Chin. Ann. of Math.
19B: 3(1998),331-340.

SOME EXTENSIONS OF PALEY-WIENNER THEOREM***

Liu Youming* Sun Qiyu** Huang Daren**

Abstract

The Shannon's sampling theorem has many extensions, two of which are to wavelet subspaces of $L^2(R)$ and to $B_\pi^2 =: \{f(x,y) \in L^2(R^2)$, supp $\hat{f} \subseteq [-\pi,\pi] \times [-\pi,\pi]\}$, where supp$\hat{f}$ denotes the support of the Fourier transform of a function f. In fact, the Paley-Wienner theorem says that each f in B_π^2 can be recovered from its sampled values $\{f(x_n, y_m)\}_{n,m}$ if (x_n, y_m) satisfies $|x_n - n| \le L < \frac{1}{4}$ and $|y_m - m| \le L < \frac{1}{4}$. Unfortunately this theorem requires strongly the product structure of sampling set $\{(x_n, y_m)\}_{m,n\in Z}$. This paper gives a sampling theorem in which the sampling set has a general form $\{(x_{nm}, y_{nm})\}$. In addition, G. Walter's sampling theorem is extended to wavelet subspaces of $L^2(R^2)$ and irregular sampling with the general sampling set $\{(x_{nm}, y_{nm})\}$ is considered in the same spaces. All results in this work can be written similarly in n-dimensional case for $n \ge 2$.

Keywords Sampling, Paley-Wienner theorem, Wavelets
1991 MR Subject Classification 94A11, 42C15
Chinese Library Classification O174.2. O236

§1. Introduction and Motivation

The classical Shannon's sampling theorem holds due to the following two reasons:
(i) $q(t,u) = \frac{\sin \pi(t-u)}{\pi(t-u)}$ is the reproducing kernel of B_π, i.e.,

$$f(t) = \int_{-\infty}^{+\infty} q(t,u) f(u) du$$

for each $f \in B_\pi$, where $B_\pi = \{f \in L^2(R),\ \mathrm{supp}\hat{f} \subseteq [-\pi,\pi]\}$;
(ii) $q(t,n) = \frac{\sin \pi(t-n)}{\pi(t-n)}$ is an orthonormal basis of B_π.

In fact, $q(t,n)$ being an orthonormal basis implies that $f(t) = \sum\limits_n \langle f(\cdot), q(\cdot,n)\rangle q(t,n)$ and furthermore

$$f(t) = \sum_n f(n) \frac{\sin \pi(t-n)}{\pi(t-n)} \tag{1.1}$$

by (i). A standard argument shows that the series in (1.1) converges uniformly, which gives the Shannon's sampling theorem. Furthermore since $q(t,t_n)$ is still a Riesz basis if $|t_n - n| \le L < \frac{1}{4}$, one obtains the Paley-Wienner Theorem $f(t) = \sum\limits_n f(t_n) S_n(t)$, where $S_n(t)$ is the biothogonal Riesz basis of $q(t,t_n)$.

Manuscript received October 6, 1995.

*Departement of Mathematics, Beijing Polytechnic University, Beijing 100022, China.
**Center for Mathematical Sciences, Zhejiang University, Hangzhou 310027, China.

***Project supported by the Science Foundation of New Stars of Beijing.

It turns out that the above Paley-Wienner theorem can be extended to 2-dimensional case by a similar argument. In fact by denoting

$$B_\pi^2 = \{f(x,y) \in L^2(R^2),\ \operatorname{supp}\hat{f} \subseteq [-\pi,\pi]^2\},$$

$$q(x,y,u,v) = \frac{\sin\pi(x-u)}{\pi(x-u)}\frac{\sin\pi(y-v)}{\pi(y-v)},$$

we have the following easily understandable lemmas:

Lemma 1.1. *The space B_π^2 is the tensor product of B_π and B_π; that is, B_π^2 is generated by linear closure of $f(x)g(y)$, where $f, g \in B_\pi$. In general we denote this fact by $B_\pi^2 = B_\pi \bigotimes B_\pi$.*

Lemma 1.2. *The function $q(x,y,u,v)$ is the reproducing kernel of B_π^2.*

Lemma 1.3. *If $\{e_n(x)\}$ is a Riesz basis of B_π, then so is $\{e_n(x)e_m(y)\}_{n,m}$ of B_π^2.*

By Lemmas 1.1–1.3, we have the following Paley-Wienner theorem in 2-dimensional case.

Theorem 1.1. *If $|x_n - n| \le L < \frac{1}{4}$ and $|y_m - m| \le L < \frac{1}{4}$, then there exists $\{S_{n,m}(x,y)\}_{n,m} \subseteq B_\pi^2$ such that*

$$f(x,y) = \sum_{n,m} f(x_n, y_m) S_{n,m}(x,y)$$

for each $f \in B_\pi^2$.

