Chin. Ann. of Math.
19B: 3(1998),349-358.

THE ABSTRACT CAUCHY PROBLEM AND A GENERALIZATION OF THE LUMER-PHILLIPS THEOREM

Li Yangrong*

Abstract

For injective, bounded operator C on a Banach space X, the author defines the C-dissipative operator, and then gives Lumer-Phillips characterizations of the generators of quasi-contractive C-semigroups, where a C-semigroup $T(\cdot)$ is quasi-contractive if $\|T(t)x\| \leq \|Cx\|$ for all $t \geq 0$ and $x \in X$. This kind of generators guarantee that the associate abstract Cauchy problem $u'(t,x) = Au(t,x)$ has a unique nonincreasing solution when the initial data is in $C(D(A))$ (here $D(A)$ is the domain of A). Also, the generators of quasi-isometric C-semigroups are characterized

Keywords Semigroups of operators, C-semigroups, Dissipative operators, Abstract Cauchy problems

1991 MR Subject Classification 47D03, 47D06

Chinese Library Classification O177.2

§1. Introduction

Recently the theory of C-semigroups has extensively devolopoed by many authors (see [2-8, 14]), but a natural and important question asked by R. deLaubenfels in [3] (page 60, Open question 6.10) remains open: Does there exist an analogue of the Lumer-Phillips theorem, for C-semigroups? Can the numerical range be generalized in such a way as to play the same role for C-semigroups that it does for c_0-semigroups?

Our main purpose of this paper is to investigate above question. We first, in section 2, generalize the numerical range and define the C-dissipative operator. We then see, in Section 3, that the generalized numerical range just plays the same role for C-semigroups as the numerical range does for c_0-semigroups. We obtain several analogues of the Lumer-Phillips Theorem, which give some partial answers to the above question. These results allow us to consider the nonincreasing solution of the abstract Cauchy problem

$$u'(t,x) = Au(t,x), \quad u(0,x) = x. \tag{ACP}$$

In Section 4, we introduce the C-conservative operator and disscuss the isometric solution of (ACP), where "isometric" means that $\|u(t,x)\| = \|x\|$ for all $t \geq 0$.

Throughout this paper, all operators are linear on a complex Banach space X. For an operator A, we write $\rho(A)$, $D(A)$, ImA for its resolvent set, domain and range respectively, and $[D(A)]$ for the Banach space with the graph norm $\|x\|_{[D(A)]} = \|x\| + \|Ax\|$ for $x \in D(A)$. C will always be an injective, bounded operator.

Manuscript received September 19, 1995. Revised November 24, 1996.

*Department of Mathematics, Southwest Normal University, Chongqing 630715, China.

§2. C-Dissipative Operators

Let X^* be the dual of X. For $x \in X$ we define the C-duality set $F_C(x) \subset X^*$ by

$$F_C(x) \equiv \{x^* \in X^* : \ \langle x, x^* \rangle = \|Cx\|, \ \ |\langle y, x^* \rangle| \le \|Cy\| \text{ for all } y \in X\}. \tag{2.1}$$

From the Hahn-Banach Theorem it follows that $F_C(x)$ is nonempty for all $x \in X$, and it is not hard to show that $F_I(x) = F(x)/\|x\|$, where $F(x) = \{x^* \in X^* : \ \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$ is the duality set (see [13]).

Definition 2.1. *An operator A on X is C-dissipative if for all $x \in D(A)$ there exists an $x^* \in F_C(x)$ such that* $\mathrm{Re}\langle Ax, x^* \rangle \le 0$.

Proposition 2.1. *An operator A on X is C-dissipative if and only if*

$$\|C(\lambda - A)x\| \ge \lambda \|Cx\| \quad \text{for all} \quad x \in D(A) \quad \text{and} \quad \lambda > 0. \tag{2.2}$$

Proof. It is analogous to the situation of strongly continuous semigroups (see [13]).

Proposition 2.2. *If $CA \subset AC$, then A is C-dissipative if and only if for each $x \in C(D(A))$ there exists an $x^* \in F(x)$ such that* $\mathrm{Re}\langle Ax, x^* \rangle \le 0$.

Proof. Assume that A is C-dissipative. Let $x \in C(D(A))$ and let $y \in D(A)$ with $x = Cy$. Then there exists a $y^* \in F_C(y)$ such that $\mathrm{Re}\langle Ay, y^* \rangle \le 0$. Define $f : \ \mathrm{Im}C \to \mathbb{C}$ by

$$f(Cz) = \langle z, y^* \rangle \quad (z \in X).$$

Clearly f is well-defined (because C is injective) and linear on $\mathrm{Im}C$. Since $|f(Cz)| = |\langle z, y^* \rangle| \le \|Cz\|$, it follows that f is bounded on $\mathrm{Im}C$ with norm ≤ 1. Hence, by Hahn-Banach Theorem, there exists an $x_0^* \in X^*$ such that $\|x_0^*\| \le 1$ and $\langle Cz, x_0^* \rangle = f(Cz) = \langle z, y^* \rangle$ for all $z \in X$. The second assertion implies that $C^* x_0^* = y^*$. Let now $x^* = \|x\| x_0^*$, then $\mathrm{Re}\langle Ax, x^* \rangle = \mathrm{Re}\langle Ay, C^* x^* \rangle = \|x\| \mathrm{Re}\langle Ay, y^* \rangle \le 0$. Since

$$\langle x, x^* \rangle = \langle Cy, \|x\| x_0^* \rangle = \|x\| \langle y, C^* x_0^* \rangle = \|x\| \langle y, y^* \rangle = \|x\| \|Cy\| = \|x\|^2,$$
$$\|x^*\| \le \|x\| \|x_0^*\| \le \|x\|,$$

it follows that $x^* \in F(x)$, which concludes the proof of the necessity.

