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A CLASS OF HOMOGENEOUS SEMISIMPLE SPACES***

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Abstract

The semisimple structure, which generalizes the complex and the paracomplex structures, is considered. The authors classify all the homogeneous semisimple spaces whose underlying spaces are $G/C(W)_0$, where G is a real simple Lie Group, $W \in \mathfrak{g}$,$C(W)_0$ is the identity component of the centralizer $C(W)$ of W in G.

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§1. Introduction

Consider a 1-1 tensor field I on a smooth manifold M. The natural generalization of the complex structure and para-complex structure is called the semisimple structure, which just forgets $I^2 = \pm 1$, precisely:

Definition 1.1. *I is called semisimple if*

(1) *$\exists$ a real polynomial $f(x)$, which has no multiple root, such that $f(I) = 0$,*

(2) *I satisfies the integrability condition:*

$$I^2[X,\ Y] - I[I\,X,\ Y] - I[X,\ I\,Y] + [I\,X,\ I\,Y] = 0, \quad \forall\ X, Y \in \mathcal{X}(M),$$

where $\mathcal{X}(M)$ is the set of all vector fields of M.

If $I^2 = \pm 1$, (2) coincides with the integrability condition of (para-)complex structure. So both complex structure and paracomplex structure are semisimple structures.

Proposition 1.1. *Given smooth manifold M, there exists a semisimple structure I if and only if $\mathcal{X}(M)^{\mathbb{C}} = \bigoplus_{a=1}^{m} \mathcal{X}_a$ as vector space over $\mathbb{C}$, such that*

(1) $[\mathcal{X}_a,\ \mathcal{X}_b] \subset \mathcal{X}_a + \mathcal{X}_b, \quad \forall\ 1 \le a, b \le m,$ (2) *$\forall\ a$, $\exists b$ such that $\overline{\mathcal{X}_a} = \mathcal{X}_b$,*
where $\mathcal{X}(M)^{\mathbb{C}}$ is the complexifocation of $\mathcal{X}(M)$.

Proof. Consider I as the endmorphism of $\mathcal{X}(M)^{\mathbb{C}}$. Since I is semisimple, we can get $\mathcal{X}(M)^{\mathbb{C}} = \bigoplus_{a=1}^{m} \mathcal{X}_a$ as the eigenspace decomposition of I, the rest is easy to check. Conversely,

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given such a decomposition: $\mathcal{X}(M)^{\mathbb{C}} = \bigoplus_{a=1}^{m} \mathcal{X}_a$, define I acting on $\mathcal{X}_a$ by scale λ_a, then I becomes a semisimple structure on M.

Remark 1.1. Once we get a semisimple structure I, by Proposition 1.1, we can get a class of semisimple structures by changing the eigenvalues of I (i.e, all $p(I)$, $\forall$ polynomial $p(x)$). To consider the geometry of M, it is essential to consider the decomposition of Proposition 1.1.

Definition 1.2. *If we collapse the decomposition* $\mathcal{X}(M)^{\mathbb{C}} = \bigoplus_{a=1}^{m} \mathcal{X}_a$ *to* $\mathcal{X}(M)^{\mathbb{C}} = \bigoplus_{a'=1}^{m'} \mathcal{X}'_a$ *by letting* $\mathcal{X}_{a'}$ *be the sum of some terms* $\mathcal{X}_a$, *so that the property* $\forall\ a'\ \exists\ b$ *such that* $\overline{\mathcal{X}_{a'}} = \mathcal{X}_{b'}$ *still holds, then we get a new semisimple structure on* M. *We say a semisimple structure is maximal if it can not be obtained by some collapsion of another semisimple structure.*

Since $m = 2$ is just the case of complex structure or (partly) paracomplex structure, which is more or less clear, here we only consider the case $m \geq 3$. To make the expression simpler, we define the semisimple manifold as:

Definition 1.3. *A semisimple manifold* M *is a smooth manifold* M *with a maximal semisimple structure* I *with* $m \geq 3$.

Definition 1.4. *The semisimple structure* I *over* G/U *is called homogeneous if* $d\tau(g)\circ I = I \circ d\tau(g)$, $\forall g \in G$, *where* $\tau(g)$ *is the left multiplication by* g.

Definition 1.5. *A homogeneous semisimple space* G/U *is the homogeneous space* G/U *with a maximal homogeneous semisimple structure* I *with* $m \geq 3$.

Remark 1.2. G/U is semisimple if and only if G/U_0 is semisimple, and $u{\cdot}I(X) = I(u{\cdot}X)$, $\forall\ u \in U, X \in T_U(G/U)$.

In this paper, we found all the homogeneous semisimple spaces whose underlying space is $G/C(W)_0$ where G is a real simple Lie Group, $W \in \mathfrak{g}$ is a semisimple element (Theorems 4.2, 4.3). Above Remark helps us to give all $G/C(W)$. For general $G/C(W)$, where G is semisimple and $C(W)$ is not connected, we have locally decomposition (Corollary 2.1).