A disadvantage of Theorem 1.1 is the restrictivity of the product structure of the sampling set $\{(x_n, y_m)\}$. Butzer and Hinsen considered the case in which the sampling set has the form $\{(x_{nm}, y_n)\}_{m,n\in Z}$ (see [1, Theorem 3.8]) while K. Gröchenig estabished a sampling theorem for more general sampling set but with oversampling[2]. In Section 2 we shall give a sampling theorem in B_π^2 with the form of sampling set $\{(x_{nm}, y_{nm})\}_{m,n\in Z}$. On the other hand, G. Walter extended the Shannon's sampling theorem (1.1) to wavelet subspaces (see [3]) as follows.

Let $\varphi(t) \in L^2(R) \cap C(R)$ satisfy $\varphi(t) = O(|t|^{-a})$ at $t = \infty$ for some $a > 1$. If $\{\varphi(t-n)\}_n$ constitutes an orthonormal basis of some subspace V_0 of $L^2(R)$ and if $\hat{\varphi}^\star(\omega) =: \sum\limits_n \varphi(n)e^{in\omega}$ has no zeros on the real line, then there exists $S_n(t)$ such that

$$f(t) = \sum_n f(n) S_n(t) \tag{1.2}$$

for each $f \in V_0$. In wavelet analysis, this V_0 is called a wavelet subspace. In Section 3, we shall find both regular and irregular sampling theorems for general sampling set $\{(x_{nm}, y_{nm})\}$ in 2-dimensional wavelet subspaces.

§2. Irregular Sampling in B_π^2

In this section, we shall remove the limitation of the product structure of the sampling set in Theorem 1.1. We begin with a simple lemma[4].

Lemma 2.1. *Let $\{f_n\}$ be an orthonormal basis of a Hilbert space H and $\{g_n\} \subseteq H$. If there exists $0 < \theta < 1$ such that*

$$\Big\| \sum_{n=1}^{k} c_n(f_n - g_n) \Big\| \le \theta \Big(\sum_{n=1}^{k} c_n^2 \Big)^{\frac{1}{2}}$$

for any $c_1, c_2, \cdots, c_n (n = 1, 2, \cdots)$, then $\{g_n\}$ is a Riesz basis of H.

Now we are ready to state

Theorem 2.1. *If* $|t_{mn} - m| \le \delta_1, |s_{mn} - n| \le \delta_2$ *with* $\delta =: \delta_1 + \delta_2 < \frac{\ln 2}{\pi}$, *then there exists* $\{S^{\star}_{mn}(x,y)\} \subseteq B^2_\pi$ *such that*

$$f(x,y) = \sum_{m,n} f(t_{mn}, s_{mn}) S^{\star}_{mn}(x,y) \tag{2.1}$$

holds uniformly on R^2 *for each* $f \in B^2_\pi$.

Proof. Since $q(x,y,t,s) = \frac{\sin \pi(x-t)}{\pi(x-t)} \cdot \frac{\sin \pi(y-s)}{\pi(y-s)}$ is the reproducing kernel of B^2_π by Lemma 1.2, one only needs to prove that $q(x,y,t_{mn},s_{mn})$ is a Riesz basis of B^2_π to have (2.1) in the sense of $L^2(R^2)$. Now define

$$g_{mn}(u,v) =: \frac{1}{2\pi} e^{it_{mn}u + is_{mn}v} K_{[-\pi,\pi]^2}(u,v),$$

where $K_{[-\pi,\pi]^2}(u,v)$ is the characteristic function of the square $[-\pi,\pi]^2$ in R^2. Then it is easy to find $\hat{g}_{mn}(\cdot,\cdot) = q(\cdot,\cdot,t_{mn},s_{mn})$. Hence one needs only to show that g_{mn} constitutes a Riesz basis of $L^2[-\pi,\pi]^2$ to have (2.1) in the sense of $L^2(R^2)$.

It is known that $\frac{1}{2\pi} e^{inu+imv}$ is an orthonormal basis of $L^2[-\pi,\pi]^2$. By Lemma 2.1, it is sufficient to prove that there exists $0 < \theta < 1$ such that

$$I = \frac{1}{2\pi} \Big\| \sum_{m,n} c_{mn} (e^{it_{mn}u + is_{mn}v} - e^{inu+imv}) \Big\|_{L^2[-\pi,\pi]^2} \le \theta \Big(\sum_{m,n} c^2_{mn} \Big)^{\frac{1}{2}} \tag{2.2}$$

for any finite sequence $\{c_{mn}\}_{m,n}$. Now one can re-express

$$I = \frac{1}{2\pi} \Big\| \sum_{m,n} (e^{i\delta_\alpha u + i\sigma_\alpha v} - 1) c_{mn} e^{inu+imv} \Big\|,$$