Conversely, let $x \in D(A)$ and let $y = Cx \in C(D(A))$. By hypothesis, there exists a $y^* \in F(y)$ such that $\mathrm{Re}\langle Ay, y^* \rangle \le 0$. Let $x^* = C^* y^* / \|y^*\|$. Then $\mathrm{Re}\langle Ax, x^* \rangle = \mathrm{Re}\langle CAx, y^* / \|y^*\| \rangle = \mathrm{Re}\langle Ay, y^* \rangle / \|y^*\| \le 0$, and clearly $x^* \in F_C(x)$. Thus A is C-dissipative.

We conclude this section with some properties of C-dissipative operators.

Proposition 2.3. (a) *If A is C-dissipative, then $\lambda - A$ is injective for all $\lambda > 0$.*

(b) *If A is C-dissipative and closable, then the closure $\overline{A}$ is also C-dissipative.*

(c) *If A is C-dissipative and densely defined, then A is closable.*

Proof. (a) Suppose that $x \in D(A)$ satisfies $(\lambda - A)x = 0$ for some $\lambda > 0$. Then $C(\lambda - A)x = 0$. From the C-dissipativity of A it follows that $\lambda \|Cx\| \le \|C(\lambda - A)x\| = 0$, which implies that $Cx = 0$. Thus $x = 0$ by the injectivity of C.

(b) Let $x \in D(\overline{A})$ and $\lambda > 0$. Then there exist $x_n \in D(A) \to x$ and $\overline{A}x = \lim\limits_{n \to \infty} Ax_n$. Thus $\lambda \|Cx\| = \lim\limits_{n \to \infty} \lambda \|Cx_n\| \le \lim\limits_{n \to \infty} \|C(\lambda - A)x_n\| = \|C(\lambda - \overline{A})x_n\|$. Therefore, by Proposition 2.1, $\overline{A}$ is C-dissipative.

(c) Assume that $x_n \in D(A)$, $x_n \to 0$ and $Ax_n \to y$. For $z \in D(A)$ and $\lambda > 0$, from Proposition 2.1 it follows that $\lambda \|C(x_n + \frac{1}{\lambda} z)\| \le \|C(\lambda - A)(x_n + \frac{1}{\lambda} z)\|$. Letting $n \to \infty$

yields $\|Cz\| \le \|C(z - y + \frac{1}{\lambda}z)\|$. Letting $\lambda \to \infty$ yields $\|Cz\| \le \|C(z-y)\|$. Note that A is densely defined. Letting $z \to y$ yields $\|Cy\| \le 0$, which implies that $y = 0$.

§3. Lumer-Phillips Characterizations for C-Semigroups

In this section, we use the C-dissipative operators to characterize generators of quasi-contractive C-semigroups. In order to state our results we need some preliminaries.

The strongly continuous family of bounded operators $\{T(t)\}_{t\ge 0}$ on X is called a C-semigroup if $T(0) = C$ and $T(t)T(s) = CT(t+s)$ for all $t, s \ge 0$. The generator A of the C-semigroup $\{T(t)\}_{t\ge 0}$ is defined by

$$D(A) = \{x : \lim_{t\to 0} \frac{1}{t}(T(t) - Cx) \text{ exists and is in } \operatorname{Im} C\},$$
$$Ax = C^{-1} \lim_{t\to 0} \frac{1}{t}(T(t) - Cx).$$

We refer to the monograph[4] for the theory of C-semigroups.

Definition 3.1.[14] *Suppose that $S(\cdot)$ is a C-semigroup, A is a closed operator. If*

(1) *$S(t)A \subset AS(t)$ for $t \ge 0$;*

(2) *for $x \in X$, we have $\int_0^t S(r)x\,dr \in D(A)$ and $S(t)x - Cx = A\int_0^t S(r)x\,dr$,*

then $S(\cdot)$ is said to be a C-semigroup for A.

Remark 3.1. There exists at most one C-semigroup for A. If $S(\cdot)$ is the C-semigroup for A, then the extension $C^{-1}AC$ of A is the generator of the C-semigroup $S(\cdot)$.

Definition 3.2. *By a* (*classical*) *solution of the Cauchy problem* (ACP), *we mean $u \in C([0,\infty),[D(A)]) \cap C^1([0,\infty), X)$ satisfying* (ACP). *By a mild solution of* (ACP), *we mean $u \in C([0,\infty), X)$ satisfying*

$$\int_0^t u(s)\,ds \in D(A) \quad \text{and} \quad u(t) = A\int_0^t u(s)\,ds + x$$

for all $t \ge 0$. We say a solution (*or a mild solution*) *$u(\cdot, x)$ is nonincreasing if $\|u(t,x)\| \le \|x\|$ for all $t \ge 0$.*

Definition 3.3. *A C-semigroup $T(\cdot)$ on X is of quasi-contraction if $\|T(t)x\| \le \|Cx\|$ for all $t \ge 0$ and $x \in X$.*

Remark 3.2. (1) A quasi-contractive I-semigroup is just a contractive c_0-semigroup. Conversely, if A generates a contractive c_0-semigroup $S(\cdot)$, and C commutes with $S(\cdot)$, then $T(\cdot) \equiv CS(\cdot)$ is a quasi-contractive C-semigroup. This is a typical example of quasi-contractive C-semigroup, more examples are given in the last section.