§2. Homogeneous Semisimple Space $G/C(W)_0$

Consider the manifold $M = G/C(W)_0$, where G is a connected real semisimple Lie Group, W is a semisimple element of $\mathfrak{g} = \mathrm{Lie}\,(G)$, $C(W)$ is the centralizer of W in G and $U = C(W)_0$ its identity component of $G/C(W)$. Denote $\mathfrak{u} = \mathrm{Lie}\,(U)$. Then $T_U(G/U) \cong \mathfrak{g}/u$. Find a subspace $\mathfrak{p}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ and $[\mathfrak{u},\ \mathfrak{p}] \subset \mathfrak{p}$. Then $\mathfrak{p} \cong \mathfrak{g}/\mathfrak{u} \cong T_U(G/U)$. View I as the automorphism of $\mathfrak{p}$, define the associated Koszul operator J: $\mathfrak{g} \to \mathfrak{g}$ as $J|_{\mathfrak{u}} = 0$, $J|_{\mathfrak{p}} = I|_{\mathfrak{p}}$. We identify the complexifocation of J with J. Then

Proposition 2.1.[1] *Suppose* I *is a homogeneous semisimple structure on* G/U. *Then* I *satisfies the integrability condition* (*Definition* 1.1) *if and only if* J *satisfies*

$$J^2[X,\ Y] - J[JX,\ Y] - J[X,\ JY] + [JX,\ JY] \subset \mathfrak{u}, \quad \forall\ X, Y \in \mathfrak{g}. \tag{2.1}$$

Theorem 2.1. G/U *is a homogeneous semisimple space if and only if, as a complex vector space,* $\mathfrak{g}^{\mathbb{C}} = \sum_{a=1}^{m} \mathfrak{g}_a$ *satisfies:*

(1) $[\mathfrak{g}_a,\ \mathfrak{g}_b] \subset \mathfrak{g}_a + \mathfrak{g}_b, \quad 1 \le a, b \le m;$ (2) $\mathfrak{g}_a \cap \mathfrak{g}_b = \mathfrak{u}^{\mathbb{C}}, \quad 1 \le a, b \le m;$

(3) $\forall\ a\ \exists\ b$ *such that* $\overline{\mathfrak{g}_a} = \mathfrak{g}_b$.

Proof. "only if": By Remark 1.1, suppose I has eigenvalues $\lambda_1, \cdots, \lambda_m$ with no $\lambda_a = 0$. Then J has eigenvalues: $0, \lambda_1, \cdots, \lambda_m$ with corresponding eigenspaces: $\mathfrak{u}^{\mathbb{C}}, \mathfrak{p}_1, \cdots, \mathfrak{p}_m$. Let $X \in \mathfrak{p}_a, \quad Y \in \mathfrak{p}_b$ by equation (2.1). Then

$$(J - \lambda_a)(J - \lambda_b)[X,\ Y] \subset \mathfrak{u}^{\mathbb{C}} \Rightarrow [X,\ Y] \subset \mathfrak{u}^{\mathbb{C}} + \mathfrak{p}_a + \mathfrak{p}_b.$$

If let $X \in \mathfrak{u}^{\mathbb{C}}, \quad Y \in \mathfrak{p}_a$ by equation (2.1), then

$$J(J - \lambda_a)[X,\ Y] \subset \mathfrak{u}^{\mathbb{C}} \Rightarrow [X,\ Y] \subset \mathfrak{u}^{\mathbb{C}} + \mathfrak{p}_a.$$

Now, define $\mathfrak{g}_a = \mathfrak{u}^{\mathbb{C}} + \mathfrak{p}_a$. Then $\mathfrak{g}^{\mathbb{C}} = \mathfrak{u}^{\mathbb{C}} + \bigoplus_{a=1}^{m} \mathfrak{p}_a = \sum_{a=1}^{m} \mathfrak{g}_a$ satisfying (1),(2). (3) comes from the fact that J is real.

"if" Define $J|_{\mathfrak{p}^{\mathbb{C}} \cap \mathfrak{g}_a} = \lambda_a$, $1 \le a \le m$.

Lemma 2.1. *Suppose* $\mathfrak{l}$ *is an ideal of* $\mathfrak{g}$, *let* $\mathfrak{l}_a = \mathfrak{l}^{\mathbb{C}} \cap \mathfrak{g}_a, \quad 1 \le a \le m$. *Then* $\mathfrak{l}^{\mathbb{C}} = \sum_{a=1}^{m} \mathfrak{l}_a$ *satisfies:*

(1) $[\mathfrak{l}_a,\ \mathfrak{l}_b] \subset \mathfrak{l}_a + \mathfrak{l}_b, \quad 1 \le a, b \le m;$ (2) $\mathfrak{l}_a \cap \mathfrak{l}_b = (\mathfrak{l} \cap \mathfrak{u})^{\mathbb{C}}, \quad 1 \le a, b \le m;$

(3) $\forall\ a\ \exists\ b$ *such that* $\overline{\mathfrak{l}_a} = \mathfrak{l}_b$.

Proof. Since $\mathfrak{g}$ is semisimple, $\mathfrak{l}$ is an ideal and W is a semisimple element of $\mathfrak{g}$, it is possible to find an element Y such that $\mathfrak{l} = \{X \in \mathfrak{g} \mid [X,\ Y] = 0\}$ and $[Y,\ W] = 0$. Recall

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{u}^{\mathbb{C}} \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_m,$$
$$\forall X \in \mathfrak{l},\ X = X_0 + X_1 + \cdots + X_m.$$

$[Y,\ *]$ acts on both sides. Since $Y \in \mathfrak{u}$, $0 = [Y,\ X] = [Y,\ X_0] + [Y,\ X_1] + \cdots + [Y,\ X_m]$ is still a decomposition of type $\mathfrak{g}^{\mathbb{C}} = \mathfrak{u}^{\mathbb{C}} \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_m$. Then, $[Y,\ X_i] = 0, \quad X_i \in \mathfrak{l}$ hence. Since $\mathfrak{l}$ is a real subalgebra of $\mathfrak{g}$, (1), (2), (3) are easy to check. This Lemma also holds when $\mathfrak{l}$ is a Levi subgroup of $\mathfrak{g}$ and contains a Cantan subalgebra $\mathfrak{h}$ such that $W \in \mathfrak{h}$.