where $\alpha = (m,n)$ and $\delta_\alpha = t_{mn} - n, \sigma_\alpha = s_{mn} - m$. By using the Taylor's formula for $e^{i\delta_\alpha u + i\sigma_\alpha v}$ at $u = v = 0$, one obtains

$$e^{i\delta_\alpha u + i\sigma_\alpha v} - 1 = \sum_{k=1}^{\infty} \frac{1}{k!} (i\delta_\alpha u + i\sigma_\alpha v)^k = \sum_{k=1}^{\infty} \sum_{l=0}^{k} \frac{1}{k!} C^l_k i^k (\delta_\alpha u)^l (\sigma_\alpha v)^{k-l},$$

where $C^l_k = \frac{l!(k-l)!}{k!}$. Therefore it follows that

$$\begin{aligned} I &= \frac{1}{2\pi} \Big\| \sum_{m,n} \sum_{k=1}^{\infty} \sum_{l=0}^{k} \frac{1}{k!} C^l_k i^k (\delta_\alpha u)^l (\sigma_\alpha v)^{k-l} c_{mn} e^{inu+imv} \Big\| \\ &\le \frac{1}{2\pi} \sum_{k=1}^{\infty} \sum_{l=0}^{k} \frac{C^l_k}{k!} \Big\| \sum_{m,n} \delta^l_\alpha \sigma^{k-l}_\alpha u^l v^{k-l} c_{mn} e^{inu+imv} \Big\| \\ &\le \sum_{k=1}^{\infty} \sum_{l=0}^{k} \frac{C^l_k}{k!} \pi^k \Big[\sum_{m,n} (\delta^l_\alpha \sigma^{k-l}_\alpha c_{mn})^2 \Big]^{\frac{1}{2}} \\ &\le \sum_{k=1}^{\infty} \frac{\pi^k}{k!} (\delta_1 + \delta_2)^k \Big(\sum_{m,n} c^2_{mn} \Big)^{\frac{1}{2}} \\ &= (e^{\pi\delta} - 1) \Big(\sum_{m,n} c^2_{mn} \Big)^{\frac{1}{2}}. \end{aligned}$$

Now one may take $\theta = e^{\pi\delta} - 1$. It is obvious that $0 < \theta < 1$ due to the assumption $\delta < \frac{\ln 2}{\pi}$, which proves (2.2). The uniform convergence follows from a standard argument (see [1, p.254]), since the kernel function $q(x,y,t,s)$ is bounded.

Theorem 2.1 is derived essentially by considering the pertubation of the known orthonormal basis. In fact we can do a little more by considering the pertubation of a known frames.

Definition 2.1. *A sequence $\{g_n\}$ of a Hilbert space H is called a frame if there exist $0 < m \le M < \infty$ such that*

$$m||f||^2 \le \sum_n |\langle f, g_n\rangle|^2 \le M||f||^2$$

for each $f \in H$.

It is known that a Riesz basis is always a frame. The following well-known frame operator theorem is needed in our discussion.

Lemma 2.2. *If $\{g_n\}$ is a frame of H, then there exists another frame $\{h_n\}$, called dual frame, such that*

$$f = \sum_n \langle f, g_n\rangle h_n = \sum_n \langle f, h_n\rangle g_n$$

for each $f \in H$.

Remark 2.1. By Lemma 2.2 we know that it is sufficient that $q(x, y, t_{mn}, s_{mn})$ is a frame to have a sampling theorem for a sampling set $\{(t_{mn}, s_{mn})\}$ in B^2_π, where

$$q(x, y, t, s) = \frac{\sin \pi(x-t)}{\pi(x-t)} \cdot \frac{\sin \pi(y-s)}{\pi(y-s)}$$

is the kernel of B^2_π. Hence in order to have a sampling theorem in B^2_π, we only need the following condition: there exist $0 < A < B < \infty$ such that

$$A||f||^2 \le \sum_{n,m} |f(t_{mn}, s_{mn})|^2 \le B||f||^2$$

for each $f \in B^2_\pi$.

Theorem 2.2. *Let $A||f||^2 \le \sum\limits_{n,m} |f(t_{mn}, s_{mn})|^2 \le B||f||^2$ for each $f \in B^2_\pi$. If*

$$e_{mn} =: u_{mn} - t_{mn} \quad \text{and } d_{mn} =: v_{mn} - s_{mn}$$

satisfy $|e_{mn}| \le e, |d_{mn}| \le d$ and

$$\delta =: \sqrt{e^2 + d^2} < \sqrt{\ln \frac{A + B^2(e^{2\pi} - 1)}{B^2(e^{2\pi} - 1)}},$$

then there exist $0 < m_0 < M_0 < \infty$ such that

$$m_0||f||^2 \le \sum_{m,n} |f(u_{mn}, v_{mn})|^2 \le M_0||f||^2$$

for each $f \in B^2_\pi$.

