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CARLESON MEASURES AND THE FRACTIONAL DERIVATIVES OF HOLOMORPHIC FUNCTIONS IN THE UNIT BALL OF $\mathbb{C}^n$**

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Abstract

The author uses certain integral inequalities involving the fractional derivatives of holomorphic functions to characterize the extended Carleson measures in the unit ball of $\mathbb{C}^n$.

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§1. Introduction

The Carleson measure in the unit disk of the complex plane $\mathbb{C}$ was first introduced by Carleson in the study of interpolation problems in H^p spaces[2]. Later on, the extended Carleson measures were studied by many authors and they were close connected with the functions (together with their derivatives) in H^p and in some other spaces (see [3, 6] and the references therein). On the other hand, it is not difficult to find that little effort has been made on the connection between Carleson measure and the fractional derivatives of holomorphic functions. The purpose of this paper is to do some analysis in this line and our results will generalize [3,6]. We will also present a theorem on bounded symmetric domains which generalizes the known result in [4, 11].

Let B be the unit ball of $\mathbb{C}^n$, $B = \{z \in \mathbb{C}^n; |z| < 1\}$. We denote by dm and $d\sigma$ the Lebesgue value element of $\mathbb{C}^n$ and the $(2n-1)$-dimension surface measure on $\partial B = \{z \in \mathbb{C}^n; |z| = 1\}$ respectively which satisfy $\int_B dm = 1$, $\int_{\partial B} d\sigma = 1$. Let $H(B)$ be the family of all holomorphic functions in B. For $0 < p < \infty$, $0 < r < 1$ and $f \in H(B)$, set

$$M_p(f,r) = \Big\{ \int_{\partial B} |f(r\zeta)|^p d\sigma(\zeta) \Big\}^{1/p}.$$

The Hardy space $H^p(B)$ is defined by

$$H^p(B) = \{f \in H(B);\ \|f\|_{H^p} = \sup_{0<r<1} M_p(f,r) < \infty\}.$$

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For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space $L_a^p(dm_\alpha)$ consists of all $f \in H(B)$ for which

$$\|f\|_{L^p(dm_\alpha)} = \Big\{ \int_B |f(z)|^p (1-|z|)^\alpha dm(z) \Big\}^{1/p} < \infty.$$

It is well-known that every function $f \in H(B)$ has a series expansion $f(z) = \sum\limits_{k,v} a_{kv}\varphi_{kv}(z)$ (see [8]). For $\beta \geq 0$ the β-th fractional derivative and the β-th fractional integral of f are defined, respectively, by

$$f^{[\beta]}(z) = \sum_{k,v} \frac{\Gamma(k+\beta+1)}{\Gamma(k+1)} a_{kv}\varphi_{kv}(z),$$

$$f_{[\beta]}(z) = \sum_{k,v} \frac{\Gamma(k+1)}{\Gamma(k+\beta+1)} a_{kv}\varphi_{kv}(z).$$

Both $f^{[\beta]}$ and $f_{[\beta]}$ are holomorphic in B.

For $\zeta \in \partial B$ and $0 < t < 1$, set $\theta(\zeta, t) = \{\xi \in \partial B; |1 - \langle \xi, \zeta\rangle| < t\}$ and

$$S(\zeta, t) = \{z \in B; 1-t < |z| < 1,\ z/|z| \in \theta(\zeta, t)\},$$

where $\langle \cdot, \cdot \rangle$ is the complex inner product in $\mathbb{C}^n$. If μ is a finite positive Borel measure on B, we call μ a Carleson measure of order s if there is a positive constant C such that $\mu(S(\zeta, t)) \leq Ct^s$ for all $\zeta \in \partial B$ and $t \in (0,1)$.

In what follows C stands for a positive constand which does not depend on the functions being considered but may change in each occurance. The expression "$A(f) \simeq B(f)$" means $B(f)/C \leq A(f) \leq CB(f)$.