Corollary 2.1. *The homogeneous semisimple space* $G/C(W)$ *can be locally decomposed into the product of some generalized homogeneous semisimple spaces* $G^i/C(W^i)$ (*i.e* m *can be* $1, 2$ *or* ≥ 3), *where* G^i *is the simple factor of* G, *and* W^i *is the factor of* W *in every simple component* $\mathfrak{g}^i$.

Theorem 2.2. *If we identify semisimple homogeneous space* $G/C(W)_0$ *with* $(\mathfrak{g}, \mathfrak{u})$, *and if* $\mathfrak{g}$ *is simple real Lie algebra, then we have the following two cases:*

(1) $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^1 + \overline{\mathfrak{g}^1}$ *as Complex Lie algebra. Let* $\mathfrak{g}^1_a = \mathfrak{g}_a \cap \mathfrak{g}^1$. *Then*

(i) $\mathfrak{g}^1 = \sum_{a=1}^{m} \mathfrak{g}^1_a;$ (ii) $[\mathfrak{g}^1_a,\ \mathfrak{g}^1_b] \subset \mathfrak{g}^1_a + \mathfrak{g}^1_b, \quad \forall\ 1 \le a, b \le m;$

(iii) $\mathfrak{g}^1_a \cap \mathfrak{g}^1_b = \emptyset, \quad \forall\ 1 \le a \ne b \le m;$ (iv) $\mathfrak{g}^1_a \cap \overline{\mathfrak{g}^1_b} = \emptyset, \quad \forall\ 1 \le a, b \le m.$

(2) $\mathfrak{g}^{\mathbb{C}}$ *is simple. Then*

(i) $[\mathfrak{g}_a,\ \mathfrak{g}_b] \subset \mathfrak{g}_a + \mathfrak{g}_b, \quad 1 \le a, b \le m;$ (ii) $\mathfrak{g}_a \cap \mathfrak{g}_b = \mathfrak{u}^{\mathbb{C}}, \quad 1 \le a, b \le m;$

(iii) $\forall\ a\ \exists\ b$ *such that* $\overline{\mathfrak{g}_a} = \mathfrak{g}_b$.

Proof. (2) is trivial. For (1), since $\mathfrak{g}^1$ is a complex ideal of $\mathfrak{g}^{\mathbb{C}}$, similar proof of Lemma 2.1 shows (i), (ii) are true. (iii) is because $\mathfrak{g}^1 \cap \overline{\mathfrak{g}^1} = \emptyset$.

In the following sections, we will find all such pairs $(\mathfrak{g},\ \mathfrak{u})$.

§3. Semisimple Decomposition of Complex Simple Lie Algebras

Definition 3.1. *Given a complex simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$, a semisimple decomposition of $\mathfrak{g}^{\mathbb{C}}$ is a complex vector space decomposition:* $\mathfrak{g}^{\mathbb{C}} = \sum_{a=1}^{m} \mathfrak{g}_a$ *(with* $m \geq 3$*) satisfies:*

(1) $[\mathfrak{g}_a,\ \mathfrak{g}_b] \subset \mathfrak{g}_a + \mathfrak{g}_b,\quad 1 \leq a, b \leq m;$ (2) $\mathfrak{g}_a \cap \mathfrak{g}_b = \mathfrak{u}^{\mathbb{C}},\quad 1 \leq a, b \leq m.$

In this section, we will find all possible semisimple decompositions of $\mathfrak{g}^{\mathbb{C}}$. Let $\mathfrak{h}^{\mathbb{C}}$ be a Cartan subalgebra containing W, then $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{u}^{\mathbb{C}}$. Denote

$$\triangle\ = \triangle(\mathfrak{g}^{\mathbb{C}},\ \mathfrak{h}^{\mathbb{C}}),\quad \triangle_0 = \triangle(\mathfrak{u}^{\mathbb{C}},\ \mathfrak{h}^{\mathbb{C}}),\quad \triangle_a = \triangle(\mathfrak{p}_a,\ \mathfrak{h}^{\mathbb{C}}).$$

Theorem 3.1. *$\mathfrak{g}^{\mathbb{C}}$ has a semisimple decomposition if and only if* $\triangle = \triangle_0 \bigcup \Big(\bigcup_{a=1}^{m} \triangle_a \Big)$ *satisfies:*

(1) $\forall\ \alpha \in \triangle_0,\ \beta \in \triangle_a,\ \alpha + \beta \in \triangle \Rightarrow \alpha + \beta \in \triangle_a;$

(2) $\forall\ \alpha \in \triangle_a,\ \beta \in \triangle_b,\ \alpha + \beta \in \triangle \Rightarrow \alpha + \beta \in \triangle_0 \cup \triangle_a \cup \triangle_b;$

(3) $\forall\ \alpha \in \triangle_0,\ \beta \in \triangle_0,\ \alpha + \beta \in \triangle \Rightarrow \alpha + \beta \in \triangle_0;$

(4) $-\triangle_0 = \triangle_0.$

Proof. (3) and (4) are because $\mathfrak{u}^{\mathbb{C}}$ is a Levi subalgebra, (1) is because $\mathfrak{g}_a$ is a subalgebra, (2) is obtained by $[\mathfrak{g}_a,\ \mathfrak{g}_b] \subset \mathfrak{g}_a + \mathfrak{g}_b$.

