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ON THE NON-EXISTENCE OF LIMIT CYCLES OF CERTAIN QUADRATIC SYSTEMS**

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Abstract

In §1 and §3, two conjectures mentioned by Ye Yanqian are studied. In §2, by use of elementary methods the author proves some non-existence theorems of limit cycles (LC, for abbreviation) for quadratic differential systems obtained recently by H. Giacomini, J. Llibre and M. Viano.

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§1.

For the system

$$\begin{cases} \dot{x} = -y + \delta x + lx^2 + ny^2 = P(x,y), \\ \dot{y} = x(1 + ax - y) = Q(x,y), \end{cases} \tag{1.1}$$

we can find in [1] the following:

Conjecture I. Assume[1]

$$a < 0,\ n > 1,\ n + l > 0,\ na^2 + l < 0,\ na^2 < (n-1)(l+n)^2. \tag{1.2}$$

Then around the anti-saddle $S_1(x_1, y_1)$ $(x_1 > 0, y_1 < 1)$ lying on $1 + ax - y = 0$, there exists no LC for any δ.

From Theorem 1 in [1], we know that $a^2 < 4(n-1)(1-l)$ under condition (1.2). This means that S_1 can never be on the line of divergence

$$\operatorname{div} = \delta + (2l-1)x = 0$$

for any δ, so S_1 is always a stable node or focus. The non-existence of LC around S_1 when $\delta \le 0$ can be proved by using the Dulac function $B(x,y) = (1-y)^{2l-1}$, because

$$\frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} = (1-y)^{2l-2}[\delta(1-y) + a(1-2l)x^2]$$

and the function in [ ] is of constant sign (≤ 0) below the line y = 1. The phase-portraits of $(1)_{\delta\le 0}$ in the neighbourhood of S_1 is shown in Fig.1, where l_1 is the separatrix entering the saddle $N(0, \frac{1}{n})$ from below, and L its tangent at N. S_1 lies always above L, since on L

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[1]Under (1.2), system (1.1) has three anti-saddles.

there can be no finite critical point other than N, although y_1 may become negative. This is the reason why Conjecture I was made.

Fig.1

Now, if we try to prove this conjecture for $\delta > 0$ by using the Dulac function

$$B(x,y) = (1-y)^{-\frac{\delta}{k_1}-1+2l}\Big(y - \frac{1}{n} - k_1 x\Big)^{\frac{\delta}{k_1}},$$

where $k_1 < 0$ is a negative root of

$$nk^2 + n\delta k + 1 - n = 0$$

and $y - \frac{1}{n} = k_1 x$ is the equation of L, then we have

$$\begin{aligned}\frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} &= \Big(y - \frac{1}{n} - k_1 x\Big)^{\frac{\delta}{k_1}-1}(1-y)^{-\frac{\delta}{k_1}+2l-2}\Big\{\Big[-\frac{\delta}{n}(1-ny)^2 \\ &\quad -\delta\Big(l+1-\frac{a}{k_1}\Big)x^2\Big](1-y) + a\Big(\frac{\delta}{k_1}+1-2l\Big)x^2\Big(y-\frac{1}{n}-k_1 x\Big)\Big\}. \qquad (1.3)\end{aligned}$$

Notice that in the $\{\}$ of (1.3), since

$$k_1 = \frac{-n\delta - \sqrt{n^2\delta^2 + 4n(n-1)}}{2n},$$

we have $\frac{\delta}{k_1} > -\frac{2n\delta}{2n\delta} = -1$ and $\frac{\delta}{k_1} + 1 - 2l > 0$. If we assume

$$l + 1 - \frac{a}{k_1} > 0, \qquad (1.4)$$

then the polynomial in $\{\}$ of (1.3) is negative everywhere in the region $x > 0,\ y < 1,\ y - \frac{1}{n} - k_1 x > 0$ containing S_1. This ensures the non-existence of LC around S_1.

Unfortunately, condition (1.4) is too strong; it is equivalent to

$$l > -1 \text{ and } a^2 n < (n-1)(1+l)^2 - na\delta(1+l). \qquad (1.5)$$

When $0 \le \delta << 1$, (1.5) is stronger than the last inequality in (1.2).