(2) It is well-known that if A (or an extension of A) generates a C-semigroup, then (ACP) has a unique solution for all $x \in C(D(A))$ (for instance see [3, 4]). If the C-semigroup is of quasi-contraction, it is not hard to see that the solution of (ACP) is nonincreasing.

We now state our results as follows. In Theorems 3.1-3.3, C is an injective, bounded operator on X. As we point out in Section 1, A is an operator on X. We emphasize, in Theorem 3.3, $D(A)$ is not necessarily dense.

Theorem 3.1. *Suppose that A is densely defined and closed, C has a dense range, $CA \subset AC$. Then the following are equivalent:*

(i) *There exists a quasi-contractive C-semigroup $T(\cdot)$ for A.*

(ii) *A is C-dissipative and*

$$\mathrm{Im}C \subset \bigcap_{n\in\mathbb{N}} \mathrm{Im}(\lambda - A)^n \quad \textit{for all} \quad \lambda > 0. \tag{3.1}$$

(iii) *A is C-dissipative and*

$$\mathrm{Im}C \subset \bigcap_{n\in\mathbb{N}} \mathrm{Im}(\lambda_0 - A)^n \quad \textit{for some} \quad \lambda_0 > 0. \tag{3.2}$$

(iv) *For all* $x \in \mathrm{Im}C$, (ACP) *has a unique nonincreasing mild solution* $u(\cdot, x)$.

Remark 3.3. If $C = I$, Theorem 3.1 reduces to the well-known Lumer-Phillips theorem.

Theorem 3.2. *Suppose that A is densely defined and has a nonempty resolvent set, $CA \subset AC$, then the following are equivalent:*

(i) *A generates a quasi-contractive C-semigroup $S(\cdot)$.*

(ii) *A is C-dissipative and* $\mathrm{Im}(\lambda - A) = X$ *for all* $\lambda > 0$.

Theorem 3.3. *If $0 < r \in \rho(A)$, $n \in \mathbb{N}\cup\{0\}$ and A is $(r-A)^{-n}$-dissipative. Then A generates a quasi-contractive $(r-A)^{-(n+1)}$-semigroup. In this case,* (ACP) *has a unique nonincreasing solution for all $x \in D(A^{n+2})$.*

To prove these theorems, we need some lemmas.

When A has no eigenvalues in $(0, \infty)$, the Hille-Yosida space Z_0 for A is defined by

$$Z_0 = \{x \in X : \text{ for } x, \text{(ACP) has a bounded uniformlycontinuous mild solution } \ u(\cdot, x)\},$$

$\|x\|_{Z_0} = \sup_{t\geq 0} \|u(t,x)\|$ for $x \in Z_0$.

The weak Hille-Yosida space Y for A is defined by

$$Y = \{x \in X : \ x \in \mathrm{Im}(s-A)^n \ \text{ for all } \ s > 0, n \in \mathbb{N} \ \text{ and}$$
$$\|x\|_Y = \sup\{s^n\|(s-A)^{-n}x\|; s > 0, n+1 \in \mathbb{N}\} < \infty\}.$$

Lemma 3.1.[4] (a) *$Z_0 \subset Y$ and $\|x\|_{Z_0} = \|x\|_Y$ for all $x \in Z_0$.*

(b) *Z_0 is the closure, in Y, of $D(A|_Y)$.*

Lemma 3.2.[14] *If A is closed, $CA \subset AC$, then* (ACP) *has a unique mild solution $u(\cdot,x)$ for all $x \in \mathrm{Im}C$ if and only if there exists a C-semigroup $T(\cdot)$ for A. In this case $T(t)x = u(t, Cx)$ for all $t \geq 0$ and $x \in X$.*

Lemma 3.3. *If there exists a quasi-contractive C-semigroup $T(\cdot)$ for A, then A is C-dissipative.*

Proof. Since $CA \subset AC$, we can use Proposition 2.2. Let $x \in C(D(A))$ and $x^* \in F(x)$. Let $y \in D(A)$ with $x = Cy$. By hypothesis we have

$$|\langle T(t)y, x^*\rangle| \leq \|T(t)y\|\cdot\|x^*\| \leq \|Cy\|\|x^*\| = \|x\|^2$$

for all $t > 0$. Thus

$$\mathrm{Re}\langle T(t)y - Cy, x^*\rangle = \mathrm{Re}\langle T(t)y, x^*\rangle - \mathrm{Re}\langle Cy, x^*\rangle = \mathrm{Re}\langle T(t)y, x^*\rangle - \|x\|^2 \leq 0.$$

Since the generator of $T(\cdot)$ is the extension $C^{-1}AC$ of A,

$$\mathrm{Re}\langle CAy, x^*\rangle = \lim_{t\to 0} \mathrm{Re}\Big\langle \frac{T(t)y - Cy}{t}, x^*\Big\rangle \leq 0,$$

that is, Re $\langle Ax, x^*\rangle \leq 0$. Therefore, by Proposition 2.2, A is C-dissipative.

Proof of Theorem 3.1. (i) $\Leftrightarrow$ (iv) follows from Lemma 3.2 and Remark 3.2 (2) of Definition 3.3. (ii) $\Longrightarrow$ (iii) is obvious.

(i) $\Longrightarrow$ (ii). From Lemma 3.3 it follows that A is C-dissipative. Hence, by Proposition 2.2, A has no eigenvalues in $(0,\infty)$. Since $D(A)$ is dense in X, for $x \in X$, the mild solution $u(t, Cx) \equiv T(t)x$ is bounded uniformly continuous (see [4, Remark 5.9]), that is, $Cx \in Z_0$, the Hille-Yosida space. Thus, by Lemma 3.1, $Cx \in Y$ and (3.1) holds at once.