Definition 3.2. *Define* $\triangle_{ab} = \{\alpha \in \triangle_a \mid -\alpha \in \triangle_b,\ \forall\ 1 \leq a, b \leq m\}.$

Theorem 3.2. *$\mathfrak{g}^{\mathbb{C}}$ has a semisimple decomposition if and only if* $\triangle = \triangle_0 \bigcup \big(\bigcup_{a,b=1}^{m} \triangle_{ab} \big)$ *with*

(1) $\alpha \in \triangle_{ab}, \beta \in \triangle_{cd}, \alpha + \beta \in \triangle \Rightarrow a = d$ *or* $b = c$ *or "*$a = c$ *and* $b = d$*";*

(2) $\alpha \in \triangle_{ab}, \beta \in \triangle_{bc}, \alpha + \beta \in \triangle \Rightarrow \alpha + \beta \in \triangle_{ac}\ \forall a \neq c;$

(3) $\alpha \in \triangle_0\ , \beta \in \triangle_{ab}, \alpha + \beta \in \triangle \Rightarrow \alpha + \beta \in \triangle_{ab};$

(4) $\alpha \in \triangle_{ab}, \beta \in \triangle_{ba}, \alpha + \beta \in \triangle \Rightarrow \alpha + \beta \in \triangle_0 \cup \triangle_{aa} \cup \triangle_{bb} \cup \triangle_{ab} \cup \triangle_{ba};$

(5) $\alpha \in \triangle_{ab}, \beta \in \triangle_{ab}, \alpha + \beta \in \triangle \Rightarrow \alpha + \beta \in \triangle_{ab};$

(6) $-\triangle_{ab} = \triangle_{ba}, -\triangle_0 = \triangle_0.$

Proof. "$\Longrightarrow$" (3) and (6) are trivial. Let us prove the following claim:

"if $\alpha \in \triangle_a, -\beta \in \triangle_b, a \neq b,$ and $\alpha + \beta \in \triangle$, then $\alpha + \beta \in \triangle_{ab}$."

Since $\alpha = (\alpha + \beta) - \beta,\ \alpha \in \triangle_a,\ -\beta \in \triangle_b,$ and $a \neq b$ by Theorem 3.1, it forces $\alpha + \beta \in \triangle_a$; symmetrically, $-(\alpha + \beta) \in \triangle_b$; hence $\alpha + \beta \in \triangle_{ab}$.

By this claim, (2) and (5) hold. For (1), if $a \neq d$ and $b \neq c$, we get $\alpha + \beta \in \triangle_{ad} \cap \triangle_{cb} \neq \emptyset$; this forces $a = c$ and $b = d$. (4) can be obtained directly by Theorem 3.1.

"$\Longleftarrow$" Since $\triangle_a = \bigcup_{b=1}^{m} (\triangle_{ab})$, calculate directly.

Corollary 3.1. *In Theorem* 3.2*, define* $\triangle_0' = \triangle_0 \bigcup \big(\bigcup_{a=1}^{m} \triangle_{aa} \big)$*. Then*

$$\triangle = \triangle_0' \bigcup \Big(\bigcup_{1 \leq a \neq b \leq m} \triangle_{ab} \Big)$$

still satisfies the conditions of Theorem 3.2*.*

Definition 3.3. *If* $\bigcup_{a=1}^{m} \triangle_{aa} = \emptyset$*, we say the decomposition* $\triangle = \triangle_0 \bigcup \big(\bigcup_{a \neq b} \triangle_{ab} \big)$ *is reduced.*

Theorem 3.3. *For type* A_l, $\mathfrak{g}^{\mathbb{C}} = sl(l+1,\mathbb{C})$, $\triangle = \{e_i - e_j \mid 1 \le i \ne j \le l+1\}$. *For any reduced semisimple decomposition* $\triangle = \triangle_0 \bigcup \big(\bigcup\limits_{a\ne b} \triangle_{ab}\big)$ *up to an isomorphism of the root system* $\triangle$, *there exists a partition of* $N \overset{\text{def.}}{=} \{1,2,\cdots,l+1\}$: $N = \bigcup\limits_{a=1}^{m} N_a$ *such that*

$$\triangle_{ab} = \{e_i - e_j \mid i \in N_a,\ j \in N_b\}, \quad \forall\ 1 \le a \ne b \le l+1,$$
$$\triangle_0 \ = \bigcup_{a=1}^{m} \{e_i - e_j \mid i,j \in N_a,\ i \ne j\}.$$

Proof. Fix $e_{i_0} - e_{j_0} \in \triangle_{a_0 b_0}$ $a_0 \ne b_0$. Since $e_{i_0} - e_1, \cdots, e_{i_0} - e_{l+1}$ generate $\triangle$, and $m \ge 3$, it follows that $\exists\ k_0$ such that $e_{i_0} - e_{k_0} \notin \triangle_0 \cup \triangle_{a_0 b_0} \cup \triangle_{b_0 a_0}$. Suppose $e_{i_0} - e_{k_0} \in \triangle_{c_0 d_0}$. Since $(e_{i_0} - e_{j_0}) + (e_{k_0} - e_{i_0}) \in \triangle$, by Theorem 3.2, we have $c_0 = a_0$ or $b_0 = d_0$. Choose $a_0 = c_0$ (if $b_0 = d_0$, change all e_i into $-e_i$), of course $b_0 \ne d_0$.

Claim. $e_{i_0} - e_i \in \triangle_0 \cup \triangle_{a_0}$, $\quad \forall\ i \ne i_0$.

Proof of the Claim. When $i = j_0$ or k_0, it is true. If $i \ne j_0$ and k_0, since $(e_{i_0} - e_i) + (e_{j_0} - e_{i_0}) \in \triangle$ and $(e_{i_0} - e_i) + (e_{k_0} - e_{i_0}) \in \triangle$, by Theorem 3.2, we get $e_{i_0} - e_i \in \triangle_0 \cup \triangle_{a_0}$.