In the following, we retain the conditions in (1.2), and try to use other methods to study Conjecture I.

Fix $\delta = \delta_1 > 0$ in (1.1):

$$\begin{cases}\dot{x} = -y + \delta_1 x + lx^2 + ny^2, \\ \dot{y} = x(1 + ax - y).\end{cases} \qquad (1.6)$$

Let us study the system:

$$\begin{cases} \dot{x} = -y + \delta_1 x + lx^2 + ny^2 + \delta_2 x(1 + ax - y), \\ \dot{y} = x(1 + ax - y), \end{cases} \tag{1.7}$$

when δ_2 decreases from zero. Notice that the finite critical points of (1.6) and (1.7) are the same. The line of divergence for (1.7) is

$$(\delta_1 + \delta_2) + (2l + 2a\delta_2 - 1)x - \delta_2 y = 0. \tag{1.8}$$

In order that S_1 lies on (8), i.e.,

$$(\delta_1 + \delta_2) + (2l + 2a\delta_2 - 1)x_1 - \delta_2(1 + ax_1) = \delta_1 + (2l + a\delta_2 - 1)x_1 = 0,$$

we must take

$$\delta_2 = \frac{(1 - 2l)x_1 - \delta_1}{ax_1}. \tag{1.9}$$

Transform the origin to S_1, (1.6) becomes

$$\begin{cases} \dot{x} = x_1 x + (2ny_1 - 1 - \delta_2 x_1)y + (l + a\delta_2)x^2 - \delta_2 xy + ny^2, \\ \dot{y} = ax_1 x - x_1 y + ax^2 - xy. \end{cases} \tag{1.10}$$

Then make the following change of variables:

$$\frac{dt}{d\tau} = \frac{1}{b}; x = -\frac{1}{ab_1}\eta - \frac{1}{ax_1}\xi, y = -\frac{1}{b_1}\eta \quad \text{or} \quad \xi = -ax_1 x + x_1 y, \eta = -b_1 y, \tag{1.11}$$

where

$$b_1 = \sqrt{-x_1[(2l + 2na^2)x_1 + \delta_1 + (2n - 1)a]} \tag{1.12}$$

and $\pm ib_1$ are characteristic roots of the linear part of (1.10) at (0,0).

Since x_1 is a root of

$$(l + na^2)x^2 + (\delta_1 + a(2n - 1))x + n - 1 = 0, \tag{1.13}$$

we have $b_1 = \sqrt{\delta_1 x_1 + a(2n - 1)x_1 + 2(n - 1)}$. Under (1.11), the system (1.10) becomes

$$\begin{cases} \dfrac{d\xi}{d\tau} = -\eta + \dfrac{1 - l - a\delta_2}{ab_1 x_1}\xi^2 + \dfrac{1 - 2l - a\delta_2}{a{b_1}^2}\xi\eta - \dfrac{x_1}{a{b_1}^3}(l + na^2)\eta^2 \\ \qquad = -\eta + L\xi^2 + M\xi\eta + N\eta^2, \\ \dfrac{d\eta}{d\tau} = \xi - \dfrac{1}{a{x_1}^2}\xi^2 - \dfrac{1}{ab_1 x_1}\xi\eta = \xi(1 + Ax_1 + B\eta). \end{cases} \tag{1.14}$$

The first focal quantity of the weak focus O(0,0) of (1.14) (namely, the critical point $S_1(x_1, y_1)$ of (1.7)) is $\overline{W_1} = M(L + N) - A(B + 2L)$. Numerical examples show that $\overline{W_1}$ may be positive as well as negative. In case $\overline{W_1} < 0$, we can prove Conjecture I by reductio ad absurdum.