Proof. Denote $\alpha = (m, n), t_\alpha = t_{mn}$, $e_\alpha = e_{mn}$, etc. By applying Taylor's formula and Cauchy inequality, one has

$$\begin{aligned} I_\alpha =: |f(u_\alpha, v_\alpha) - f(t_\alpha, s_\alpha)| &= \Big| \sum_{k=1}^\infty \sum_{l=0}^k \frac{C_k^l}{k!} D_x^l f D_y^{k-l} f e_\alpha^l d_\alpha^{k-l} \Big| \\ &\le \Big[\sum_{k=1}^\infty \sum_{l=0}^k \frac{C_k^l}{k!} |D_x^l f D_y^{k-l} f|^2 \Big]^{\frac12} \Big[\sum_{k=1}^\infty \sum_{l=0}^k \frac{C_k^l}{k!} |e_\alpha^{2l} d_\alpha^{2k-2l}| \Big]^{\frac12} \\ &\le (e^{\delta^2} - 1)^{\frac12} \Big[\sum_{k=1}^\infty \sum_{l=0}^k \frac{C_k^l}{k!} |D_x^l f D_y^{k-l} f|^2 \Big]^{\frac12}, \end{aligned}$$

where

$$D_x^n f =: \frac{\partial^n f}{\partial x^n}(t_\alpha, s_\alpha), \quad D_y^n f = \frac{\partial^n f}{\partial y^n}(t_\alpha, s_\alpha)$$

and $C_k^l = \frac{k!}{l!.(k-l)!}$. Hence it follows that

$$\begin{aligned}\sum_\alpha |I_\alpha|^2 &\le (e^{\delta^2} - 1)\sum_{k=1}^\infty \sum_{l=0}^k \frac{C_k^l}{k!} \sum_\alpha |D_x^l f D_y^{k-l} f|^2 \\ &\le (e^{\delta^2} - 1)\sum_{k=1}^\infty \sum_{l=0}^k \frac{C_k^l}{k!} \Big[\sum_\alpha |D_x^l f|^2 \cdot \sum_\alpha |D_y^{k-l} f|^2\Big]. \end{aligned} \tag{2.3}$$

Noticing that

$$\Big(\frac{\partial^l f}{\partial x^l}\Big)^\wedge (u,v) = (-iu)^l \hat{f}(u,v), \tag{2.4}$$

one has $\frac{\partial^l f}{\partial x^l} \in B_\pi^2$ for each $l \in Z^+$ and

$$\sum_\alpha |D_x^l f|^2 \le B\Big\|\frac{\partial^l f}{\partial x^l}\Big\|^2 \le B\pi^l ||f||^2 \le B\pi^l \tag{2.5}$$

for each $||f|| \le 1$, where the first inequality follows from the assumption of this theorem and the second one does from (2.4). Similarly one has

$$\sum_\alpha |D_y^{k-l} f|^2 \le B\Big\|\frac{\partial^{k-l} f}{\partial y^{k-l}}\Big\|^2 \le B\pi^{k-l} ||f||^2 \le B\pi^{k-l} \tag{2.6}$$

for each $f \in B_\pi^2$ with $||f|| \le 1$. Combining (2.3) with (2.5) and (2.6), one obtains

$$\sum_\alpha |I_\alpha|^2 \le (e^{\delta^2} - 1)\sum_{k=1}^\infty \sum_{l=0}^k \frac{C_k^l}{k!} B\pi^l B\pi^{k-l} = B^2(e^{\delta^2} - 1)(e^{2\pi} - 1) < A$$

due to the assumption on δ. Hence for each $f \in B_\pi^2$ the following inequality holds:

$$I^2 =: \sum_\alpha |I_\alpha|^2 = \sum_\alpha |f(u_\alpha, v_\alpha) - f(t_\alpha, s_\alpha)|^2 \le C||f||^2,$$

where $C =: B^2(e^{\delta^2} - 1)(e^{2\pi} - 1) < A$. Also since

$$\Big(\sum_{n,m} |f(t_\alpha, s_\alpha)|^2\Big)^{\frac{1}{2}} - I \le \Big(\sum_{n,m} |f(u_\alpha, v_\alpha)|^2\Big)^{\frac{1}{2}} \le \Big(\sum_{n,m} |f(t_\alpha, s_\alpha)|^2\Big)^{\frac{1}{2}} + I,$$

one may find

$$(\sqrt{A} - \sqrt{C})||f|| \le \Big(\sum_\alpha |f(u_\alpha, v_\alpha)|^2\Big)^{\frac{1}{2}} \le (\sqrt{B} + \sqrt{C})||f||.$$

The proof is completed by taking $m_0 = (\sqrt{A} - \sqrt{C})^2$ and $M_0 = (\sqrt{B} + \sqrt{C})^2$.