By the claim, we can define $N_{a_0} = \{i \in N \mid (e_{i_0} - e_i) \in \triangle_0\}$, $N_a = \{i \in N \mid (e_{i_0} - e_i) \in \triangle_{a_0 a}\}$, $1 \le a \ne a_0 \le m$. Now, $N = \bigcup\limits_{a=1}^{m} N_a$, $\forall\ a \ne a_0,\ b \ne b_0$ and $a \ne b$.

$$e_i - e_j = (e_i - e_{i_0}) + (e_{i_0} - e_j) \in \triangle_{a a_0} + \triangle_{a_0 b} \subset \triangle_{ab}, \quad \forall\ i \in N_a,\ j \in N_b;$$
$$e_i - e_k = (e_i - e_{i_0}) + (e_{i_0} - e_k) \in \triangle_{a a_0} + \triangle_{a_0 a} \subset \triangle_0 \cup \triangle_{a a_0} \cup \triangle_{a_0 a};$$
$$e_i - e_k = (e_i - e_{j_0}) + (e_{j_0} - e_k) \in \triangle_{a b_0} + \triangle_{b_0 a} \subset \triangle_0 \cup \triangle_{a b_0} \cup \triangle_{b_0 a}.$$

Therefore, $e_i - e_k \in \triangle_0$, where $k \ne i,\ k,i \in \triangle_a$. Hence,

$$\triangle_{ab} = \{e_i - e_j \mid i \in N_a,\ j \in N_b\}, \quad \forall\ 1 \le a \ne b \le l+1,$$
$$\triangle_0 \ = \bigcup_{a=1}^{m} \{e_i - e_j \mid i,j \in N_a\ i \ne j\}.$$

Theorem 3.4. *For type* B_l, $\mathfrak{g}^{\mathbb{C}} = so(2l+1,\mathbb{C})$ *has no semisimple decomposition.*

Proof. Let $\triangle = \{e_i \pm e_j, \pm e_k \mid 1 \le i \ne j \le l, \quad 1 \le k \le l\}$. Since $e_1, \cdots, e_l$ generate $\triangle$, choose $e_i \in \triangle_{ab}$ for some $a \ne b$. Then $\forall\ j \ne i$, $e_i \pm e_j \in \triangle$. By Theorem 3.2, $e_j \in \triangle_0 \cup \triangle_{aa} \cup \triangle_{bb} \cup \triangle_{ab} \cup \triangle_{ba} \subset \triangle_0 \cup \triangle_a \cup \triangle_b$. Now $\triangle \subset \triangle_0 \cup \triangle_a \cup \triangle_b \subset \triangle$, so $\triangle = \triangle_0 \cup \triangle_a \cup \triangle_b$ forces $m = 2$, a contradiction!

Theorem 3.5. *For type* $C_l, \mathfrak{g}^{\mathbb{C}} = sp(2l,\mathbb{C})$ *has no semisimple decomposition, either.*

Proof. Let $\triangle = \{\pm(e_i \pm e_j), \pm 2e_k \mid 1 \le i \le j \le l, 1 \le k \le l\}$. Since $m \ge 3$, and $\{e_i \pm e_j \mid 1 \le i \ne j \le l\}$ generates $\triangle$, without loss of generality, we can assume $\exists\ i,j,k$ such that $e_i - e_j \in \triangle_{ab}$ and $e_i - e_k \in \triangle_{ac}$ for some $a \ne b \ne\ c$. Since $(e_i + e_j) \pm (e_i - e_j) \in \triangle$, by Theorem 3.2, we have $e_i + e_j \in \triangle_0' \cup \triangle_{ab} \cup \triangle_{ba}$. But $(e_i + e_j) - (e_i - e_k) \in \triangle$ implies $e_i + e_j \in \triangle_0' \cup \triangle_{ab}$. Thus $2e_i = (e_i + e_j) + (e_i - e_j) \in \triangle_{ab}$. Symmetrically we have $e_i + e_k \in \triangle_0' \cup \triangle_{ac}$, so $2e_i = (e_i + e_k) + (e_i - e_k) \in \triangle_{ac}$. But $\triangle_{ab} \cap \triangle_{ac} = \emptyset$, a contradiction!

Theorem 3.6. *For type* D_l $(l \ge 4)$, $\mathfrak{g}^{\mathbb{C}} = so(2l,\mathbb{C})$, $\triangle = \{\pm(e_i \pm e_j) \mid 1 \le i < j \le l\}$ *up to an isomorphism of* $\triangle$, *Every semisimple decomposition is given by:*

$$\triangle_{12} = \{e_i + e_j \mid 2 \le i < j \le l\}, \quad \triangle_{23} = \{e_1 - e_i \mid 2 \le i \le l\},$$
$$\triangle_{13} = \{e_1 + e_i \mid 2 \le i \le l\}, \quad \triangle_0 = \{e_i - e_j \mid 2 \le i \ne j \le l\},$$

specially, $m = 3$.

Proof. First assume the semisimple decomposition is reduced. Find a root. Without loss of generality, suppose $e_{l-1} + e_l \in \triangle_0$ (if does not exist such root, let $\triangle'' = \triangle$). Since $\triangle' \overset{\text{def.}}{=} \{e_i - e_j) \mid 1 \le i < j \le l\}$ and $e_{l-1} + e_l$ generate $\triangle$, by Theorem 3.2, $\triangle'$ has a reduced semisimple decomposition by restriction. Because $\triangle'$ is of type A_{l-1}, by Theorem 3.3, there exists a partition of $N = \{1, 2, \cdots, l\} = \bigcup_{a=1}^{m} N_a$. For each a, choose an element $i_a \in N_a$, to generate a type D_m root system $\triangle'' \overset{\text{def.}}{=} \{\pm(e_{i_a} \pm e_{i_b}) \mid 1 \le a < b \le m\}$. Then $\triangle''$ has also a semisimple decomposition (with the same m). If $m \ge 4$, by Theorem 3.2

$$\left.\begin{array}{l} e_{i_1} - e_{i_3} \in \triangle''_{13} e_{i_2} - e_{i_3} \in \triangle''_{23} \Rightarrow e_{i_3} + e_{i_4} \in \triangle''_0 \cup \triangle''_3 \\ e_{i_1} - e_{i_4} \in \triangle''_{14} e_{i_2} - e_{i_4} \in \triangle''_{24} \Rightarrow e_{i_3} + e_{i_4} \in \triangle''_0 \cup \triangle''_4 \end{array}\right\} \Rightarrow e_{i_3} + e_{i_4} \in \triangle''_0.$$