§1. Main Theorem

Let $\beta(\cdot,\cdot)$ denote the Bergman distance on B. For $z \in B$ and $r > 0$, the Bergman ball $E(z, r)$ with centre z and radius r is defined by

$$E(z, r) = \{w \in B; \beta(w, z) < r\}.$$

Lemma 1.1. *Let μ be a finite positive Borel measure on B and $s > n$. Then the following two statements are equivalent.*

(i) *μ is Carleson measure of order s.*

(ii) *For $r > 0$ there exists a positive constant C such that $\mu(E(z,r)) \leq C(1-|z|)^s$, $z \in B$.*

Proof. For $r > 0$ fixed and $z \in B$, we choose $t > 0$ to be the smallest number such that $E(z, r) \subseteq S(z/|z|, t)$. As mentioned by Luecking in [7], when z is sufficiently near ∂B, say $c \leq |z| < 1$, t is proportional to $1 - |z|$. Hence

$$\mu(E(z,r)) \leq \mu(S(z/|z|, t)) \leq Ct^s \leq C(1-|z|)^s, \quad \text{for } \ c \leq |z| < 1.$$

That $\mu E(z,r)) \leq C(1-|z|)^s$ for $|z| < c$ is obvious. Therefore (i) implies (ii).

Conversely, if μ is a finite positive Borel measure and (ii) is valid with $s > n$, for $\alpha = s - (n+1) > -1$ we do the analysis as in the proof of Proposition 1 in [7] to obtain

$$\int_B |f(z)| d\mu(z) \leq C\|f\|_{L^1(dm_\alpha)}, \quad \text{for } \ f \in H(B).$$

This implies (i) at once.

Lemma 1.2. *Let $0 < p < \infty, \alpha > -1$, and $\beta \geq 0$. Then for $f \in H(B)$,*

$$\|f\|_{L^p(dm_\alpha)} \simeq \|f^{[\beta]}\|_{L^p(dm_{\alpha+\beta p})}.$$

Proof. Using the polar coordinates in the the integral of $\|f\|_{L^p(dm_\alpha)}$ and applying Theorem 5 in [10], we get

$$\begin{aligned}\|f\|^p_{L^p(dm_\alpha)} &= \int_0^1 r^{2n-1} M_p^p(f,r)(1-r)^\alpha dr \leq \int_0^1 M_p^p(f,r)(1-r)^\alpha dr \\ &\leq C\int_0^1 M_p^p(f^{[\beta]},r)(1-r)^{\alpha+\beta p} dr \\ &\leq C\int_0^1 r^{2n-1} M_p^p(f^{[\beta]},r)(1-r)^{\alpha+\beta p} dr \\ &= C\|f^{[\beta]}\|^p_{L^p(dm_{\alpha+\beta p})}.\end{aligned}$$

The last two inequalities come from the monotonicity of $M_p(f^{[\beta]},r)$. Similarly we can also have $\|f^{[\beta]}\|_{L^p(dm_{\alpha+\beta p})} \leq C\|f\|_{L^p(dm_\alpha)}$. The proof is completed.

Theorem 1.1. *Let μ be a finite positive Borel measure on B, and suppose $0 < p \leq q < \infty$, $\alpha > -1$ and $\beta \leq 0$. Then in order that*

$$\Big\{\int_B |f^{[\beta]}(z)|^q d\mu(z)\Big\}^{1/q} \leq C\|f\|_{L^p(dm_\alpha)} \quad \textit{for} \quad f \in H(B), \tag{1.1}$$

it is necessary and sufficient that μ is a Carleson measure of order $(n+1+\alpha)q/p + \beta q$.