As a particular case, if $t_{mn} = m$ and $s_{mn} = n$, we have

Corollary 2.1. *If* $|u_{mn} - m| \le e$ *and* $|v_{mn} - n| \le d$ *with*

$$\delta =: \sqrt{e^2 + d^2} < \sqrt{2\pi - \ln(e^{2\pi} - 1)},$$

then there exists $\{S_{mn}^0\} \subset B_\pi^2$ *such that*

$$f(x,y) = \sum_{m,n} f(u_{mn}, v_{mn}) S_{mn}^0(x,y)$$

holds uniformly on R^2 *for each* $f \in B_\pi^2$.

The proof of Corollary 2.1 follows from Theorem 2.2 and the Remark following Lemma 2.2.

§3. Sampling in Wavelet Subspaces of $L^2(R^2)$

In this part, we shall extend G. Walter's sampling theorem (1.2) to wavelet subspaces of $L^2(R^2)$ and derive a desired irregular sampling theorem in those spaces, for which we need

Lemma 3.1. *Let $\varphi(x,y) \in L^2(R^2) \cap C(R^2)$ satisfy*

(i) $\{\varphi(x-n, y-m)\}_{n,m}$ *is an orthonormal system of $L^2(R^2)$;*

(ii) $|\varphi(x,y)| \le C(1+x^2+y^2)^{-1-\delta}$ *for some $C, \delta > 0$.*

Define

$$V_0 = \Big\{ f(x,y) \big| f(x,y) = \sum_{n,m} a_{nm}\varphi(x-n, y-m), \{a_{nm}\} \in l^2(Z^2) \Big\},$$
$$q(x,y,u,v) = \sum_{m,n} \varphi(x-n, y-m)\varphi(u-n, v-m).$$

Then

(1) $\sum\limits_{n,m} a_{nm}\varphi(x-n, y-m)$ *converges both uniformly and in $L^2(R^2)$;*

(2) *the kernel function $q(x,y,u,v)$ is well-defined on R^4 and $q(x,y,u_0,v_0) \in V_0$ for any $u_0, v_0 \in R$;*

(3) *for each $f \in V_0$,*

$$f(x,y) = \int_{R^2} f(u,v) q(x,y,u,v) dudv$$

holds for each $x, y \in R$.

Proof. (1) It is obvious that $\sum\limits_{n,m} a_{nm}\varphi(x-n, y-m)$ converges in $L^2(R^2)$ since $\{\varphi(x-n, y-m)\}$ is an orthonormal system and $\{a_{nm}\} \in l^2(Z^2)$. The continuity and decay condition (ii) on φ imply the uniform convergence of the periodic function $\sum\limits_{m,n} |\varphi(x-n, y-m)|^2$ and furthermore the boundedness. Furthermore it follows that $\sum\limits_{n,m} a_{nm}\varphi(x-n, y-m)$ converges uniformly from

$$\Big| \sum_{n,m} a_{nm}\varphi(x-n, y-m) \Big|^2 \le \sum_{n,m} |a_{nm}|^2 . \sum_{n,m} |\varphi(x-n, y-m)|^2.$$

(2) The function $q(x,y,u,v)$ is well-defined due to the assumptions (i) and (ii). The decay condition on φ implies $\varphi(u_0-n, v_0-m) \in l^2(Z^2)$ and therefore $q(x,y,u_0,v_0) \in V_0$ by the definition of V_0.

(3) It is obvious that $f(x,y) = \sum\limits_{n,m} \langle f(u,v), \varphi(x-n, y-m)\varphi(u-n, v-m) \rangle$ in $L^2(R^2)$-sense. Since

$$\int \sum_{n,m} |f(u,v)\varphi(x-n, y-m)\varphi(u-n, v-m)| dudv \le \sum_{n,m} |\varphi(x-n, y-m)| \cdot ||f|| \cdot ||\varphi|| < \infty$$

for each $x, y \in R$, one has

$$f(x,y) = \int_{R^2} f(u,v) q(x,y,u,v) dudv$$

by the Lebesgue dominated convergence theorem.

