Chin. Ann. of Math.
19B: 3(1998),305-310.

D_∞-APPROXIMATION OF QUADRATIC VARIATIONS OF SMOOTH ITÔ PROCESSES***

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Abstract

The purpose of this paper is to prove that the quadratic variations of smooth Itô process in the sense of Malliavin-Nualart can be approximated in Sobolev spaces over the Wiener space by its discrete quadratic variations.

Keywords D_∞-approximation, Quadratic variations, Smooth Itô Processes
1991 MR Subject Classification 60H07, 60G44
Chinese Library Classification O211.6

§1. Introduction

In [5], Malliavin and Nualart defined the notion of (real) smooth martingales and we then proved in [6] that the process of quasi sure quadratic variation of a smooth martingale can be approximated by its processes of discrete quadratic variation quasi surely.

On the other hand, we know the following famous result of Millar in the classical theory of martingales: If $p \geq 2$, then for any continuous L^p-martingale, its process of quadratic variation can be obtained as the $L^{p/2}$-limit of its discrete quadratic variation as the meshes of divisions tend to zero. So it is natural to ask if the convergence takes place in the intersection of all Sobolev spaces over Wiener space when we consider smooth martingales instead of L^p-martingales. It is the very aim of the present paper to give an affirmative answer to this question. Compared with the case of quasi-sure convergence, the difficulty here is in that we must show that the approximating series converges in every Sobolev space whereas it suffices to prove its boundedness in every Sobolev space to establish the quasi sure convergence. Our main result is stated in Section 2. In Section 3 we shall consider manifold-valued case and extend our previous results in [6].

Now we explain basic notions and notations we shall use. Our fundamental probability space will be the classical Wiener space (X, H, μ) of d-dimensional Brownian motion, i.e., X is the space of continuous maps from $[0,1]$ to $\mathbf{R}^d$, null at zero; H is the Cameron-Martin space and μ the standard Wiener measure. $D_{p,2r}$ stands for the $(p, 2r)$-Sobolev space over X and D_∞ their projective limit. In $D_{p,2r}$ we have two equivalent norms (Meyer's equivalence):

$$\|F\|_{p,2r} = \|(I-L)^r F\|_p, \quad \|F\|_{p,2r} = \|\nabla^{2r} F\|_p + \|F\|_p.$$

Manuscript received November 28, 1995. Revised November 4, 1996.

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***Project supported by the National Natural Science Foundation of China (19631030).

Here L is the Ornstein-Uhlenbeck operator, ∇ is the gradient operator and we will use δ to denote the divergence operator (cf., e.g., [3]). The $(p,2r)$-capacity is then a set function on X defined by $C_{p,2r}(O)=\inf\{\|F\|_{p,2r}; F\geq 0, F(x)\geq 1, \mu\text{-a.e on } O\}$ for any open set O and $C_{p,2r}(A)=\inf\{C_{p,2r}(O); A\subset O\}$ for an arbitrary set $A\subset X$.

Throughout the paper C will denote a constant whose value is independent of n but may be different from one expression to another.

§2. Real Smooth Itô Processes

In order to be able to define smoothness in the manifold-valued case, we need to consider real smooth Itô processes instead of smooth martingales. First we give the definition.

Let $\{w(\cdot)\}$ be the canonical realization of the d-dimensional Brownian motion on (X,H,μ) and we denote by $\mathcal{F}_t$ the σ-algebra generated by the paths of $w(\cdot)$ up to time t. Then a real Itô process is a semimartingale represented in the form

$$x_t=\sum_{j=1}^{d}\int_0^t a_s^j dw_s^j+\int_0^t b_s ds, \quad 0\leq t\leq 1. \tag{2.1}$$

Definition 2.1. *An Itô process $\{x(\cdot)\}$ represented as (2.1) is called smooth if*

$$\sum_{j=1}^{d}\int_0^1 \|a_s^j\|_{p,2r}^p ds+\int_0^1 \|b_s\|_{p,2r}^p ds<\infty, \quad \forall p>1, r>1. \tag{2.2}$$

