dim \mathrm{ad}\, y(L) \leq \dim L_{[s-3]} + L_{[s-2]} + L_{[s-1]} + L_{[s]}.$$

By simple computation we have $2\dim\ \mathrm{ad}y(L) < \dim L$.

We call the following elements the standard generators of the graded Lie algebras of Cartan type: $x^\alpha D_i$, $1 \leqslant i \leqslant m$ and $0 \leqslant \alpha \leqslant \tau$ for type W, $\mathcal{D}_{ij}(x^\alpha)$, $1 \leqslant i, j \leqslant m$ and $0 \leqslant \alpha \leqslant \tau$ for type S, $\mathcal{D}_H(x^\alpha)$, $0 \leqslant \alpha < \tau$ for type H and $\mathcal{D}_K(x^\alpha)$, $0 \leqslant \alpha < \tau$ if $n+3 \equiv 0$ mod p and $0 \leqslant \alpha \leqslant \tau$ if $n+3 \not\equiv 0 \pmod p$ for type K.

Lemma 2.3. *Let $E_s \in L_{[s]}$ be a standard generator of L. $E = D_i$, $1 \leqslant i \leqslant m$ for $X = W, S, H$ and $E = \mathcal{D}_K(x_{\bar{i}})$, $0 \leqslant i \leqslant m-1$ for $X = K$. Then*

(1) $[E, L_{[1]}] = L_{[0]}$.

(2) $[E, L_{[0]}] = L_{[-1]}$.

(3) $[E_s, L_{[0]}] = FE_s$.

Proof. This follows from [10, Chapter 4].

Lemma 2.4. *There exist $y, z \in L_{[s-1]} + L_{[s]}$ so that* $\dim\ \mathrm{ad}z(L) \neq \dim\ \mathrm{ad}y(L)$. *Thereby* $\exp\ \mathrm{ad}z$ *and* $\exp\ \mathrm{ad}y$ *are not conjugate in* $G(Y(\mathfrak{L}_1),\mathrm{ad})$.

Proof. Let $z = E_s \in L_{[s]}$, a standard generator, $y = [E, E_s]$ where E is the same as in Lemma 2.3. We will prove $\dim[z, L] < \dim[y, L]$, which is divided into two cases. For $X = W, S$ and H,

$$[z, L] = [z, L_{[-1]}] \oplus [z, L_{[0]}] \text{ and } [y, L] = [y, L_{[-1]}] \oplus [y, L_{[0]}] \oplus [y, L_{[1]}].$$

Moreover by Lemma 2.2,

$$[y, L_{[0]}] = [[E, L_{[0]}], E_s] + [E, [E_s, L_{[0]}]] = [z, L_{[-1]}],$$
$$[y, L_{[1]}] = [[E, E_s], L_{[1]}] = [[E, L_{[1]}], E_s] = [L_{[0]}, E_s] = [z, L_{[0]}]$$

and $[y, L_{[-1]}] \neq 0$, so the inequality holds. For $X = K$,

$$[z, L] = [z, L_{[-2]}] \oplus [z, L_{[-1]}] \oplus [z, L_{[0]}]$$

and

$$[y, L] = [y, L_{[-2]}] \oplus [y, L_{[-1]}] \oplus [y, L_{[0]}] \oplus [y, L_{[1]}].$$

By the argument as above,

$$[y, L_{[0]}] = [z, L_{[-1]}] \text{ and } [y, L_{[1]}] = [z, L_{[0]}].$$

In addition,

$$\dim [y, L_{[-1]}] > 1 = \dim [z, L_{[-2]}],$$

hence $\dim[z, L] < \dim[y, L]$. Note that

$$(\mathrm{ad}\, z)^2 = (\mathrm{ad}\, y)^2 = 0.$$

If there exists $w \in G(Y(\mathfrak{L}_1),\mathrm{ad})$ such that

$$w(\exp \mathrm{ad}\, y)w^{-1} = \exp \mathrm{ad}\, z,$$

then

$$w(\exp \mathrm{ad}\, y - 1)w^{-1} = \exp \mathrm{ad}\, z - 1,$$

which would lead to $\dim \mathrm{ad} z(L) = \dim \mathrm{ad} y(L)$, a contradiction.