Example 1.1. Take in (1.7) $a = -\frac{1}{3},\ n = 3,\ l = -\frac{5}{3},\ \delta_1 = 5$. Then

$$l + na^2 = -\frac{4}{3} < 0, \quad n + l = \frac{4}{3} > 0, \quad na^2 - (n - 1)(l + n)^2 = \frac{-29}{9} < 0.$$

So by (1.13), (1.9), (1.12) and (1.14) we have $x_1 = 3,\ \ y_1 = 0,\ \ \delta_2 = -8,\ \ b_1 = \sqrt{14}$,

$$L = 0, \quad M = \frac{-5}{14}, \quad N = \frac{-6}{7\sqrt{14}}, \quad A = \frac{1}{3}, \quad B = \frac{1}{\sqrt{14}},$$

and finally,

$$\overline{W_1} = \frac{15}{49\sqrt{14}} - \frac{1}{3\sqrt{14}} = \frac{-4}{147\sqrt{14}} < 0.$$

This shows that

$$\begin{cases} \dot{x} = -y + 5x - \dfrac{5}{3}x^2 + 3y^2, \\ \dot{y} = x(1 - \dfrac{1}{3}x - y) \end{cases} \tag{1.15}$$

has no LC around $S_1(3,0)$. For otherwise, if there exists $\Gamma_2 \supset \Gamma_1 \supset S_1$, where Γ_1 is unstable, Γ_2 is stable (may be $\Gamma_1 = \Gamma_2 = \overline{\Gamma}$ a semi-stable LC). When δ_2 decreases from zero in the system

$$\begin{cases} \dot{x} = -y + 5x - \dfrac{5}{3}x^2 + 3y^2 + \delta_2 x(1 - \dfrac{1}{3}x - y), \\ \dot{y} = x(1 - \dfrac{1}{3}x - y), \end{cases} \tag{1.16}$$

Γ_2 will expand, but Γ_1 will contract to S_1 at $\delta_2 = -8$, and then S_1 becomes an unstable weak focus, which contradicts $\overline{W_1} < 0$. It is easy to see that (1.16) will have a stable LC around S_1 when $\delta_2 < -8$. Notice that this cannot be proved by (1.3), since now $l + 1 - \frac{a}{k_1} < 0$.

Example 1.2. Take in (1.7)

$$a = -\frac{1}{3}, \quad n = \frac{5}{4}, \quad l = -\frac{1}{3}, \quad \delta_1 = 0.65.$$

Then

$$l + na^2 = -\frac{1.75}{3} < 0, \quad n + l > 0, \quad na^2 - (n-1)(l+n)^2 < 0,$$

x_1 satisfies the quadratic equation $1.75x^2 - 0.45x - 0.75 = 0$, so

$$x_1 \doteq 0.796, \quad y_1 \doteq 0.735, \quad \delta_2 \doteq -2.55, \quad b_1 \doteq 0.787,$$

$$L \doteq -2.3165, \quad M \doteq -3.955, \quad N \doteq -2.8576, \quad A \doteq 4.735, \quad B \doteq 0.2088$$

and $A(B + 2L) < 0,\ L(M + N) > 0$. We get

$$\overline{W_1} > 0. \tag{1.17}$$

Since now for the system

$$\begin{cases} \dot{x} = -y + 0.65x - \dfrac{1}{3}x^2 + \dfrac{5}{4}y^2, \\ \dot{y} = x(1 - \dfrac{1}{3}x - y), \end{cases} \tag{1.18}$$

we have

$$k_1 \doteq -0.878 \tag{1.19}$$

and (1.4) is valid, so we can prove the non-existence of LC of (1.18) around S_1 by (1.3).

Conjecture I is true in both examples.

§2.

In paper [2], the authors got two new criteria for the study of non-existence, existence and uniqueness of limit cycles of planar vector fields, and applied these criteria to some families of quadratic and cubic polynomial vector fields. We find that by using more elementary methods we also can get the same results for all the families of quadratic vector fields in [2]. The following are our proofs.

(1) For the system

$$\begin{cases} \dot{x} = \delta x - y + x^2 + mxy + ny^2 = P(x,y), \\ \dot{y} = x + bxy = Q(x,y) \end{cases} \tag{2.1}$$

under condition

$$b = \frac{(m + mn + n\delta)\delta + (1+n)^2}{n\delta^2}$$

and $\delta m(1+n) \neq 0$, it is easy to prove that $y = \frac{1}{n} + \frac{1+n}{n\delta}x$ is an invariant straight line. Since $1 + by = 0$ is also an invariant straight line, so this system has no limit cycle.[2]

(2) For the system

$$\begin{cases} \dot{x} = \dfrac{1+c^2}{c}x - y + \dfrac{a}{c}x^2 + \dfrac{b}{c}xy, \\ \dot{y} = x + ax^2 + bxy, \end{cases} \tag{2.2}$$

since $y = cx$ is an invariant straight line, and the only two critical points O(0,0) and $N(\frac{-1}{a+bc}, \frac{-c}{a+bc})$ are on this line, this system has no LC.