It is a classical result that the process of quadratic variation of $\{x(\cdot)\}$ is given by

$$[x]_t=\sum_{j=1}^{d}\int_0^t (a_s^j)^2 ds, \quad 0\leq t\leq 1. \tag{2.3}$$

We shall denote $\eta_i^n=2^{-n}i$ and put $t_i^n=t\wedge\eta_i^n$. The main result of this paper is the following:

Theorem 2.1. *Suppose that $x(\cdot)$ is a smooth Itô process represented as (2.1). Put $S_n(t)=\sum\limits_{i=0}^{2^n-1}(x(t_{i+1}^n)-x(t_i^n))^2$. Then $S_n(\cdot)\to[x](\cdot)$ in D_∞.*

Proof. We shall in fact prove a stronger result:

$$\sup_{t\in[0,1]}\|S_n(t)-[x](t)\|_{p,2r}\longrightarrow 0, \quad \forall p>1,\ r\geq 0. \tag{2.4}$$

Since (2.4) is trivial for $r=0$, we only need to look at the case $r\geq 1$. First, for $r=1$ we have

$$\begin{aligned} L\Big(\sum_{i=0}^{2^n-1}(x(t_{i+1}^n)-x(t_i^n))^2\Big)&=\sum_{i=0}^{2^n-1}2(x(t_{i+1}^n)-x(t_i^n))(Lx(t_{i+1}^n)-Lx(t_i^n))\\ &\quad+\sum_{i=0}^{2^n-1}2\|\nabla(x(t_{i+1}^n)-x(t_i^n))\|_H^2:=I_1+I_2. \end{aligned} \tag{2.5}$$

Using Stroock's commutation relation $\delta(L-I)=L\delta$ and standard argument in Itô calculus we easily see that

$$\sup_{t\in[0,1]}\Big\|I_1(t)-2\sum_{j=1}^{d}\int_0^t a_s^j(La_s^j-a_s^j)ds\Big\|_p\longrightarrow 0. \tag{2.6}$$

For I_2 we can again use a formula of Stroock, i.e., formula (7.9) in [7] to obtain

$$\begin{aligned}I_2(t) &= 4\sum_{j=1}^{d}\sum_{i=0}^{2^n-1}\int_{t_i^n}^{t_{i+1}^n}\langle\nabla(x(s)-x(t_i^n)),\nabla a_s^j\rangle dw_s^j\\&\quad+4\sum_{j=1}^{d}\sum_{i=0}^{2^n-1}\int_{t_i^n}^{t_{i+1}^n}\langle\nabla(x(s)-x(t_i^n)),\nabla b_s\rangle ds\\&\quad+2\sum_{j=1}^{d}\sum_{i=0}^{2^n-1}\int_{t_i^n}^{t_{i+1}^n}\|\nabla a_s^j\|^2ds+2\sum_{j=1}^{d}\sum_{i=0}^{2^n-1}\int_{t_i^n}^{t_{i+1}^n}|a_s^j|^2ds=\sum_{k=1}^{4}I_{2k}.\end{aligned}\tag{2.7}$$

Applying Burkholder's inequalities for (both discrete and continuous) martingales we have