By the same reason, $e_{i_2} + e_{i_3} \in \triangle''_0$. Now $e_{i_2} - e_{i_4} = (e_{i_2} + e_{i_3}) - (e_{i_3} + e_{i_4}) \in \triangle''_0$, but in fact $(e_{i_2} - e_{i_4}) \in \triangle''_{24}$, a contradiction!

Hence $m = 3$. Now come back to $\triangle'$. By Theorem 3.3, we can suppose the partition of $N = N_a \cup N_b \cup N_c$ since $l \ge 4$. Without loss of generality, suppose $|N_c| \ge 2$, choose $k, k' \in N_c$.

Case 1. if $|N_b| \ge 2$, and $|N_a| \ge 2$, choose $i, i' \in N_a$; $\quad j, j' \in N_b$. Now

$$\left.\begin{array}{l} (e_k + e_{k'}) + (e_i - e_k) \in \triangle \\ (e_k + e_{k'}) + (e_j - e_{k'}) \in \triangle \\ (e_i - e_k) + (e_k + e_{k'}) + (e_j - e_{k'}) \in \triangle \end{array}\right\} \overset{\text{Theorem 3.2}}{\Longrightarrow} (e_k + e_{k'}) \in \triangle_{ca} \cup \triangle_{cb},$$

$$\left.\begin{array}{l} (e_k + e_{k'}) + (e_i - e_k) \in \triangle \\ (e_k + e_{k'}) + (e_j - e_{k'}) \in \triangle \\ (e_{j'} - e_k) + (e_k + e_{k'}) + (e_j - e_{k'}) \in \triangle \end{array}\right\} \overset{\text{Theorem 3.2}}{\Longrightarrow} (e_k + e_{k'}) \in \triangle_0 \cup \triangle_{cb},$$

$$\left.\begin{array}{l} (e_k + e_{k'}) + (e_i - e_k) \in \triangle \\ (e_k + e_{k'}) + (e_j - e_{k'}) \in \triangle \\ (e_{i'} - e_k) + (e_k + e_{k'}) + (e_i - e_{k'}) \in \triangle \end{array}\right\} \overset{\text{Theorem 3.2}}{\Longrightarrow} (e_k + e_{k'}) \in \triangle_0 \cup \triangle_{ca}.$$

Now, $(e_k + e_{k'}) \in (\triangle_0 \cup \triangle_{ca}) \cap (\triangle_0 \cup \triangle_{cb}) \cap (\triangle_{ca} \cup \triangle_{cb}) = \emptyset$, a contradiction!

Case 2. Without loss of generality, assume $|N_a| = 1$. If $|N_b| \ge 2$, above calculation shows $e_k + e_{k'} \in \triangle_{cb}$; if $|N_b| = 1$, we get $e_k + e_{k'} \in \triangle_{cb} \cup \triangle_{ca}$. Since a, b are symmetric in this case, without loss of generality, assume $e_k + e_{k'} \in \triangle_{cb}$ and $N_a = \{1\}$. It is easy to get:

$$e_1 + e_k \in \triangle_{ab}, \ \ \forall k \in N_c; \quad e_j + e_k \in \triangle_0, \ \ \forall j \in N_b, \ k \in N_c;$$
$$e_1 + e_j \in \triangle_{ac}, \ \ \forall j \in N_b; \quad e_j + e_{j'} \in \triangle_{bc}, \ \ \forall j \ne j' \in N_b;$$

plus what we have already known for the roots of $\triangle'$:

$$e_1 - e_j \in \triangle_{ab}, \ \ \forall j \in N_b; \quad e_j - e_k \in \triangle_{bc}, \ \ \forall j \in N_b, \ k \in N_c;$$
$$e_1 - e_k \in \triangle_{ac}, \ \ \forall k \in N_c; \quad e_j - e_{j'} \in \triangle_0, \ \ \forall j \ne j' \in N_b;$$
$$e_k - e_{k'} \in \triangle_0, \ \ \forall k \ne k' \in N_b.$$

Now take an isomorphism $\triangle \longrightarrow \triangle$ by

$$e_1 \longrightarrow -e_1; \quad e_j \longrightarrow \quad e_j, \ \ \forall\ j \in N_b; \quad e_k \longrightarrow -e_k, \ \ \forall\ k \in N_c.$$

Let $b=1,\ c=2,\ a=3$. Rewrite the result as

$$\triangle_{12}=\{e_i+e_j \mid 2\le i<j\le l\};\quad \triangle_{23}=\{e_1-e_i \mid 2\le i\le l\};$$
$$\triangle_{13}=\{e_1+e_i \mid 2\le i\le l\};\quad \triangle_0\ =\{e_i-e_j \mid 2\le i\ne j\le l\}.$$

For general case, since $\triangle_0'=\{e_i-e_j \mid 2\le i\ne j\le l\}$, $\triangle_{12},\ \triangle_{23},\ \triangle_{13}$ are given above. Hence, $\forall\ \alpha\in\triangle_0',\ \alpha+\triangle_{12}\ne\emptyset$, we get $\triangle_{33}=\emptyset$; for the same reason, $\triangle_{22}=\triangle_{11}=\emptyset$, i.e., the semisimple decomposition is automatically reduced.