Proof. For function $f \in H(B)$ and $p > 0$, we have from [7]

$$\begin{aligned}|f(z)|^p &\leq C|E(z,r)|^{-1}\int_{E(z,r)} |f(w)|^p dm(w) \\ &\leq C(1-|z|)^{-(n+1)}\int_{E(z,r)} |f(w)|^p dm(w).\end{aligned} \tag{1.2}$$

Suppose that $\mu(E(z,r)) \leq C(1-|z|)^{(n+1+\alpha)q/p+\beta q}$. Then we obtain as that in [6, pp. 99–91]

$$\begin{aligned}\int_B |f(z)|^q d\mu(z) &\leq C\int_B \Big[(1-|z|)^{-(n+1)}\int_B \chi_{E(z,r)}(w)|f(w)|^p dm(w)\Big]^{\frac{q}{p}} d\mu(z) \\ &\leq C\Big\{\int_B |f(w)|^p dm(w)\Big[\int_B (1-|z|)^{-(n+1)q/p}\chi_{E(z,r)}(w) d\mu(z)\Big]^{\frac{p}{q}}\Big\}^{\frac{q}{p}} \\ &\leq C\Big\{\int_B |f(w)|^p (1-|w|)^{-(n+1)}\mu(E(w,r))^{p/q} dm(w)\Big\}^{q/p} \\ &\leq C\Big\{\int_B |f(w)|^p (1-|w|)^{\alpha+\beta p} dm(w)\Big\}^{\frac{q}{p}}.\end{aligned} \tag{1.3}$$

This implies

$$\Big\{\int_B |f^{[\beta]}(z)|d\mu(z)\Big\}^{1/q} \leq C\|f^{[\beta]}\|_{L^p(dm_{\alpha+\beta p})}. \tag{1.4}$$

Now Lemma 1.2 gives the estimate (1.1).

To prove the necessity, for $a \in B$ we take

$$g(z) = (1-\langle z,a\rangle)^{-k}, \quad f(z) = g_{[\beta]}(z), \tag{1.5}$$

where $k > 0$ is large enough. By Proposition 1.4.10 in [9] we get

$$\|f\|_{L^p(dm_\alpha)} \simeq \|g\|_{L^p(dm_{\alpha+\beta p})} \simeq (1-|a|)^{-(kp-(n+1+\alpha+\beta p))/p}.$$

Then the hypothesis implies

$$(1-|a|)^{-(kp-(n+1+\alpha+\beta p))/p} \geq C\Big\{\int_B |f^{[\beta]}(z)|^q d\mu(z)\Big\}^{1/q}$$
$$\geq C \max_{z\in E(a,r)} |g(z)|\mu(E(a,r))^{1/q} \geq C(1-|a|)^{-k}\mu(E(a,r))^{1/q}.$$

This, together with Lemma 1.1, tells us that μ is a Carleson measure of order $(n+1+\alpha)q/p+\beta p$. The proof is completed.

Theorem 1.2. *Let μ be a finite positive Borel measure on B; suppose $2\leq p\leq q<\infty$, or $0<p<q<\infty$, and $\beta\geq 0$. Then in order that*

$$\Big\{\int_B |f^{[\beta]}(z)|^q d\mu(z)\Big\}^{1/q} \leq C\|f\|_{H^p}, \quad \text{for } \ f\in H(B), \tag{1.6}$$

it is necessary and sufficient that μ is a Carleson measure of order $nq/p+\beta q$.

Proof. The case that $p=q$ and $\beta=0$ has been considered by Cima, Wogen[3] and Luecking[6]. We suppose $p<q$ or $\beta>0$. Then $nq/p+\beta q>n$. In this case we know from Lemma 1.1 that we need to prove that (1.6) is valid if and only if

$$\mu(E(z,r)) \leq C(1-|z|)^{nq/p+\beta q} \quad \text{for } \ z\in B. \tag{1.7}$$

The proof of the sufficiency will be divided into two steps.

Case 1. $p=q\geq 2$. Notice that $n+\beta p=n+1+(\beta p-1)$. By Theorem 1.1 we have

$$\Big\{\int_B |f^{[\beta]}(z)|^p d\mu(z)\Big\}^{1/p} \leq C\|f^{[\beta]}\|_{L^p(dm_{\beta p-1})} \quad \text{for } f\in H(B). \tag{1.8}$$

Since $(f^{[\beta]})^{[1]}=(f^{[1]})^{[\beta]}$, Lemma 1.2 gives

$$\|f^{[\beta]}\|_{L^p(dm_{\beta p-1})} \simeq \|(f^{[\beta]})^{[1]}\|_{L^p(dm_{\beta p-1+p})} \simeq \|f^{[1]}\|_{L^p(dm_{p-1})}. \tag{1.9}$$

Meanwhile, from Theorem 2.5 in [1] we get

$$\|f^{[1]}\|_{L^p(dm_{p-1})} \leq C\|f\|_{H^p}. \tag{1.10}$$

Combine (1.8), (1.9) and (1.10) to obtain (1.6).