(iii) $\Longrightarrow$ (ii). Let $\Lambda = \{\lambda \in (0,\infty) : \ \mathrm{Im}C \subset \bigcap_{n\in N} \mathrm{Im}(\lambda - A)^n\}$. We have to prove that $\Lambda = (0,\infty)$. By hypothesis, we may assume that $0 < r \in \Lambda$. We shall show that $(r - r_0, r + r_0) \subset \Lambda$ for any r_0 with $0 < r_0 < r$. To this end, for $x \in X$ we define the series

$$R_1(s) = \sum_{k=0}^{\infty} (r-s)^k (r-A)^{-(k+1)} Cx, \quad |s-r| < r_0, \tag{3.3}$$

and we claim that the series (3.3) converges uniformly on $\{s \in \mathbb{C} : \ |s-r| < r_0\}$. It suffices to show that

$$\|(r-A)^{-k} Cx\| \le \frac{1}{r^k} \|Cx\|; \quad k = 1, 2, \cdots . \tag{3.4}$$

Let $Cx = (r-A)^k y$. Then $C^2 x = C(r-A)^k y = (r-A)^k Cy$. It follows from Proposition 2.1 that

$$\begin{aligned} \|C^2 x\| &= \|C(r-A)(r-A)^{k-1} y\| \ge r \|C(r-A)^{k-1} y\| \\ &\ge r^2 \|C(r-A)^{k-2} y\| \ge \cdots \ge r^k \|Cy\| = r^k \|(r-A)^{-k} C^2 x\|. \end{aligned} \tag{3.5}$$

Since $\mathrm{Im}C$ is dense in X, there exists $x_n \in X$, $n = 1, 2, \cdots$ such that $Cx_n \to x$ as $n \to \infty$. By (3.5) we have

$$\|C^2 x_n\| \ge r^k \|(r-A)^{-k} C^2 x_n\|, \quad k, n = 1, 2, \cdots . \tag{3.6}$$

Since A is closed, $(r-A)^{-k} C$ is bounded. Letting $n \to \infty$ in (3.6) we obtain (3.4), and the series (3.3) converges uniformly on $\{s \in \mathbb{C} : \ |s-r| < r_0\}$.

Note that each partial sum in (3.3) is in $D(A)$ and the series

$$\begin{aligned} &\sum_{k=0}^{\infty} A(r-s)^k (r-A)^{-(k+1)} Cx \\ =& \sum_{k=0}^{\infty} (r-(r-A))(r-s)^k (r-A)^{-(k+1)} Cx \\ =& \sum_{k=0}^{\infty} r(r-s)^k (r-A)^{-(k+1)} Cx - \sum_{k=0}^{\infty} (r-s)^k (r-A)^{-k} Cx = sR_1(s) - Cx \end{aligned}$$

converges. Thus, since A is closed, $R_1(s) \in D(A)$ with $(s-A)R_1(s) = Cx$ for $|s-r| < r_0$. This proves that $\mathrm{Im}C \subset \mathrm{Im}(s-A)$ for $|s-r| < r_0$.

Fix now $s > 0$ with $|s-r| < r_0$. We claim that

$$\begin{gathered} R_1(s) \in D((r-A)^{-m}) \equiv \mathrm{Im}(r-A)^m, \\ \|(r-A)^{-m} R_1(s)\| \le \frac{M}{r^m}, \ m = 1, 2, \cdots , \end{gathered} \tag{3.7}$$

where $M > 0$ is constant and independent of m. Indeed, from (3.4) it follows that the series

$$\sum_{k=0}^{\infty} (r-A)^{-1} (r-s)^k (r-A)^{-(k+1)} Cx$$

converges uniformly on $|s-r|<r_0$. Thus, since $(r-A)^{-1}$ is closed, $R_1(s)\in D((r-A)^{-1})$ with

$$(r-A)^{-1}R_1(s)=\sum_{k=0}^{\infty}(r-A)^{-1}(r-s)^k(r-A)^{-(k+1)}Cx.$$

Let $M\equiv\sum\limits_{k=0}^{\infty}\frac{(r-s)^k}{r^{k+1}}\|Cx\|$. Then by (3.4) we have

$$\|(r-A)^{-1}R_1(s)\|\le\sum_{k=0}^{\infty}|r-s|^k\|(r-A)^{-(k+2)}Cx\|\le\frac{M}{r}.$$

This proves that (3.7) is valid for $m=1$. Iterating this inequality yields (3.7) for all $m=2,3,\cdots$. Hence we can define

$$R_2(t)=\sum_{j=0}^{\infty}(r-t)^j(r-A)^{-(j+1)}R_1(s),\quad |t-r|<r_0, \tag{3.8}$$

and the series (3.8) converges uniformly in $\{t>0:\ |t-r|<r_0\}$. Similar to the previous argument, we have

$$R_2(t)\in D(A)\ \text{ and }\ (t-A)R_2(t)=R_1(s),\quad |t-r|<r_0. \tag{3.9}$$

In particular, taking $t=s$ in (3.9) we obtain

$$(s-A)R_2(s)=R_1(s)\text{ for }|s-r|<r_0. \tag{3.10}$$

Thus $Cx=(s-A)R_1(s)=(s-A)^2R_2(s)\in\mathrm{Im}(s-A)^2$ for $|s-r|<r_0$. Repeating this proceeding we obtain $Cx\in\mathrm{Im}(s-A)^n$ for $|s-r|<r_0,\ n\in\mathbb{N}$. Since $x\in X$ is arbitrary, we have $\{s>0:\ |s-r|<r_0\}\subset\Lambda$; and by the arbitrariness of r_0 with $0<r_0<r$, we have $(0,2r)\subset\Lambda$. Thus $\frac{3}{2}r\in\Lambda$, which implies that $(0,3r)\subset\Lambda$; and $\frac{5}{2}r\in\Lambda$, which implies that $(0,5r)\subset\Lambda,\cdots$. Therefore $(0,\infty)\subset\Lambda$, and (3.1) holds.