(3) For the system

$$\begin{cases} \dot{x} = \dfrac{1+c^2}{c}x - y + 2xy - cy^2, \\ \dot{y} = x, \end{cases} \tag{2.3}$$

since, for any $c \neq 0$, $\left|\frac{1+c^2}{c}\right| \geq 2$, the unique anti-saddle $O(0,0)$ is a node, this system has no LC.

(4) For the system

$$\begin{cases} \dot{x} = cx - y - x^2 + \dfrac{2}{c}xy = P(x,y), \\ \dot{y} = x - \dfrac{2}{c}x^2 + xy = Q(x,y), \end{cases} \tag{2.4}$$

without loss of generality, we can assume $c > 0$. When $0 < c \leq 2$, there exists only one critical point $O(0,0)$; when $c > 2$, there exist another two critical points

$$N_{1,2}\Big(\frac{c(4-c^2) - c^2 \pm \sqrt{c^2-4}}{4-c^2}, \frac{\pm c\sqrt{c^2-4}}{4-c^2}\Big).$$

It is easy to prove that N_1 and N_2 are saddles. When $c < 2$, by using the Dulac function $B(x,y) = \frac{c}{c-2x}$, we can get

$$\frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} = \frac{c^2(c-x)}{(c-2x)^2}.$$

Notice that $c - 2x = 0$ is a straight line without contact. Therefore

$$\frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} > 0$$

on the left side of $x = \frac{c}{2}$, and so no LC can appear around $O(0,0)$. Moreover, $O(0,0)$ is a node when $c \geq 2$. Hence, this system has no LC for any c.

[2]It is well-known that for fixed m, n and b, when $\delta m(1+n) < 0$ and $0 < |\delta| << 1$, (2.1) has a unique LC around O(0,0). The above result shows that if we fix $\delta = \delta_1$ in Q(x,y) and let δ in P(x,y) vary from zero to δ_1, the LC will expand and become finally an infinite separatrix cycle which has two finite parts, L_1: a part of $y = \frac{1}{n} + \frac{1+n}{n\delta}x$ and L_2: a part of $y = -\frac{1}{b}$.

(5) For the system

$$\begin{cases} \dot{x} = \dfrac{1+c^2}{c}x - y + x^2 - \dfrac{2}{c}xy, \\ \dot{y} = x + \dfrac{c}{2}x^2 - xy, \end{cases} \tag{2.5}$$

since $\left|\frac{1+c^2}{c}\right| \geq 2$, no LC can appear around $O(0,0)$, and the other critical point $N(\frac{2c}{c^2-2}, \frac{2c^2-2}{c^2-2})$ is a saddle, so this system has no LC.

(6) For the system

$$\begin{cases} \dot{x} = \dfrac{1+c^2}{c}x - y, \\ \dot{y} = x - \dfrac{2+c^2}{2c}x^2 + xy, \end{cases} \tag{2.6}$$

the proof is similar to (5).

(7) For the system

$$\begin{cases} \dot{x} = \delta x - y + (n-1)x^2 - n\delta xy + ny^2 = P(x,y), \\ \dot{y} = x(1-y) = Q(x,y), \end{cases} \tag{2.7}$$

since $y = 1$ is an integral line, no LC can appear around the critical points on this line. The two critical points on y-axis are $O(0,0)$ and $N(0, \frac{1}{n})$. Now, we need to study two cases.

(a) When $n < 0$ or $n \geq 1$, N is a saddle or a high-order critical point. We know already that, when $\delta = 0$, this system has $O(0,0)$ as a center. When δ changes, denoting

$$\Delta = \begin{vmatrix} P & Q \\ P'_\delta & Q'_\delta \end{vmatrix},$$

we can get $\Delta = x^2(1-ny)(y-1)$. It is easy to see that the sign of Δ is fixed in the region $\frac{1}{n} < y < 1$ $(n < 0)$, or $y < \frac{1}{n}$ $(n > 1)$. Hence, by the theory of rotated vector fields, there is no LC around $O(0,0)$ for $\delta \neq 0$.

