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ON COMPLETE SPACE-LIKE SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR**

Shen Yibing* Dong Yuxing*

Abstract

Let M^n be a complete space-like submanifold with parallel mean curvature vector in an indefinite space form $N_p^{n+p}(c)$. A sharp estimate for the upper bound of the norm of the second fundamental form of M^n is obtained. A generalization of this result to complete space-like hypersurfaces with constant mean curvature in a Lorentz manifold is given. Moreover, harmonic Gauss maps of M^n in $N_p^{n+p}(c)$ in a generalized sense are considered.

Keywords Pseudo-Riemannian manifold, Space-like submanifolds, Parallel mean curvature vector, Second fundamental form

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§1. Introduction

Let N_p^{n+p} be an $(n+p)$-dimensional connected pseudo-Riemannian manifold of index p. If N_p^{n+p} is complete and has constant sectional curvature c, then it is called the indefinite space form, denoted by $N_p^{n+p}(c)$. The indefinite space forms $N_1^{n+1}(c)$ of index 1 are called the Lorentz space forms, which contain the de Sitter space $S_1^{n+1}(c)$, where $c > 0$, the Minkowski space R_1^{n+1} and the anti-de Sitter space $H_1^{n+1}(c)$, where $c < 0$.

Generalizing Cheng-Yau's result[4],Ishihara[8] and Nishikawa[11] have shown the Bernstein-type property for maximal space-like submanifolds in $N_p^{n+p}(c)$ with $c \geq 0$ and for maximal space-like hypersurfaces in a locally symmetric N_1^{n+1}, respectively. An entire space-like hypersurface with constant mean curvature in R_1^{n+1} is investigated respectively by Goddard[7], Treibergs[17] and Choi-Treigergs[3]. For a complete space-like hypersurface M in $S_1^{n+1}(c)$ with constant mean curvature H, it is seen by Akutagawa[1] and Ramanathan[14] that M is totally umbilical if either $H^2 \leq c$ for $n = 2$ or $n^2H^2 < 4(n-1)c$ for $n \geq 3$. This statement has been generalized by Q. Cheng[2] to complete space-like submanifolds in $N_p^{n+p}(c)$ with parallel mean curvature vector.

A pseudo-hyperbolic space form $H_p^{n+p}(c)$ of constant negative curvature $c(< 0)$ and of index p can be realized as a hyperquadric in a pseudo-Euclidean $(n+p+1)$-space R_{p+1}^{n+p+1}

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*Department of Mathematics, Hangzhou University, Hangzhou 310028, China.

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of index $p+1$ given by

$$H_p^{n+p}(c) = \Big\{x \in R_{p+1}^{n+p+1} \Big| \langle x, x\rangle \equiv \sum_{i=1}^{n} x_i^2 - \sum_{\alpha=n+1}^{n+p+1} x_\alpha^2 = \frac{1}{c}\Big\}. \tag{1.1}$$

Let $H^k(c_k)$, $c_k < 0$, be the component of $H_0^k(c_k)$ through $(0, \cdots, 0, \frac{1}{\sqrt{-c_k}})$. Suppose that $(k_1, \cdots, k_{p+1})$ are positive integers satisfying $\sum\limits_{i=1}^{p+1} k_i = n$. Let x_i be the point of $H^{k_i}(\frac{nc}{k_i})$ for $c < 0$ and $1 \le i \le p+1$. Then, $x = (x_1, \cdots, x_{p+1})$ is a point in R_{p+1}^{n+p+1} with $\langle x, x\rangle = \frac{1}{c}$. The following product manifold[8]

$$H_{k_1 \cdots k_{p+1}}(c) = H^{k_1}\Big(\frac{nc}{k_1}\Big) \times \cdots \times H^{k_{p+1}}\Big(\frac{nc}{k_{p+1}}\Big) \tag{1.2}$$

is a maximal space-like submanifold of dimension n in $H_p^{n+p}(c)$. It is proved in [8] that the square S of the norm of the second fundamental form of a complete maximal space-like submanifold M^n in $H_p^{n+p}(c)$ satisfies $0 \le S \le -pnc$. Moreover, the submanifolds $H_{k_1 \cdots k_{p+1}}(c)$ described as in (1.2) are the only complete connected maximal space-like submanifolds in $H_p^{n+p}(c)$ with $S = -pnc$.