(ii) $\Longrightarrow$ (i). Similar to the proof of (3.4), we have

$$\|s^n(s-A)^{-n}Cx\|\le\|Cx\|\ \text{ for all }\ s>0,\quad n\in\mathbb{N},\quad x\in X,$$

that is, $\mathrm{Im}C\subset Y$, the weak Hille-Yosida space, and $\|Cx\|_Y\le\|Cx\|$ for all $x\in X$. Since A is densely defined, for $x\in X$, there exists a sequence $\{x_n\}\subset D(A)$ such that $x_n\to x$ as $n\to\infty$, and hence $\|Cx_n-Cx\|_Y\le\|C(x_n-x)\|\to0$, i.e. $Cx_n\to Cx$ in the topology of Y. This together with $Cx_n\in D(A|_Y)$ implies that $Cx\in\overline{D(A|_Y)}=Z_0$ by Lemma 3.1. Therefore $\mathrm{Im}C\subset Z_0$ and from Lemma 3.2 it follows that there exists a C-semigroup $T(\cdot)$ for A with $T(t)x=u(t,Cx)$ for all $x\in X$ and $t\ge0$. Moreover, for $t\ge0$, we have $\|T(t)x\|\le\sup\limits_{s\ge0}\|u(s,Cx)\|=\|Cx\|_{Z_0}=\|Cx\|_Y\le\|Cx\|$, concluding the proof.

Proof of Theorem 3.2. (a) $\Longrightarrow$ (b). The C-dissipativity of A follows from Lemma 3.3, and by [3, Theorem 5.2], $(0,\infty)\subset\rho(A)$.

(b) $\Longrightarrow$ (a). Since $\rho(A)\ne\emptyset$, A is closed. For $\lambda>0$, by Proposition 2.3, $\lambda-A$ is injective. Thus $(0,\infty)\subset\rho(A)$. We claim that

$$\|\lambda^k(\lambda-A)^{-k}Cx\|\le\|Cx\|\ \text{ for }\ \lambda>0,\quad k\in\mathbb{N}\ \text{ and }x\in X. \tag{3.11}$$

Indeed, since $CA\subset AC$, it follows that $(\lambda-A)^{-1}C=C(\lambda-A)^{-1}$ for all $\lambda>0$. Note that $(\lambda-A)^{-1}x\in D(A)$. By Proposition 2.1, we have

$$\lambda\|(\lambda-A)^{-1}Cx\|=\lambda C(\lambda-A)^{-1}x\|\le\|C(\lambda-A)(\lambda-A)^{-1}x\|=\|Cx\|.$$

This proves (3.11) for $k = 1$. Iterating above inequality yields (3.11) for all $k = 2, 3, \cdots$.

Note that A is densely defined. Similar to the proof of (ii) $\Longrightarrow$ (i) in Theorem 3.1, from (3.11) it follows that there exists a C-semigroup $S(\cdot)$ for A such that $\|S(t)x\| \le \|Cx\|$ for all $x \in X$ and $t \ge 0$. Moreover, since $\rho(A)$ is nonempty, it follows from [4, Proposition 3.9] that A is exactly the generator of the C-semigroup $S(\cdot)$.

Proof of Theorem 3.3. We first prove that $(0, \infty) \subset \rho(A)$. Let

$$\Lambda = \{\lambda \in (0, \infty) : \ \mathrm{Im}(\lambda - A) = X\}.$$

Since $\rho(A)$ is open, Λ is an open subset of $(0, \infty)$. We have only to show that Λ is closed in $(0, \infty)$. To this end, let $\lambda_k \in \Lambda$, $\lambda_k \to \lambda_0$, $\lambda_0 > 0$. For any $x \in X$, there exists a sequence $\{y_k\} \subset D(A)$ such that $x = (\lambda_k - A)y_k$, $k = 1, 2, \cdots$. Since A is $(r - A)^{-n}$-dissipative, we have

$$\|(r - A)^{-n} y_k\| \le \frac{1}{\lambda_k} \|(r - A)^{-n}(\lambda_k - A)y_k\| = \frac{1}{\lambda_k} \|(r - A)^{-n} x\| \le M \quad (k \in \mathbb{N})$$

and for $k_1, k_2 \in \mathbb{N}$,

$$\begin{aligned}\lambda_{k_1} \|(r - A)^{-n} y_{k_1} - (r - A)^{-n} y_{k_2}\| &\le \|(\lambda_{k_1} - A)(r - A)^{-n}(y_{k_1} - y_{k_2})\| \\ &= |\lambda_{k_1} - \lambda_{k_2}| \|(r - A)^{-n} y_{k_2}\|.\end{aligned}$$

Thus $\{(r - A)^{-n} y_k\}_{k=1}^{\infty}$ is a Cauchy sequence. Writing $(r - A)^{-n} y_k \to z$ as $k \to \infty$, we have

$$A(r - A)^{-n} y_k = \lambda_k (r - A)^{-n} y_k - (r - A)^{-n} x \to \lambda_0 z - (r - A)^{-n} x \quad (\text{as } k \to \infty).$$