Corollary 3.2. *For type A_l, as a complex semisimple space, every maximal semisimple space is* 1-1 *corresponding to the reduced semisimple decomposition of $\mathfrak{g}^{\mathbb{C}}$, which is given by Theorem* 3.3. *For type D_l, every semisimple decomposition is automatically maximal.*

Theorem 3.7. *For types G_2,F_4, there are no semisimple decompositions.*

Proof. Since the Dykin diagram for G_2 is

the only possible case ($m\ge 3$) is $\alpha_1\in\triangle_{ab}$, $\alpha_2\in\triangle_{bc}$ for some $a\ne b\ne c$. But $\alpha_1+\alpha_2\in\triangle_{ac}$, $\alpha_1+2\alpha_2=(\alpha_1+\alpha_2)+\alpha_2\in\triangle_{ac}+\triangle_{bc}=\emptyset$, a condiction!

For type F_4, the Dykin diagram is

which contains two subsystems

One is of type B_3, and the other is of type C_3. By Theorems 3.4, 3.5, both of them have $m\le 2$, so the possible case might be: $\alpha_1\in\triangle_{ab},\ \alpha_2\in\triangle_0,\ \alpha_3\in\triangle_0,\ \alpha_4\in\triangle_{bc}$ for some $a\ne b\ne c$. Consider the subroot system C_3, which has the restricted semisimple

decomposition with $m=3$, a contradiction to Theorem 3.5!

Theorem 3.8. *For types E_7 and E_8, there are no semisimple decompositions with $m\ge 3$. But for type E_6, up to an isomorphism, there is a unique semisimple decomposition, given by: consider the Dykin diagram of E_6:*

with $\alpha_1\in\triangle_{ab}\quad \alpha_2,\alpha_3,\alpha_4,\alpha_6,\in\triangle_0\quad \alpha_5\in\triangle_{bc}$, then

$$\triangle_{ab}=\{\textit{all positive roots containing }\alpha_1,\textit{ not containing }\alpha_5\},$$
$$\triangle_{bc}=\{\textit{all positive roots containing }\alpha_5,\textit{ not containing }\alpha_1\},$$
$$\triangle_{ac}=\{\textit{all positive roots containing both }\alpha_5\textit{ and }\alpha_1\},$$
$$\triangle_0\ =\{\textit{all roots generated by }\alpha_2,\alpha_3,\alpha_4,\alpha_6,\}.$$

Proof. Since $\mathfrak{g}^{\mathbb{C}}$ contains a subalgebra of type D_5, by Theorem 3.6, $\exists\ \alpha\in\triangle_0$. Find a subalgebra $\mathfrak{l}$ of type D_5,D_6 or D_7 which does not contain α. Then $\mathfrak{l}$ has a semisimple decomposition from that of $\mathfrak{g}^{\mathbb{C}}$. By Theorem 3.6 and the fact that $\mathfrak{l}$ and α generate $\mathfrak{g}^{\mathbb{C}}$, if $\mathfrak{g}^{\mathbb{C}}$ has such a decomposition, then $m=3$. We list in the left the possible cases that the simple roots belong to. By Theorem 3.2, it is possible to give some limit to all roots. Write the roots which make this case impossible in the right. For E_7:

For E_8, there are also three possible cases:

Two possible cases for E_6:

The only remainding case is the last case for E_6, which can be conjugated to:

By Theorem 3.2, we can write down all roots of $\triangle_{ab}$ $\triangle_{bc}$ $\triangle_{ac}$ and $\triangle_0$ as the theorem gives.

§4. Real Homogeneous Semisimple Space $G/C(W)_0$

By Theorem 2.2, there are two classes of semisimple pairs $(\mathfrak{g}, \mathfrak{u})$, each pair of which arises to a real semisimple homogeneous space $G/C(W)_0$. For the first class, every semisimple decomposition (here we allow $m = 2$) $\mathfrak{g}^1 = \sum\limits_{a=1}^{m} \mathfrak{g}_a$ induces the semisimple decomposition

$$\mathfrak{g}^{\mathbb{C}} = \sum_{a=1}^{m} (\mathfrak{g}_a + \mathfrak{u}^{\mathbb{C}} \cap \overline{\mathfrak{g}^1}) + \sum_{a=1}^{m} (\overline{\mathfrak{g}_a} + \mathfrak{u}^{\mathbb{C}} \cap \mathfrak{g}^1),$$

and $f(x)$, the minimal polynomial of I, has $2m$ imaginary roots. Conversely, by Theorem 2.2, every semisimple decomposition of $\mathfrak{g}^{\mathbb{C}}$ arises in above way.

Theorem 4.1 (see [1, Chapter 3]). *Every reduced semisimple decomposition with* $m = 2$ *is given by* $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_1 + \mathfrak{g}_2$ *where* $\mathfrak{g}_1$, $\mathfrak{g}_2$ *are the opposited parabolic subalgebra with Levi subalgebra* $\mathfrak{u}^{\mathbb{C}}$.

Corollary 4.1. *Given a fixed prime root system* Γ, *up to an isomorphism of root system, every semisimple decomposition of* $\triangle$ *with* $m = 2$ *is given by* $\triangle = \triangle_0 \cup \triangle_{11} \cup \triangle_{22} \cup \triangle_{12} \cup \triangle_{21}$, *where*

(1) $\triangle$ *is generated by the subset* Γ_0 *of* Γ, $\triangle_0'$ *is generated by the subset* Γ_0', $\Gamma_0 \subset \Gamma_0' \subset \Gamma$,

(2) $\triangle_{11} \cup \triangle_{22} = \triangle_0' \setminus \triangle_0, \quad \triangle_{11} + \triangle_{22} = \emptyset$,

(3) $\triangle_{12} \subset \triangle^+, \quad \triangle_{12} = -\triangle_{21}, \quad \triangle_{11} = -\triangle_{11}, \quad \triangle_{22} = -\triangle_{22}$.