Theorem 3.1. *In addition to the assumptions of Lemma* 3.1*, if*

$$\hat{\varphi}^{\star}(u,v)=\sum_{n,m}\varphi(n,m)e^{inu+imv}$$

has no zeros on R^2*, then there exists* $S(x,y)\in V_0$ *such that for each* $f\in V_0$

$$f(x,y)=\sum_{n,m}f(n,m)S(x-n,y-m) \tag{3.1}$$

holds uniformly on R^2*.*

Proof. First one shows that $q(x,y,k,l)=\sum\limits_{n,m}\varphi(k-n,l-m)\varphi(x-n,y-m)$ is a Riesz basis of V_0. It is clear that

$$\begin{aligned}\hat{q}(u,v,k,l)&=\sum_{n,m}\int_{R^2}\varphi(k-n,l-m)\varphi(x-n,y-m)e^{-ixu-iyv}dxdy\\&=\sum_{n,m}\varphi(k-n,l-m)\hat{\varphi}(u,v)e^{-inu-imv}\\&=\hat{\varphi}^{\star}(u,v)\hat{\varphi}(u,v)e^{-iku-ilv}.\end{aligned} \tag{3.2}$$

Since $\{\varphi(x-k,y-l)\}$ is an orthonormal system and $0<m_0\le|\hat{\varphi}^{\star}(u,v)|\le M_0<+\infty$, one has

$$m_0^2\sum_{k,l}c_{kl}^2\le\int\Big|\sum_{k,l}c_{kl}\hat{\varphi}^{\star}(u,v)\hat{\varphi}(u,v)e^{-iku-ilv}\Big|^2dudv\le M_0^2\sum_{k,l}c_{kl}^2;$$

that is, $\hat{q}(u,v,k,l)$ is a Riesz basis of V_0 and so is $q(u,v,k,l)$.

Let $S_{nm}(x,y)$ be the bi-orthogonal Riesz basis of $q(x,y,n,m)$, then for each $f\in V_0$,

$$f(x,y)=\sum_{n,m}\langle f(u,v),q(u,v,n,m)\rangle S_{nm}(x,y)=\sum_{n,m}f(n,m)S_{nm}(x,y)$$

holds in the sense of $L^2(R^2)$ by (3) of Lemma 3.1. Next one needs to show that $S_{nm}(x,y)$ can be generated from one function S in the sense that $S_{nm}(x,y)=S(x-n,y-m)$. In fact define $S(x,y)$ by $\widehat{S}(u,v)=\frac{\hat{\varphi}(u,v)}{\hat{\varphi}^*(u,v)}$. Then $S(u,v)$ is well-defined and $S\in V_0$ by the assumptions of φ and $\hat{\varphi}^*$. Furthermore it is easy to see that

$$S(.-n,.-m)^{\wedge}(u,v)=\hat{S}(u,v)e^{-inu-imv}=\frac{\hat{\varphi}(u,v)e^{-inu-imv}}{\hat{\varphi}^*(u,v)}.$$

Hence one obtains

$$\langle S(x-n,y-m)^{\wedge}(u,v),\hat{q}(u,v,k,l)\rangle=\delta_{nk}.\delta_{ml}$$

by (3.2) and the fact that $\{\varphi(x-n,y-m)\}$ is an orthonormal system. Since the bi-orthogonal Riesz basis of $q(x,y,m,n)$ is unique, one can conclude that $S_{nm}(x,y)=S(x-n,y-m)$.

Finally the uniform convergence can be obtained as follows: denoting the partial sum of $\sum\limits_{n,m}f(n,m)S_{nm}(x,y)$ by $g_{kl}(x,y)$, one has the following estimate that

$$\begin{aligned}|f(x,y)-g_{kl}(x,y)|&=\Big|\int[f(u,v)-g_{kl}(u,v)]q(x,y,u,v)dudv\Big|\\&\le||f-g_{kl}||\,||q(x,y,u,v)||_{u,v}\\&=||f-g_{kl}||\Big[\sum_{n,m}|\varphi(x-n,y-m)|^2\Big]^{\frac{1}{2}}.\end{aligned}$$

It is known that $\sum\limits_{n,m} |\varphi(x-n, y-m)|^2$ is bounded from the proof of (1) in Lemma 3.1. Hence one obtains the uniform convergence, which completes the proof of the theorem.

Remark 3.1. Comparing with Walter's theorem in one dimensional case, the function $\hat{\varphi}^\star(u,v)$ can be thought as a nature extension of $\hat{\varphi}^\star(\omega)$. It should be pointed out that the scaling function in Theorem 3.1 is not even necessarily orthogonal. Instead if φ is a Riesz basis of V_0 and $\hat{\varphi}^\star(u,v) \neq 0$, we still have a sampling theorem. In fact $\varphi(x,y)$ can be orthogonalized to $\Phi(x,y)$ given by

$$\widehat{\Phi}(u,v) = \frac{\hat{\varphi}(u,v)}{\sum\limits_{k,j} |\hat{\varphi}(u+2\pi k, v+2\pi j)|^2}.$$

It is easy to show that $V_0(\varphi) = V_0(\Phi)$ and

$$\widehat{\Phi}^\star(u,v) = \sum_{k,j} \widehat{\Phi}(u+2\pi k, v+2\pi j) = \frac{\hat{\varphi}^\star(u,v)}{\sum\limits_{k,j} |\hat{\varphi}(u+2\pi k, v+2\pi j)|^2}.$$

Hence $\hat{\Phi}^\star(u,v) \neq 0$ if and only if $\hat{\varphi}^\star(u,v) \neq 0$.