Thus, since $\rho(A) \ne \emptyset$, A is closed, we have $z \in D(A)$ and $Az = \lambda_0 z - (r - A)^{-n} x$, which implies that $x = (r - A)^n (\lambda_0 - A)z = (\lambda_0 - A)(r - A)^n z \in \mathrm{Im}(\lambda_0 - A)$. Hence $\lambda_0 \in \Lambda$ and Λ is closed in $(0, \infty)$. Therefore $\Lambda = (0, \infty)$. Since A is $(r - A)^{-n}$-dissipative, it follows that $\|\lambda^k (\lambda - A)^{-k} (r - A)^{-n} x\| \le \|(r - A)^{-n} x\|$ for $\lambda > 0$, $k \in \mathbb{N}$ and $x \in X$. Hence $Im(r - A)^{-n} \subset Y$, and for $y \in X$ we have

$$\begin{aligned}(r - A)^{-(n+1)} y &= (r - A)^{-n}(r - A)^{-1} y \in D(A) \cap Y, \\ A(r - A)^{-(n+1)} y &= (r - A)^{-n} A (r - A)^{-1} y \in Y.\end{aligned}$$

This implies that $D(A^{n+1}) = \mathrm{Im}(r - A)^{n+1} \subset D(A|_Y) \subset Z_0$. Thus by Lemma 3.2 there exists an $(r - A)^{-(n+1)}$-semigroup $T(\cdot)$ for A. Note that $((r - A)^{-(n+1)})^{-1} A (r - A)^{-(n+1)} = A$, A is exactly the generator of $T(\cdot)$. Finally, by Lemma 3.1, we have

$$\begin{aligned}\|T(t)x\| &= \|u(t, (r - A)^{-(n+1)} x)\| \le \|(r - A)^{-(n+1)} x\|_{Z_0} \\ &= \|(r - A)^{-(n+1)} x\|_Y \le \|(r - A)^{-(n+1)} x\|\end{aligned}$$

for all $t > 0$ and $x \in X$. Therefore the $(r - A)^{-(n+1)}$-semigroup is quasi-contractive.

§4. Quasi-Isometric C-Semigroups and C-Conservative Operators

If an operator A generates an isometric c_0-semigroup, then for $x \in D(A)$, (ACP) has a unique isometric solution (that is, $\|u(t, x)\| = \|x\|$). In the case of C-semigroups, to guarantee that (ACP) has also an isometric solution (with initial data in $C(D(A))$), one must have

$$\|T(t)x\| = \|Cx\| \quad \text{for} \ \ x \in X, \quad t \ge 0. \tag{4.1}$$

In this section, we characterize the generators of C-semigroups satisfying (4.1).

Definition 4.1. *A C-semigroup $T(\cdot)$ is said to be quasi-isometric if $T(\cdot)$ satisfies* (4.1).

Definition 4.2. *An operator A is said to be C-conservative if for every $x \in D(A)$ there exists an $x^* \in F_C(x)$ such that* $\mathrm{Re}\langle Ax, x^*\rangle = 0$.

Similar to the proof of Proposition 2.2, we have the following characterization of C-conservative operators. The proof is omitted.

Proposition 4.1. *If $CA \subset AC$, then A is C-conservative if and only if for each $x \in C(D(A))$ there exists an $x^* \in F(x)$ such that* $\mathrm{Re}\langle Ax, x^*\rangle = 0$.

Theorem 4.1. *Assume that A is a densely defined, closed operator, C has a dense range, $CA \subset AC$. Then the following are equivalent:*

(a) *There exists a quasi-isometric C-semigroup $T(\cdot)$ for A.*

(b) *A is C-conservative and*

$$\mathrm{Im}C \subset \bigcap_{n\in N} \mathrm{Im}(\lambda - A)^n \quad \textit{for some} \quad \lambda > 0. \tag{4.2}$$

(c) *For all $x \in \mathrm{Im}C$,* (ACP) *has a unique mild solution $u(\cdot, x)$ such that $\|u(t,x)\| = \|x\|$ for all $t \geq 0$.*

Proof. (a) $\Leftrightarrow$ (c) follows from Lemma 3.3.

(a) $\Longrightarrow$ (b). (4.2) follows from Theorem 3.1. We show only that A is C-conservative. Let $z \in D(A)$, $t > 0$ and $z_t^* \in F(T(t)y)$. Then the scalar function $\phi(s) \equiv \mathrm{Re}\langle T(s)z, z_t^*\rangle$ $(s \geq 0)$ is continuously differentiable and has a relative maximum at $s = t$. This deduces

$$\mathrm{Re}\langle T(t)Az, z_t^*\rangle = 0, \quad \text{for} \quad z \in D(A), \quad t > 0 \text{ and } z_t^* \in F(T(t)z). \tag{4.3}$$

Let now $x \in C(D(A))$ and let $y \in D(A)$ with $x = Cy$. Let $\{t_n\}$ be a sequence of positive number tending to zero and $x_n^* \in F(T(t_n)y)$, $n = 1, 2, \cdots$. Since $\{x_n^*\}$ is bounded $(\|x_n^*\| = \|T(t_n)y\| = \|Cy\|)$, by passing, if necessary, to a subsequence, we can assume that $x_n^* \to x^*$ weakly. We claim that $x^* \in F(x)$ and $\mathrm{Re}\langle Ax, x^*\rangle = 0$. Indeed, $\mathrm{Re}\langle Ax, x^*\rangle = \mathrm{Re}\langle CAy, x^*\rangle = \lim_{n\to\infty} \langle T(t_n)Ay, x_n^*\rangle = 0$ in view of (4.3), and since $\langle x, x^*\rangle = \langle Cy, x^*\rangle = \lim_{n\to\infty} \langle T(t_n)y, x_n^*\rangle = \lim_{n\to\infty} \|T(t_n)y\|^2 = \|Cy\|^2 = \|x\|^2$ and $\|x^*\| \leq \lim_{n\to\infty} \|x_n^*\| \leq \|Cy\| = \|x\|$, $x^* \in F(x)$ holds at once. Therefore, by Proposition 4.1, A is C-conservative.