Theorem 4.2. *All the semisimple homogeneous spaces* $(\mathfrak{g}, \mathfrak{u})$ *of first class whose* $f(x)$ *has* 4 *imaginary roots are determined by Theorem* 2.2, *Corollary* 4.1 (*too many, including all the complexifocation of complex homogeneous spaces and paracomplex homogeneous spaces* $G/C(W)_0$, *we do not list here*). *Other case* ($f(x)$ *has* $2s$ *imaginary roots and* $s \geq 3$) *is given by*

(1) $\big(sl(n,\mathbb{C}), s\big(\sum\limits_{a=1}^{m} gl(n_a,\mathbb{C})\big)\big)$, *with* $2s = 2m, \quad s \geq 3$. *The semismiple decomposition is determined by Theorem* 3.3.

(2) $(so(2n,\mathbb{C}), \mathbb{C} + gl(n-1,\mathbb{C}))$, *where* $\mathbb{C} + gl(n-1,\mathbb{C}) \hookrightarrow gl(n,\mathbb{C}) \hookrightarrow so(2n,\mathbb{C})$, *with* $2s = 6$. *The semisimple decomposition is determined by Theorem* 3.6.

(3) $(e_6, \mathbb{C}^2 + so(8,\mathbb{C}))$ *with* $2s = 6$, *where the semisimple decomposition is determined by Theorem* 3.8.

For the second type, $\mathfrak{g}^{\mathbb{C}}$ is simple, and the semisimple decompositions of $\mathfrak{g}^{\mathbb{C}}$ are given by Theorems 3.3–3.8. We only need to find all real forms $\mathfrak{g}$, such that the associated conjugation σ respects above decomposition, i.e $\forall\ a,\ \exists\ b$ such that $\sigma(\mathfrak{g}_a) = \mathfrak{g}_b$. Choose Cartan subalgebra $\mathfrak{h} \subset \mathfrak{u}$ as we did in Section 3. Now the action of σ on roots satisfies: $\sigma(\triangle_0) = \triangle_0$ and $\sigma(\triangle_{ab}) = \triangle_{\sigma(a)\sigma(b)}$, where $\sigma(a)$ is defined by $\sigma(\triangle_a) = \triangle_{\sigma(a)}$.

Lemma 4.1. *All* σ *which respects the semisimple decomposition* (*in Theorems* 3.3, 3.4 *and* 3.8) *arise in the following way:*

(1) *For type* A_l, *let* σ *be an involutive permutation of* $N = \{1, 2, \cdots, n\}$, *such that* $\forall\ a\ \exists\ b$ *such that* $\sigma(N_a) = N_b$, *define* $\sigma(e_i - e_j) = e_{\sigma(i)} - e_{\sigma(j)}$, $\forall\ 1 \leq i \neq j \leq n$.

(2) *For type* D_l, *let* σ *be an involutive permutation of* $N' = \{2, \cdots, n\}$, *define* $\sigma(e_1) = e_{\sigma(1)}, \quad \sigma(e_i) = \pm e_{\sigma(i)}, \quad \forall\ i \in N'$.

(3) *For type* E_6, *given a member* w *of Wyel Group of* $\triangle_0$, *viewed as the element of the Wyle Group of* $\triangle$, *then* $\sigma = w$ *or* $\sigma = -w\rho$, *where* ρ *is the regular outer isomorphism* (*see* [2]) *of* $\triangle$ (*i.e the Graph isomorphism of Dykin diagram* E_6).

We omit the proof, it is a corollary of Theorems 3.3, 3.4 and 3.8.

Theorem 4.3. *The semisimple homogeneous spaces* $(\mathfrak{g}, \mathfrak{u})$ *of the second class are given by following table: Table* 1. *Here we suppose* I *has a minimal polynomial* $f(x)$, *denote the number of real roots of* $f(x)$ *by* r *and the number of complex roots of* $f(x)$ *by* $2s$.

Table 1. Real Homogeneous Semisimple Space

Proof. As [2] or [4] did, we start from a compact real form $\mathfrak{g}_C$, choose some isomorphism θ of $\mathfrak{g}_C$, and give the real form $\mathfrak{g}$. For type A_l, since σ is determined by above Lemma, it forces θ to be an outer isomorphism, so $\mathfrak{g}$ must be $sl(n, \mathbb{R})$ or $su^*(2n)$. Find all W, calculate directly. For type D_l, if $\sigma(e_i) = -e_{\sigma(i)}$ and l is odd, θ is inner; if $\sigma(e_i) = -e_{\sigma(i)}$ and l is even, or if $\sigma(e_i) = e_{\sigma(i)}$, then θ is outer. For every case, find all W such that the root system admits such decomposition (Theorem 3.6) and σ (above Lemma). For type E_6, θ is always outer, so $\mathfrak{g}$ must be $e_6(6)$ or $e_6(-26)$. Find all W such that the root system admits such decomposition (Theorem 3.8) and σ (above Lemma).

First, although on $G/C(W)_0$ there is a natural symplectic form, it is difficult to define a metric on the semisimple homogeneous space such that the symplectic form, metric and the semisimple structure fit nicely like the Kähler manifold or para-Kähaler manifold. Second, the semisimple manifold seems more "rigid" than the complex manifold, so the semisimple homogeneous space is much less than the complex homogeneous space.

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