(b) When $0 < n < 1$, N is a focus. Notice that, at $N(0, \frac{1}{n})$,

$$P_x + Q_y = \delta + (2n-3)x - n\delta y = \delta - \delta = 0.$$

So N is a weak focus. It is well-known that a quadratic polynomial vector field with an invariant straight line and a weak focus has no LC. Therefore, for any n and δ, this system has no LC.

Similarly, we can get the same result for the system

$$\begin{cases} \dot{x} = \delta x - y + (n+1)x^2 - n\delta xy + ny^2, \\ \dot{y} = x(1+y). \end{cases} \tag{2.8}$$

In fact, for the system

$$\begin{cases} \dot{x} = \delta x - y + lx^2 - n\delta xy + ny^2, \\ \dot{y} = x(1 \pm y), \end{cases} \tag{2.9}$$

the above proof remains valid for any δ, n and l.

In conclusion, in order to show the power of the new criteria got in [2], other examples should be suggested.

§3.

In [3] or [4] a conjecture was given as follows:

Conjecture II. For the system

$$\begin{cases} \dot{x} = -y + lx^2 + mxy + ny^2 = P(x,y), \\ \dot{y} = x(1 + ax - y) = Q(x,y), \end{cases} \tag{3.1}$$

let

$$\begin{cases} W_1 = m(l+n) - a(-1+2l), \\ W_2 = ma(5a-m)[(l+n)^2(n-1) - a^2(-1+2l+n)], \\ W_3 = ma^2[2a^2 + n(l+2n)][(l+n)^2(n-1) - a^2(-1+2l+n)]. \end{cases} \tag{3.2}$$

If $W_1 = 0$ and $W_2W_3 > 0$, then (3.1) has no LC.

It is easily seen that Conjecture II cannot be proved in general by the Dulac function method, because from $W_1 = 0$ we see that the curve $\operatorname{div}(BP, BQ) = 0$ will pass through $O(0,0)$ for many Dulac functions $B(x,y)$ (see [5, §16. (16.29)] and [6]). This is the reason why in [6, Theorem 6] the author only proved the absence of LC around $S_1 \neq 0$ under condition $W_1 = 0$.[3] Also, by this theorem, concerning the truth of conjecture II, we need only to examine the existence of LC around $O(0,0)$ case by case with the help of the techniques recently developed in [1], [6] and [7]. We limit our investigation of Conjecture II only under the conditions $n > 1, \quad l < 0$.

Case 1. Assume[4]

$$a < 0, \quad l < 0, \quad n > 1, \quad n + l > 0, \quad W_1 = 0 \tag{3.3}$$

and

$$C := na^2 + am(l+n) - (n-1)(l+n)^2 < 0. \tag{3.4}$$

Then

$$m = m_1 = \frac{a(-1+2l)}{n+l} > 0, \quad 5a - m_1 < 0, \tag{3.5}$$

and hence

$$\begin{aligned} \overline{W} &:= (n-1)(l+n)^2 - a^2(n+2l-1) = -C > 0, \\ G &:= m^2(n+2l-1) - (n-1)(1-2l)^2 = \frac{-(-1+2l)^2\overline{W}}{(n+l)^2} < 0, \\ D &:= (1-l)m^2 + am(1-2l) + (n-1)(1-2l)^2 = -G > 0. \end{aligned}$$

For the meaning of C, $\overline{W}$, G and D, and the relations between them, see [6]. From (3.4) we can get $a - (n+l)k_1 > 0, \quad a - (n+l)k_2 < 0$, where

$$k_1 = \frac{-m - \sqrt{m^2 + 4n(n-1)}}{2n} < 0, \quad k_2 = \frac{-m + \sqrt{m^2 + 4n(n-1)}}{2n} > 0$$

are roots of the equation $nk^2 + mk + 1 - n = 0$. Moreover, we have, for the second and third focal quantities of (3.1) at O, $W_2 > 0$ and $W_3 > 0$. So from [6, Theorem 3], we have Fig.2,

[3] $S_1(x_1, y_1)$ is an anti-saddle of (3.1) on $1 + ax - y = 0$ with $x_1 > 0, y_1 < 0$.