$$\begin{aligned}\|I_{21}(t)\|_p^p &= CE\Big|\sum_{j=1}^{d}\sum_{i=0}^{2^n-1}\int_{t_i^n}^{t_{i+1}^n}\langle\nabla(x(s)-x(t_i^n)),\nabla a_s^j\rangle dw_s^j\Big|^p\\&\le CE\Big|\sum_{j=1}^{d}\sum_{i=0}^{2^n-1}\Big(\int_{t_i^n}^{t_{i+1}^n}<\nabla(x(s)-x(t_i^n)),\nabla a_s^j\rangle dw_s^j\Big)^2\Big|^{p/2}\\&\le C2^{\frac{n(p-2)}{2}}E\sum_{j=1}^{d}\sum_{i=0}^{2^n-1}\Big|\int_{t_i^n}^{t_{i+1}^n}\langle\nabla(x(s)-x(t_i^n)),\nabla a_s^j\rangle dw_s^j\Big|^p\\&\le C2^{\frac{n(p-2)}{2}}E\sum_{j=1}^{d}\sum_{i=0}^{2^n-1}\Big(\int_{t_i^n}^{t_{i+1}^n}\langle\nabla(x(s)-x(t_i^n)),\nabla a_s^j\rangle^2ds\Big)^{p/2}\\&\le CE\sum_{j=1}^{d}\sum_{i=0}^{2^n-1}\int_{t_i^n}^{t_{i+1}^n}\langle\nabla(x(s)-x(t_i^n)),\nabla a_s^j\rangle^pds\\&\le C\Big(\sum_{j=1}^{d}\sum_{i=0}^{2^n-1}\Big(\int_{t_i^n}^{t_{i+1}^n}E(\|\nabla(x(s)-x(t_i^n))\|^p\|\nabla a_s^j\|^p\Big)ds\Big)\\&\le C\sum_{j=1}^{d}\sum_{i=0}^{2^n-1}\int_{t_i^n}^{t_{i+1}^n}(E\|\nabla(x(s)-x(t_i^n))\|^{2p})^{\frac12}(E\|\nabla a_s^j\|^{2p})^{\frac12}ds\\&\le C\sum_{j=1}^{d}\sum_{i=0}^{2^n-1}\int_{\eta_i^n}^{\eta_{i+1}^n}(E\|\nabla(x(s)-x(\eta_i^n))\|^{2p})^{\frac12}(E\|\nabla a_s^j\|^{2p})^{\frac12}ds.\end{aligned}\tag{2.8}$$

But, since $\nabla\delta u=\delta\nabla u+u$, we have

$$\nabla(x(s)-x(\eta_i^n))=\sum_{j=1}^{d}\Big(\int_{\eta_i^n}^{s}\nabla a_u^jdw_u^j+\int_{\eta_i^n}^{s\wedge\cdot}a_u^jdu\Big)+\int_{\eta_i^n}^{s}\nabla b_sds.$$

Hence we have

$$\begin{aligned}&\sup_i\sup_{s\in[\eta_i^n,\eta_{i+1}^n]}E\|\nabla(x(s)-x(t_i^n))\|^{2p}\\&\le\sum_{j=1}^{d}\Big(2^{-n(p-1)}\int_0^1(\|\nabla a_u^j\|)^{2p}du+2^{-n(2p-1)}\int_0^1\|a_u^j\|^{2p}du\Big)\\&\quad+2^{-n(2p-1)}\int_0^1\|\nabla b_s\|^{2p}ds\to0,\qquad n\to\infty.\end{aligned}$$

Consequently

$$\sup_t \|I_{21}(t)\|_p \to 0, \qquad n \to \infty. \tag{2.9}$$

Similarly

$$\sup_t \|I_{22}(t)\|_p \to 0, \qquad n \to \infty. \tag{2.10}$$

Finally, since it is trivial that

$$\sup_t \Big\| I_{23}(t) + I_{24}(t) - 2\sum_{j=1}^d \int_0^t (\|\nabla a_s^j\| + (a_s^j)^2) ds \Big\| \to 0, \tag{2.11}$$

combining (2.6), (2.9), (2.10), (2.11) we arrive at

$$\sup_{\tau\in[0,1]} \Big\| L\Big(\sum_{i=0}^{2^n-1} \|x(t_{i+1}^n) - x(t_i^n)\|^2 \Big) - \Big(\int_0^t 2a_s^j L a_s^j + 2\|\nabla a_s^j\|^2 \Big) \Big\|_p \to 0$$

in L^p for any $p > 1$. Since $L[x](t) = 2\int_0^t (a_s^j L a_s^j + \|a_s^j\|^2) ds$, (2.4) is proved for $r = 1$.