Example 3.1 (Tensor Product Form). If φ_1 and φ_2 are orthonormal scaling functions of $L^2(R)$, then $\varphi(x,y) =: \varphi_1(x).\varphi_2(y)$ defines a new orthonormal scaling function of $L^2(R^2)$. In this case it is easy to show $\hat{\varphi}^\star(u,v) = \hat{\varphi_1}^\star(u).\hat{\varphi_2}^\star(v)$. Hence $\hat{\varphi}^\star(u,v)$ has no zeros if neither $\hat{\varphi_1}^\star$ nor $\hat{\varphi_2}^\star(v)$ has. A typical example is given by $\varphi(x,y) = \frac{\sin \pi x}{\pi x}.\frac{\sin \pi y}{\pi y}$, where $\hat{\varphi}^\star(u,v) = 1$. Unfortunately this function does not satisfy the decay condition of Theorem 3.1.

Example 3.2 (Space of Splines). Let

$$\Phi_\alpha(u,v) = \Big\{ \sum_{k,j} [(u-2\pi k)^2 + (v-2\pi j)^2]^{-\alpha} \Big\}^{\frac{1}{2}},$$

$$\hat{\varphi}_\alpha(u,v) = \frac{(u^2+v^2)^{-\frac{\alpha}{2}}}{\Phi_\alpha(u,v)} \qquad \text{with } \alpha > 2.$$

Then φ_α is a scaling function of $L^2(R^2)$ and satisfies all conditions of Lemma 3.1 (see [5, Proposition 6]). It is true that

$$\hat{\varphi_\alpha}^\star(u,v) = \sum_{m,n} \varphi_\alpha(m,n) e^{imu+inv} = \sum_{m,n} \hat{\varphi_\alpha}(u+2\pi m, v+2\pi n) > 0.$$

Therefore a sampling theorem

$$f(x,y) = \sum_{m,n} f(m,n) S^\alpha_{mn}(x,y)$$

follows from Theorem 3.1.

Example 3.3 (Meyer's Function). The Meyer's function $\varphi(x,y)$ is given by

$$\hat{\varphi}(u,v) = \begin{cases} 1, & |u| \le \frac{2\pi}{3} \text{ and } |v| \le \frac{2\pi}{3}, \\ g(x,y), & \text{otherwise}, \\ 0, & |u| \ge \frac{4\pi}{3} \text{ or } |v| \ge \frac{4\pi}{3} \end{cases}$$

with $\hat{\varphi} \in C^2$ and $0 \le \hat{\varphi}(u,v) \le 1$. It is easy to see that

$$0 < m_1 \le \sum_{k,j} \hat{\varphi}(u+2\pi k, v+2\pi j) \le M_1 < +\infty$$

and

$$0 < m_2 \leq \sum_{k,j} |\hat{\varphi}(u + 2\pi k, v + 2\pi j)|^2 \leq M_2.$$

Therefore one can have a sampling theorem in the space $V_0 = \overline{\text{span}}\{\varphi(x-m, y-n)\}_{n,m}$ by Theorem 3.1 and Remark 3.1.

Next we shall extend Theorem 3.1 to the irregular case. It will be shown that under the assumptions of Theorem 3.1, each $f \in V_0$ can be recoved from $f(t_{nm}, s_{nm})$ if both $|t_{nm} - m|$ and $|s_{nm} - n|$ are smaller than an pre-assigned positive number.

Theorem 3.2. *Under all assumptions of Theorem* 3.1 *if the sampling function* $S(x, y)$ *has the same decay condition as* φ *in Lemma* 3.1, *then there exist* $\delta > 0$ *and* $\{S^*_{nm}\} \subseteq V_0$ *such that for each* $f \in V_0$

$$f(x, y) = \sum_{n,m} f(t_{nm}, s_{nm}) S^*_{nm}(x, y)$$

holds uniformly on R^2 *as* $|t_{nm} - n|, |s_{nm} - m| \leq \delta$, *where* $S^*_{nm} \subseteq V_0$.