By Proposition 1.1(2), L becomes an irreducible module of simple algebraic group $G(Y(\mathfrak{L})$, ad). According to Lemma 2.1 and Lemma 2.4, we can apply Propositions 1.2 and 1.3 to obtain our main result:

Theorem 2.1. *Let L be a nonrestricted graded simple Lie algebra $X(m : \mathbf{n})^{(2)}$, $X \in \{W, S, H, K\}$. Then*

(1) *$G(Y(L + Fu),\mathrm{ad})$ is one of* $\mathrm{SL}(L)$, $\mathrm{SO}(L)$ *or* $\mathrm{Sp}(L)$.

(2) *$G(Y(L + Fu),\mathrm{ad})$ is one of the groups* $\mathrm{SO}(L)$ (*resp.* $\mathrm{Sp}(L)$) *if and only if L has a nondegenerate L-invariant symmetric* (*resp. skew-symmetric*) *bilinear form.*

Remark 2.1. (1) If $\mathbf{n} \neq 1$ and $\not> \mathbf{1}$, let for example $n_i = 1$. Choose $u = D_i$ for $X \in \{W, S, H\}$ or $u = \mathcal{D}_K(x_{\tilde{i}})$ if $i < m$ for $X = K$, $u = \mathcal{D}_K(1)$ if $i = m$ for $X = K$. Then the above group is just $G(Y(L),\mathrm{ad})$, which is one of $\mathrm{SL}(L)$, $\mathrm{SO}(L)$ or $\mathrm{Sp}(L)$.

(2) The nondegenerate bilinear form of L was discussed by R. Farnsteiner exhaustively in [2].

Corollary 1.1 shows that this group is independent of the choices of u. We have

Theorem 2.2. *Let L be a nonrestricted graded simple Lie algebra of Cartan type, $\mathfrak{L}$ the p-envelope of L in* $\mathrm{Der}\mathfrak{A}(m : \mathbf{n})$, *$\mathfrak{L}_1$ a subalgebra of $\mathfrak{L}$ containing u. Then*

$$G(Y(\mathfrak{L}_1), \mathrm{ad})) = G(Y(L + Fu), \mathrm{ad}).$$

In particular,

$$G(Y(L + u), \mathrm{ad}) = G(Y(\mathfrak{L}), \mathrm{ad}).$$

If $n_i = 1$ for some $i \in \{1, 2, \cdots, m\}$, then

$$G(Y(\mathfrak{L}), \mathrm{ad}) = G(Y(L), \mathrm{ad}).$$

Remark 2.2. Let $\mathfrak{L}$ be the p-envelope of L in $\mathrm{Der}\mathfrak{A}(m : \mathbf{n})$ and ρ an irreducible restricted representation of $\mathfrak{L}$. By [11, Corollary 1.5] ρ is graded. Hence $\rho|_L$ is graded and irreducible.

It is easily seen from [11, Theorem 1.2 and Lemma 1.3] that an irreducible graded module V of L can be extended to an irreducible restricted module of $\mathfrak{L}$ if and only if the base space V_0 of V is an irreducible restricted module of $L_{[0]}$. The irreducible graded modules of L are described in [12] for $X = W, S, H$, and in [5-6] for $X = K$. Furthermore, let $\rho|_L$ be nontrivial. Then the Premet condition for $\rho|_L$ can be verified and the discussion in the above theorems for $G(Y(L+Fu),\text{ad})$ can be generalized to $G(Y(L+Fu),\rho)$. The invariant bilinear forms on the graded irreducible modules of L have been discussed in [1].

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