Case 2. $0<p<q<\infty$. For $nq/p+\beta q=(n+1)+(nq(1/p-1/q)-1+\beta q)$ we see, by Theorem 1.1 and Lemma 1.1 again, that for all $f\in H(B)$,

$$\Big\{\int_B |f^{[\beta]}(z)|^p d\mu(z)\Big\}^{1/q} \leq C\|f^{[\beta]}\|_{L^q(dm_{nq(1/p+1/q)-1+\beta q})} \simeq C\|f\|_{L^q(dm_{nq(1/p-1/q)-1})}.$$

Now Theorem 1.4 in [8] implies (1.6) at once.

That (1.6) implies (1.7) can be proved by applying (1.6) to the function (1.5). The only thing we have to do is to estimate the H^p norm $\|f\|_{H^p}$. If $p\geq 1$, Theorem 1.1 in [10] and Proposition 1.4.10 in [9] give

$$M_p(f,r) \leq Cr^{-\beta}\int_0^r (r-\rho)^{\beta-1}M_p(g,\rho)d\rho$$
$$\leq C\int_0^1 (1-\rho)^{\beta-1}(1-\rho|a|)^{-(k-n/p)}d\rho \leq C(1-|a|)^{-(k-n/p)+\beta}.$$

Similarly, if $0<p<1$, then

$$M_p^p(f,r) \leq C\int_0^r (1-\rho)^{\beta p-1}M_p^p(g,r\rho)d\rho$$
$$\leq C\int_0^1 (1-\rho)^{\beta p-1}(1-\rho|a|)^{-(kp-n)}d\rho \leq C(1-|a|)^{-(kp-n)+\beta p}.$$

Therefore, for $0 < p < \infty$ we have $\|f\|_{H^p} \le C(1-|a|)^{-(k-n/p)+\beta}$. From this we obtain (1.7) as that in the proof of Theorm 1.1. The theorem is proved.

For $f \in H(B)$, let Rf be the radial derivative of f,

$$Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}, \quad \text{and} \quad R^m f = R(R^{m-1} f),\ m \in N.$$

As an application of Theorems 1.1 and 1.2 we have the following two theorems.

Theorem 1.3. *Let μ, p, q, α be as in Theorem* 1.1 *and m be a positive integer. Then in order that*

$$\Big\{ \int_B |R^m f(z)|^q d\mu(z) \Big\}^{1/q} \le C\|f\|_{L^p(dm_\alpha)} \quad \textit{for} \quad f \in H(B),$$

it is necessary and sufficient that μ is a Carleson measure of order $(n+1+\alpha)q/p + mq$.

Theorem 1.4. *Let μ, p, q be as in Theorem* 1.2 *and m be a positive integer. Then in order that*

$$\Big\{ \int_B |R^m f(z)|^q d\mu(z) \Big\}^{1/q} \le C\|f\|_{H^p} \quad \textit{for} \quad f \in H(B),$$

it is necessary and sufficient that μ is a Carleson measure of order $nq/p + mq$.