Now we turn to the case $r = 2$. Making use of the relation $L\nabla = \nabla L - \nabla$ we easily see that $L^2 f^2 = 2(Lf)^2 + 2fL^2 f + 4\langle \nabla f, \nabla Lf\rangle + 2L\|\nabla f\|^2$. Hence

$$\begin{aligned}
& L^2\Big(\sum_{i=0}^{2^n-1} (x(t_{i+1}^n) - x(t_i^n))^2 \Big) \\
&= 2\sum_{i=0}^{2^n-1} \Big\{ (Lx(t_{i+1}^n) - Lx(t_i^n))^2 + 2\sum_{i=0}^{2^n-1} (x(t_{i+1}^n) - x(t_i^n))(L^2(x(t_{i+1}^n) - x(t_i^n))) \\
&\quad + 4\sum_{i=0}^{2^n-1} \langle \nabla(x(t_{i+1}^n) - x(t_i^n)), \nabla L(x(t_{i+1}^n) - x(t_i^n))\rangle \\
&\quad + 2L\sum_{i=0}^{2^n-1} \|\nabla(x(t_{i+1}^n) - x(t_i^n))\|^2 \Big\} = \sum_{k=1}^4 I_{3k}.
\end{aligned} \tag{2.12}$$

It is now trivial that

$$\sup_{t\in[0,1]} \Big\| I_{31}(t) - 2\sum_{j=1}^d \int_0^t (La_s^j - a_s^j)^2 ds \Big\|_p \to 0. \tag{2.13}$$

$$\sup_{t\in[0,1]} \Big\| I_{32}(t) - 2\sum_{j=1}^d \int_0^t (a_s^j (L^2 a_s^j - 2La_s^j + a_s^j) ds \Big\|_p \to 0. \tag{2.14}$$

By polarization we can use the result for I_2 to deduce that

$$\sup_{t\in[0,1]} \Big\| I_{33}(t) - 4\sum_{j=1}^d \int_0^t \big\{ \langle \nabla L a_s^j, \nabla a_s^j\rangle_H + a_s^j L a_s^j - \|\nabla a_s^j\|_H^2 - (a_s^j)^2 \big\} ds \Big\|_p \to 0. \tag{2.15}$$

Finally, noting that $I_{34} = LI_2$ we can use the expression (2.7) and the same argument for I_2 to establish

$$\sup_{t\in[0,1]} \Big\| I_{34}(t) - 2\sum_{j=1}^d \int_0^t L(\|\nabla a_s^j\|^2 + \|a_s^j\|^2) ds \Big\|_p \to 0. \tag{2.16}$$

Combining (2.12)–(2.16) and noting that $Lf^2 = 2fLf + 2\|\nabla f\|^2$ gives the desired result.

Doing the same calculation for higher order derivatives, we can finally achieve the proof of the theorem.

Remark. More careful calculation can give the speed of the convergence which implies the quasi sure convergence of the series. Hence the above result can in fact be strengthened to cover [6]. We do not give the details.

§3. Manifold-Valued Smooth Itô Processes

The result just established above combined with that in [6] can be considered as a complete study of process of quadratic variation of Euclidean space valued smooth Itô processes. In this section we will define manifold valued smooth Itô processes and state a generalization of the result in [6].

Suppose that M is a compact manifold. We give

Definition 3.1. *Let $\{x_t, t \in [0,1]\}$ be an M-valued process. If for every $f \in C^\infty(M)$ $\{f(x.)\}$ is a real smooth Itô process, then one says that $\{x.\}$ is a smooth Itô process.*

By imbedding M into a Euclidean space, we see easily that any smooth Itô process admits an ∞-modification (cf. [5]). Moreover, we have

Proposition 3.1. *$\{x(\cdot)\}$ is a smooth Itô process if and only if there exist an integer l, an R^l-valued smooth Itô process $\{u_t = (u_t^1, \cdots, u_t^l)\}$ and vector fields X_j $(1 \le j \le l)$ such that $\{x(\cdot)\}$ solves SDE*

$$dx_t = \sum_{j=1}^{l} X_j(x_t) \circ du_t^j, \tag{3.1}$$

where $\circ$ designs the Stratonovich integral.