[4] The first three inequalities in (3.3) are the same as in [6].

where l_1 and l_2 are separatrices passing through the saddle point $N(0, \frac{1}{n})$, and

$$L: \; y = \frac{1}{n} + k_1 x, \qquad L': \; y = \frac{1}{n} k_2 x$$

are tangents of l_1 and l_2 at N, respectively.

Fig.2 Fig.3

Now, if M_1 is above M_2, then there is no or an even number of LC around O. Assume there are two LC $\Gamma_1 \supset \Gamma_2 \supset O$. Then by perturbing first m from m_1, and next adding a term δx to the right side of the first equation in (3.1) such that $0 \leq \delta << m_1 - m << 1$ and $0 < \delta << -W_1 << W_2$, we can get another two LC $\Gamma_3 \supset \Gamma_4$, while Γ_1 and Γ_2 still exist. Hence there are four LC: $0 \subset \Gamma_4 \subset \Gamma_3 \subset \Gamma_2 \subset \Gamma_1$, where O, Γ_3, Γ_1 are unstable, Γ_4, Γ_2 are stable. As δ increases, Γ_1 and Γ_2, Γ_3 and Γ_4 will come closer and closer, and finally coincide, becoming semi-stable LC and disappearing. But this shows that as δ decreases from a certain positive value to zero, semi-stable LC appear two times, which contradicts a proposition in [8]. So there is no LC around O in Fig.2.

If on the contrary, M_2 is above M_1 in Fig.2, then surely a stable LC must exist around O. We can get a contradiction as before if we notice that $\mathrm{div}|_N = \frac{m}{n} > 0$. Also this contradiction ensures that M_2 must be below M_1.

Case II. Assume (3.3) and $C > 0$ in (3.4). Then (3.5) still holds, but now $\overline{W} = -C < 0$, so $W_2 < 0, W_3 < 0$, and

$$G > 0, \;\; a - (n+l)k_1 < 0, \;\; a - (n+l)k_2 < 0.$$

By [6, Theorem 3], we have Fig.3. The non-existence of LC around O can be proved as in Case I.

Case III. Assume (3.3) and C $= 0$ in (3.4), then $a - (l+n)k_1 = \frac{m}{k_1} + 1 - 2l = 0$. By [6, (44)], the Dulac function

$$B(x, y) = (y - \frac{1}{n} - k_1 x)^{\frac{m}{nk_1}} (1-y)^{-1+2l-\frac{n}{k_1}}$$

becomes an integrating factor of (1.1), so O and S_1 are both centers.

Case IV. Assume[5]

$$a < 0, \quad n > 1, \quad n + l < 0, \quad \overline{W_1} = 0. \tag{3.6}$$

Then we have

$$m = m_2 = \frac{a(-1+2l)}{n+l} < 0.$$

[5]When $a < 0, l < 0$ and $\overline{W} = 0$, we cannot have n+l=0.

It is easy to see that $\overline{W} > 0$, so $C < 0$, and $a - (l+n)k_2 > 0, l_2$ lies above L'; also $a - (l+n)k_1 < 0, l_1$ lies above L.

(i) Assume $5a - m_2 > 0$. Then $5n+3l+1 > 0$, so $3(l+2n) > n-1 > 0$ and $W_2 > 0, W_3 < 0$, which contradicts the assumption of Conjecture II. In this case, O is unstable, a small amplitude stable LC appears around O when $W_2 << |W_3|$.

(ii) Assume $5a - m_2 < 0$ and $2a^2 + n(l+2n) < 0$. Then $W_2W_3 < 0$, which contradicts Conjecture II, too. In this case, O is stable, a small amplitude unstable LC appears around O when $|W_2| << W_3$.