The "if" part of both Theorems 1.3 and 1.4 follow from Theorems 1.1, 1.2 and the fact that

$$R^m f = f^{[m]} + c_1 f^{[m-1]} + \cdots + c_{m-1} f^{[1]} + c_m f,$$

where $c_1, c_2, \cdots, c_m$ are constants independent of f. To prove the necessity one can choose function $f(z) = (1 - \langle z, a\rangle)^{-k}$ for $a \in B, k$ a positive integer and large enough. For this function it is obvious that

$$R^m f(z) = \sum_{j=0}^{m} s_j (1 - \langle z, a\rangle)^{-(k+j)}, \quad \text{for} \quad s_j \in \mathbb{C}, \quad j = 0, 1, 2, \cdots, m.$$

§2. Generalization

Let Ω be an irreducible bounded symmetric domain in $\mathbb{C}^n$ in its standard realization (see [4]). The definition of the fractional derivative works well on Ω. Now a natural question is how to generalize Theorems 1.1 and 1.2 to such a domain. What we can solve at present is only in the case when $\beta = 0$ in Theorem 1.1 and our next theorem is also a generalization of [4, Theorem 8] and [11, Theorem 7]. To express the result, we need some more notation. It is well known that Ω is uniquely determined (up to a biholomorphic mapping) by three analytic invariants r, a and b. Set $N = a(r-1) + b + 2$ (see [5], where the letter p is used instead of N). The importance of the constant N is that there exists a polynormal $h(z, w)$ in z and $\bar{w}(z, w \in \mathbb{C}^n)$ such that the Bergman kernel $K(z, w)$ of Ω is given by $K(z, w) = \frac{1}{h(z,w)^N},\ z, w \in \Omega$. When $\Omega = B$, we have $N = n+1$ and $h(z, w) = 1 - \langle z, w\rangle$. For $\alpha \in \mathbb{R}$, define $dm_\alpha = h(z, z)^\alpha dm(z)$. It is well-known that $dm_\alpha(z)$ is a finite measure on Ω if and only if $\alpha > -1$. Now for $\alpha > -1$ and $0 < p < \infty$ we let $L_a^p(\Omega, dm_\alpha)$ denote the weighted Bergman space of holomorphic functions f in Ω such that

$$\|f\|_{L^p(\Omega, dm_\alpha)} = \Big\{ \int_\Omega |f(z)|^p dm_\alpha(z) \Big\}^{1/p} < \infty.$$

Theorem 2.1. *Let μ be a finite positive Borel measure on Ω, and suppose $0 < p \leq q < \infty$, $\alpha > -1$. Then in order that*

$$\Big\{ \int_\Omega |f(z)|^q d\mu(z) \Big\}^{1/q} \leq C \|f\|_{L^p(\Omega, dm_\alpha)} \quad \text{for } f \text{ holomorphic in } \Omega, \tag{2.1}$$

it is necessary and sufficient that for some (or any) $r > 0$ there exists a positive constant C such that

$$\mu(E(z,r)) \leq C m_\alpha(E(z,r))^{q/p}, \quad z \in B. \tag{2.2}$$

Proof. For f holomorphic in Ω, by the estimate in [11, pp. 332–333] we get

$$|f(z)|^p \leq C|E(z,r)|^{-1} \int_{E(z,r)} |f(w)|^p dm(w) \leq Ch(z,z)^{-N} \int_{E(z,r)} |f(w)|^p dm(w). \tag{2.3}$$

Suppose that μ satisfies (2.2) for some $r > 0$. Comparing (2.3) with (1.2) we can easily modify the approach in the proof of Theorem 1.1 to obtain (2.1).

The converse is obtained by applying (2.1) to the holomorphic function

$$f(z) = h(z,a)^{-k},$$

where k is a positive integer and large enough ($h(z,w) \neq 0$ for all $z, w \in \Omega$, hence f is holomorphic). Instead of using Proposition 1.4.10 in [9], we apply [5, Theorem 4.1] to obtain

$$\|f\|_{L^p(\Omega, dm_\alpha)} \leq Ch(a,a)^{-(pk-N-\alpha)/p}. \tag{2.4}$$

For any $r > 0$, now (2.2) follows from (2.4) and the fact that

$$\min_{z \in E(a,r)} |f(z,a)| \geq C|f(a,a)|.$$

The proof is completed.

Remark 2.1. Only for the simplicity's sake Ω is supposed to be irreducible. Theorem 2.1 remains true on any bounded symmetric domain.

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