Proof. By the Remark following Lemma 2.2, it is sufficient to show that

$$A||f||^2 \leq \sum_{n,m} |f(t_{nm}, s_{nm})|^2 \leq B||f||^2.$$

But the proof of Theorem 3.1 implies

$$A_0||f||^2 \leq \sum_{n,m} |f(n,m)|^2 \leq B_0||f||^2. \tag{3.3}$$

Therefore one only needs to prove that there exists $0 < \theta < \sqrt{A_0}$ such that

$$I =: \Big[\sum_{n,m} |f(t_{nm}, s_{nm}) - f(n,m)|^2\Big]^{\frac{1}{2}} \leq \theta||f||^2 \tag{3.4}$$

for each $f \in V_0$. It is known that

$$f(t, s) = \sum_{k_1,k_2 \in Z} f(k_1, k_2) S(t - k_1, s - k_2)$$

from Theorem 3.1 and also it is easy to see that $S(x, y)$ is a continuous function with $S(n, m) = \delta_{nm}$. By using

$$f(t_{nm}, s_{nm}) = \sum_{k_1,k_2 \in Z} f(k_1, k_2) S(t_{nm} - k_1, s_{nm} - k_2),$$

one has

$$\begin{aligned}
I &= \Big[\sum_{n,m} \Big| \sum_{k_1,k_2} f(k_1, k_2) S(t_{nm} - k_1, s_{nm} - k_2) - f(n,m)\Big|^2\Big]^{\frac{1}{2}} \\
&\leq \Big[\sum_{n,m} |f(n,m)|^2 |S(t_{nm} - n, s_{nm} - m) - 1|^2\Big]^{\frac{1}{2}} \\
&\quad + \Big[\sum_{n,m} \Big| \sum_{(k_1,k_2) \neq (n,m)} f(k_1, k_2) S(t_{nm} - k_1, s_{nm} - k_2)\Big|^2\Big]^{\frac{1}{2}} \\
&=: I_1 + I_2.
\end{aligned}$$

Combining (3.3) with continuity of $S(x,y)$ and $S(0,0)=1$, one obtains

$$I_1 = \Big[\sum_{n,m} |f(n,m)|^2 |S(t_{nm}-n, s_{nm}-m)-1|^2\Big]^{\frac{1}{2}} \le \frac{A_0}{4}||f|| \tag{3.5}$$

if $|t_{nm}-n|$ and $|s_{nm}-m|$ are chosen sufficiently small. Also combining the decay condition on S with the continuity of S and $S(n,m)=\delta_{nm}$, one can easily show that for any $0<\epsilon<\sqrt{\frac{A_0}{4B_0}}$

$$\sum_{(k_1,k_2)\ne(n,m)} |S(t_{nm}-k_1, s_{nm}-k_2)| \le \epsilon \tag{3.6}$$

for each $(n,m)\in Z^2$ and

$$\sum_{(n,m)\ne(k_1,k_2)} |S(t_{nm}-k_1, s_{nm}-k_2)| \le \epsilon \tag{3.7}$$

for each $(k_1,k_2)\in Z^2$ as $|t_{nm}-n|$ and $|s_{nm}-m|$ are small enough. Therefore one may conclude that

$$\begin{aligned} I_2^2 &= \sum_{n,m}\Big|\sum_{(k_1,k_2)\ne(n,m)} f(k_1,k_2)S(t_{nm}-k_1,s_{nm}-k_2)\Big|^2 \\ &\le \sum_{n,m}\Big[\sum_{(k_1,k_2)\ne(n,m)} |f(k_1,k_2)|^2|S(t_{nm}-k_1,s_{nm}-k_2)| \\ &\quad \cdot \sum_{(k_1,k_2)\ne(n,m)} |S(t_{nm}-k_1,s_{nm}-k_2)|\Big] \\ &\le \epsilon\sum_{m,n}\sum_{(k_1,k_2)\ne(n,m)} |f(k_1,k_2)|^2|S(t_{nm}-k_1,s_{nm}-k_2)| \\ &= \epsilon\sum_{k_1,k_2}\sum_{(n,m)\ne(k_1,k_2)} |f(k_1,k_2)|^2|S(t_{nm}-k_1,s_{nm}-k_2)| \\ &\le \epsilon^2 B_0||f||^2 < \frac{A_0}{4}, \end{aligned} \tag{3.8}$$

where the first inequality holds due to the Cauchy inequality and the second one does because of (3.6) and the third one follows from (3.3) and (3.7). The formula (3.8) implies $I_2 < \frac{\sqrt{A_0}}{2}||f||$ and furthermore

$$I =: I_1 + I_2 < \frac{3}{4}\sqrt{A_0} \text{ by (3.5).}$$

This shows the inequality (3.4). Similar to Theorem 3.1 the uniform convergence is obtained, which completes the proof.

References

[1] Zayed, A., Advances in Shannon's sampling theory, CRC Press, 1993.
[2] Gröchenig, K., Reconstruction algorithms in irregular sampling, *Math. Comp.,* **59** (1992), 103–125.
[3] Walter, G., A sampling theorem for wavelet subspaces, *IEEE Inform Theory,* **38** (1992), 881–884.
[4] Young, R., An introduction to non-harmonic Fourier series, Academic Press, New York, 1980.
[5] Madych, W., Some elementary properties of multiresolution analysis of $L^2(R^n)$, in Wavelets—A Tutorial in Theory and Applications, edited by C. K. Chui, Academic Press, 1992, 259–294.