(iii) Assume $5a - m_2 < 0$ (i.e., $5n + 3l + 1 < 0$), $2a^2 + n(l+2n) > 0$. Then $W_2 < 0, W_3 < 0$, O is stable, and we have Fig.4 when $na^2 + am_2 + l < 0$, (S_1 and S_2 are both anti-saddles) but Fig.5 when $na^2 + am_2 + l > 0$ (S_2 becomes a saddle). In Fig.4, l_5 and l_6 are separatrices passing through the critical points at infinity $A_1'(-x_1, -1, 0)$ and $A_2'(-x_2, -1, 0)$, respectively, where $x_1 < 0$ and $x_2 > 0$ are roots of the cubic equation

$$F(x) = ax^3 - (1+l)x^2 - mx - n = 0. \tag{3.7}$$

We can prove easily by (3.6) and the relations between coefficients and roots of (3.7) that in our case $F(\frac{1+l}{a}) > 0$, so

$$\frac{1+l}{a} < x_3, \quad x_1 + x_2 = \frac{1+l}{a} - x_3 < 0.$$

But we cannot even compare the absolute values of the slopes of l_5 and l_6 at A_1' and A_2', although we know at this moment $k_2 > |k_1| > 0$.

Fig.4 Fig.5

Let us now calculate the inner stability of the infinite separatrix Γ^* formed by l_5, L_6 and $\widehat{A_1'A_2'}$ when the former two coincide. By the formula given in [9], we have at $A_1'(-x_1, -1, 0)$

$$\rho_1 = -\frac{\rho_2^{(1)}}{\rho_1^{(1)}} = \frac{-3ax_1^2 + 2(1+l)x_1 + m}{x_1(ax_1 - 1)} > 0 \tag{3.8}$$

and at $A_2'(-x_2, -1, 0)$

$$\rho_2 = -\frac{\rho_2^{(2)}}{\rho_1^{(2)}} = \frac{-x_2(ax_2 - 1)}{3ax_2^2 - 2(1+l)x_2 - m} > 0. \tag{3.9}$$

If we can prove

$$\rho_1\rho_2 = \frac{x_2(ax_2 - 1)(3ax_1^2 - 2(1+l)x_1 - m)}{x_1(ax_1 - 1)(3ax_2^2 - 2(1+l)x_2 - m)} > 1 \tag{3.10}$$

or equivalently

$$\begin{aligned}&x_2(ax_2-1)(3ax_1^2-2(1+l)x_1-m)-x_1(ax_1-1)(3ax_2^2-2(1+l)x_2-m)\\&=(x_1-x_2)[a(2l-1)x_1x_2+ma(x_1+x_2)-m]>0,\end{aligned}\tag{3.11}$$

then Γ^* will be inner stable (see [5, §3, p.73]).

Now,

$$\begin{aligned}a(2l-1)x_1x_2+ma(x_1+x_2)-m&=\frac{n(2l-1)}{x_3}+ma\Big(\frac{1+l}{a}-x_3\Big)-m\\=\frac{n(2l-1)+mlx_3-max_3^2}{x_3}&=\frac{1}{x_3}(-ma)\Big(x_3+\frac{n}{a}\Big)\Big(x_3-\frac{l+n}{a}\Big)\end{aligned}\tag{3.12}$$

when $W_1=0$, i.e., $m=m_2=\frac{a(2l-1)}{n+l}<0$. Since $5n+3l+1<0$, we have

$$x_3>\frac{1+l}{a}>-\frac{n}{a}.\tag{3.13}$$

Moreover,

$$F\Big(\frac{l+n}{a}\Big)=\frac{(n-1)(l+n)^2}{a^2}-2l+1-n>0,$$

so

$$x_3>\frac{l+n}{a}.\tag{3.14}$$

(3.12), (3.13) and (3.14) ensure (3.10) as desired.

Now, we return to Fig.4.

(1) If l_5 goes to the right of l_6 as shown in the figure, then no LC or an even number of LC appear around $O(0,0)$; we can get a contradiction as in Case 1, if the later occurs.

(2) If $l_5=l_6$ or l_5 goes to the left of l_6, then between O and Γ^* (separatrix cycle of (3.1) or separatrix cycle of the system after m has been changed and δx has been added in the first equation of (3.1) suitably, similar to Case 1), there must be an unstable LC, so we can still get a contradiction as in Case 1. Also this contradiction ensures that the relative position of l_5 and l_6 must be that shown in Fig.4. The proof of the non-existence of LC in Fig.5 is the same as in Case II.

Conclusion: Conjecture II in [3] is true under condition $n>1,\ l<0$.

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