(b) $\Longrightarrow$ (a). If A is C-conservative, it is also C-dissipative. Hence, by Theorem 3.1, there exists a C-semigroup $T(\cdot)$ for A such that $\|T(t)x\| \leq \|Cx\|$ for all $x \in X$ and $t \geq 0$.

Suppose that $u \in C(D(A^2))$ and $u \neq 0$. We shall show that

$$\Lambda \equiv \Lambda(u) \equiv \{t \geq 0 : \ \|T(t)u\| = \|Cu\|\} = (0, \infty). \tag{4.4}$$

Note that $u \in D(A^2)$. If $t, h \geq 0$, we calculate that

$$T(t+h)u = T(t)u + hT(t)Au + h\rho(t,h), \tag{4.5}$$

where $\rho(t,h) = \frac{1}{h}\int_0^h (h-s)T(s+t)A^2u\, ds$. Since $\|T(s+t)A^2u\| \leq \|CA^2u\|$, it follows that $\rho(t,h) \to 0$ as $h \to 0$ uniformly in $t \geq 0$. Note that $T(t)u \in C(D(A))$. By Proposition 4.1, there exists an $x_t^* \in F(T(t)u)$ such that $\mathrm{Re}\langle AT(t)u, x_t^*\rangle = 0$. Thus, in view of (4.5), we have

$$\begin{aligned} \|T(t)u\|\|T(t+h)u\| &\geq |\langle x_t^*, T(t+h)u\rangle| \geq \mathrm{Re}\langle T(t+h)u, x_t^*\rangle \\ &= \|T(t)u\|^2 + h\mathrm{Re}\langle AT(t)u, x_t^*\rangle + h\mathrm{Re}\langle \rho(t,h), x_t^*\rangle \\ &\geq \|T(t)u\|^2 - h\|Cu\|\|\rho(t,h)\|. \end{aligned} \tag{4.6}$$

Suppose that $t_0 \equiv \sup\{t: \ t \in \Lambda\} < \infty$. Then there exists a sequence $\{t_n\} \subset \Lambda$ such that $t_n \to t_0$ and $\|T(t_0)u\| = \lim\limits_{n\to\infty} \|T(t_n)u\| = \|Cu\|$, which implies that $t_0 \in \Lambda$ and $\|T(t_0)u\| \neq 0$. Choose $\alpha > 0$ so small that $\|T(t)u\|$ is bounded away from zero in $t_0 \le t \le t_0 + \alpha$. For any such t we devide (4.6) by $\|T(t)u\|$; the result is

$$\|T(t+h)u\| \ge \|T(t)u\| - h\eta(t,h), \tag{4.7}$$

where η is nonnegative and $\eta(t,h) \to 0$ as $h \to 0$ uniformly in $t_0 \le t \le t_0 + \alpha$. For any $\epsilon > 0$, let $\delta > 0$ such that $|\eta(t,h)| \le \epsilon$ for $0 \le h \le \delta$ and $t_0 \le t \le t_0 + \alpha$. Let $t_0 < t_1 < \cdots < t_m = t_0 + \alpha$ be a partition of the interval $[t_0, t_0+\alpha]$ such that $t_j - t_{j-1} \le \delta$ $(1 \le j \le m)$. In view of (4.7) we calculate as follows:

$$\begin{aligned} 0 \le \|T(t_0)u\| - \|T(t_0+\alpha)u\| &= \sum_{j=1}^{m} (\|T(t_{j-1})u\| - \|T(t_j)u\|) \\ \le \sum_{j=1}^{m} (t_j - t_{j-1})\eta(t_j, t_j - t_{j-1}) &\le \sum_{j=1}^{m} (t_j - t_{j-1})\epsilon = \alpha\epsilon. \end{aligned}$$

Since ϵ is arbitrary, it follows that $\|T(t_0+\alpha)u\| = \|T(t_0)u\| = \|Cu\|$, which contradicts the fact that $t_0 = \sup\{t: \ t \in \Lambda\}$. Thus $\sup\{t: \ t \in \Lambda\} = \infty$.

Assume now $s \ge 0$. Then there exists an $s_0 \ge s$ such that $s_0 \in \Lambda$. Let $y \in D(A^2)$ such that $u = Cy$. We have

$$\|Cu\| = \|T(s_0)u\| = \|T(s_0)Cy\| = \|T(s_0 - s)T(s)y\| \le \|CT(s)y\| = \|T(s)u\| \le \|Cu\|,$$

which implies that $s \in \Lambda$. Therefore $\Lambda(u) = [0,\infty)$ for $u \in C(D(A^2))$.

We finally claim that $C(D(A^2))$ is dense in X. For each $z \in D(A)$, we have

$$\int_0^s T(r)z\,dr \in D(A^2) \text{ and } Cz = \lim_{s\to 0} \int_0^s T(r)z\,dr.$$

Thus $C(D(A)) \subset \overline{D(A^2)}$. Note that both $D(A)$ and ImC are dense in X, $C(D(A))$ and hence $D(A^2)$ are also dense in X, which implies that $C(D(A^2))$ is dense in X. By use of this fact, a standard approximation argument shows that $\Lambda(u) = [0,\infty)$ is valid for all $u \in X$, which completes the proof.