Proof. "if" part: Suppose (3.1) is fulfilled. Let $f \in M$. When M is Euclidean, the result is obvious by a simple application of Picard's iteration plus the invariance of D_∞ under composition by smooth functions null outside of compacts. We then proceed to general manifold. By definition of SDE on manifold, for any $f \in C^\infty$, writing $y_t = f(x_t)$, we have

$$dy_t = \sum_{j=1}^{l} (Xf)(x_t) \circ du_t^j. \tag{3.2}$$

Without any loss of generality we can assume that f is bijective, since otherwise we can first imbed M into a Euclidean space and then use the invariance of D_∞. But this time we can rewrite (3.1) as

$$dy_t = \sum_{j=1}^{l} Y_j(y_t) \circ du_t^j, \tag{3.3}$$

where $Y_j = df(X_j)$; we can therefore use the just established result for Euclidean spaces.

"only if" part: This part has been essentially proved by [2]. But we feel that his proof needs slight modification. In fact, one can take, for example, $l = 2m+1$, F the Whitney imbedding of M into R^l, $u_t = F(x_t)$ and

$$X_j(m) = dF_m^{-1}(\pi'_{F(m)}(e_j)),$$

where $\{e_j, j = 1, \cdots, l\}$ is the standard ONB of R^l, $\pi'_{F(m)}$ the orthogonal projection of $R^l_{F(m)} (= R^l)$ to $F(M)_{F(m)}$.

Now suppose $\{x_t\}$ and $\{y_t\}$ are two smooth Itô processes solving respectively

$$dx_t = \sum_{j=1}^{l} X_j(x_t) \circ du_t^j, \quad dy_t = \sum_{k=1}^{l'} Y_j(y_t) \circ du_t^k.$$

The following definition is taken from [2]

Definition 3.2. *The mutual quadratic variation is the* $(T^*M \times T^*M)^*$*-valued process* V *such that for* $\theta \in T^*M \times T^*M$

$$V_{x.,y.}(\theta) = \sum_{j=1}^{l}\sum_{k=1}^{l'} \int_0^t \theta(X_j, Y_k)(x_s, y_s) d[u^j, v^k]_s,$$

where TM, T^*M *are tangent bundle and cotangent bundle respectively.*

It is then easily seen that $V(\theta)$ has an ∞-modification. Suppose that $e_m \in C^\infty(M, M_m)$ satisfying the following conditions:

(1) $e_x(x) = 0$, (2) $\forall f \in C^\infty$, $f(m') - f(m) - e_m(m')f = o(|m' - m|)$,
where $|\cdot|$ is a (so any) local Euclidean metric.

Then writing $\triangle_i x. = e_{x(s_i^n)}(x(t_{i+1}^n)) - e_{x(s_i^n)}(x(t_i^n))$, we have

Theorem 3.1. $\lim\limits_{n\to\infty} \sum\limits_{i=0}^{n} \theta(\triangle_i x., \triangle_i y.) \to V_{x.,y.}(\theta)$, q.s.

Proof. Whitney's imbedding plus results on R^d-valued smooth Itô process[6].

As an example of applications of Proposition 3.1 and Theorem 3.1 we consider the quasi sure Riemannian quadratic variation of Riemannian Brownian motion.

Suppose that M is a compact Riemannian manifold with Levi-Civita connection and $x(t)$ is the Brownian motion on M. By [1] x is a martingale with Riemannian quadratic variation $\int \theta(dx, dx) = \int \mathrm{Tr}\,\theta(x)dt$. On the other hand, according to [4] x is part of the solution of a (horizontal) stochastic differential equation, and hence is a smooth Itô process by Proposition 3.1. Combining these two facts and applying Theorem 3.1 gives

Corollary 3.1.

$$\lim_{n\to\infty} \sum_{i=0}^{n} \theta(\triangle_i x., \triangle_i x.) \to \int \mathrm{Tr}\,\theta(x), \text{q.s.}$$

Moreover, as already mentioned in [1], *we can take* e_x *in the definition of* $\triangle_i x.$ *in such a way that* $g(e_x, e_y) = d^2(x, y)$ *where* g *is the Riemannian metric.*

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