§5. Examples

We present, in this section, some examples of quasi-contractive (or quasi-isometric) C-semigroups.

Example 5.1. If A satisfies the Hille-Yosida condition, that is, $(0,\infty) \subset \rho(A)$ and $\|s(s-A)^{-1}\| \le 1$ for all $s > 0$, then by Theorem 3.3 (for the case $n = 0$) A generates a quasi-contractive $(r-A)^{-1}$-semigroup, where $r > 0$. We mention that A is not necessarily densely defined (for instance, see [10]), then A does not necessarily generate a c_0-semigroup.

In particular, if A has no eigenvalues in $(0,\infty)$, Y is the weak Hille-Yosida space for A, then the part $A|_Y$, of A on Y, satisfies the Hille-Yosida condition (see [4]). Thus $A|_Y$ generates a quasi-contractive $(r - A|_Y)^{-1}$-semigroup on Y $(r > 0)$.

We say a semigroup $T(\cdot)$ of unbounded operators (see [11]) is contractive (resp. isometric) if $\|T(t)x\| \le \|x\|$ (resp. $\|T(t)x\| = \|x\|$) for all $x \in D$ and $t \ge 0$.

Proposition 5.1. *If A generates a contractive (resp. isometric) semigroup $T(\cdot)$ of unbounded closed operator, and there exists an injective, bounded operator C such that* $\mathrm{Im}C \subset D$ *and* $CT(t)C = T(t)C^2$ *for* $t \geq 0$, *then* $T(\cdot)C$ *is a quasi-contractive (resp. quasi-isometric) C-semigroup generated by an extension of A.*

Proof. It is immediate.

The following example is due to R. deLaubenfels[3]; here, we point out that the $(1-A)^{-1}$-semigroup $S(\cdot)$ generated by A is quasi-isometric.

Example 5.2. Let $X = C_0(-\infty, 0]$. Define

$$T(t)f(s) = \begin{cases} f(s+t), & s+t \leq 0, \\ 0, & s+t > 0, \end{cases}$$

for $f \in X$ and $t \geq 0$. $\{T(t)\}_{t\geq 0}$ is an isometric semigroup of unbounded closed operator on $D \equiv \{f \in C_0(-\infty, 0] : \ f(0) = 0\}$. ($T(\cdot)$ is not a C_0-semigroup, because, for $f \notin D$, $T(t)f$ is not continuous. Its generator is

$$D(A) = \{f \in C_0(-\infty, 0] \cap C_0^1(-\infty, 0] : \ f(0) = 0\},$$
$$Af = \frac{d}{dx} f \quad \text{for} \quad f \in D(A).$$

It is clear that $(0, \infty) \subset \rho(A)$. Thus, by Proposition 5.1, an extension of A generates a quasi-isometric $(1-A)^{-1}$-semigroup $S(t) \equiv T(t)(1-A)^{-1}$ defined by

$$S(t)f(s) = \begin{cases} e^{s+t} \int_{s+t}^{0} e^{-r} f(r)\, dr, & s+t \leq 0, \\ 0, & s+t > 0, \end{cases}$$

for $f \in X$. Since $\rho(A) \neq \emptyset$, A is exactly the generator.

References

[1] Arendt, W., Vector valued Laplace transforms and Cauchy problems, *Isreal J. Math.,* **59** (1987), 327–352.

[2] Davies, E. B. & Pang, M. M., The Cauchy problem and a generalization of the Hille-Yosida theorem, *Proc. London Math. Soc.,* **55** (1987), 181–208.

[3] DeLaubenfels, R., C-semigroups and the Cauchy problem, *J. Func. Anal.,* **111** (1993), 44–61.

[4] DeLaubenfels, R., Existence families, functional calculus and evolution equations, *Lecture Notes in Math. Soc.,* **1570**, 1994.

[5] DeLaubenfels, R., Existence and uniqueness families for the abstract Cauchy problem, *Semigroup Forum,* **44** (1991), 310–338.

[6] DeLaubenfels, R., Integrated semigroups, C-semigroups and the abstract Cauchy problem, *Semigroup Forum,* **41** (1990), 83–95.

[7] DeLaubenfels, R. C-semigroups and strongly continuous semigroups, *Isreal J. Math.,***81** (1993), 227–255.

[8] DeLaubenfels, R. & S.Kantorovitz, Laplace and Laplace-Stieltjies spaces, *J. Func. Anal.,* **116** (1993), 1–61.

[9] Fattorini, H. O., The Cauchy problem, Addison Wesley, Reading Mass, 1983.

[10] Goldstein, J. A., Semigroups of operators and applications, Oxford, New York, 1985.

[11] Hughes, R. J., Semigroup of unbounded linear operators in Banach spaces, *Trans. Amer. Math. Soc.,* **230** (1977), 113–145.

[12] Kantorovitz, S., The Hille-Yosida space of an arbitrary operator, *J. Math.Anal. and Appl.,* **136** (1988), 107–111.

[13] Pazy, A., Semigroups of linear operators and applications to partial differential equations, Springer, New York, 1983.

[14] Sun, G. Integrated C-semigroups, local C-semigroups, C-existence families and Cauchy problem, Ph. D. dissertation, Nanjing University, 1993.

[15] Yosida, K., Functional analysis, Springer, Berlin, 1978.