A hyperbolic cylinder in $N_1^{n+1}(c)$ is defined as a product manifold $H^1(c_1) \times M^{n-1}(c_2)$ where $M^{n-1}(c_2)$ is a sphere $S^{n-1}(c_2)$, a Euclidean space R^{n-1}, or a hyperbolic space $H^{n-1}(c_2)$, according to $c > 0$, $c = 0$ or $c < 0$ (see [10]). Here it is satisfied that $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$ when $c \ne 0$. Clearly, such hyperbolic cylinders are space-like hypersurfaces in $N_1^{n+1}(c)$ and have constant mean curvature $H = \pm(\sqrt{c - c_1} \pm (n-1)\sqrt{c - c_2})/n$ for $c \ne 0$ and $H = \pm\sqrt{-c_1}/n$ for $c = 0$. It is easy to check that the square of the norm of the second fundamental form of a hyperbolic cylinder in $N_1^{n+1}(c)$ is equal to

$$S_{H,1} = -nc + \frac{n}{2(n-1)}\{n^2H^2 + (n-2) \mid H \mid \sqrt{n^2H^2 - 4(n-1)c}\}. \tag{1.3}$$

In [10] it is shown that $S \le S_{H,1}$ where S stands for the square of the norm of the second fundamental form of a complete space-like hypersurface M^n in $N_1^{n+1}(c)$ with constant mean curvature H. Moreover, hyperbolic cylinders are the only complete space-like hypersurfaces with constant mean curvature in $N_1^{n+1}(c)$ satisfying $S = S_{H,1}$.

In this paper, we shall firstly extend the above results to higher codimension. We will put

$$S_{H,p} = pS_{H,1} - (p-1)nH^2, \tag{1.4}$$

where the constant $S_{H,1}$ is defined by (1.3). Then, we shall prove the following

Theorem 1.1. *Let M^n be an n-dimensional complete space-like submanifold in $N_p^{n+p}(c)$ with parallel mean curvature vector $\mathfrak{h}$, $\mid \mathfrak{h} \mid^2 = H^2$. If one of the following cases occurs:*

(1) *$c \le 0$,*

(2) *$c > 0$, $n = 2$ and $H^2 > c$,*

(3) *$c > 0$, $n \ge 3$ and $n^2H^2 \ge 4(n-1)c$,*

then the square S of the norm of the second fundamental form of M^n satisfies $S \le S_{H,p}$, where the constant $S_{H,p}$ is defined by (1.4). Moreover, the equality holds everywhere if and only if either

(1) *$H = 0$ and $M^n = H_{k_1 \cdots k_{p+1}}(c)$ given by (1.2); or*

(2) *$H \ne 0$, $p = 1$ and M^n is a hyperbolic cylinder in $N_1^{n+1}(c)$.*

By this theorem and the Gauss equation of M^n in $N_p^{n+p}(c)$, we have immediately the following

Corollary 1.1. *If M^n is a complete space-like n-submanifold in $N_p^{n+p}(c)$ with parallel mean curvature vector $\mathfrak{h}$, $|\mathfrak{h}|^2 = H^2$, then the scalar curvature ρ of M^n satisfies*

$$n(n-1)(c-H^2) \le \rho \le n(n-1)c - n^2H^2 + S_{H,p}, \tag{1.5}$$

where the first equality in (1.5) *holds if and only if M^n is totally umbilical in $N_p^{n+p}(c)$.*

The proof of Theorem 1.1 will be completed in §3. In §4 we shall give a generalization of Theorem 1.1 to space-like hypersurfaces with constant mean curvature in a locally symmetric Lorentz manifold, so that the results of [1, 10, 14] are involved. Finally, in §5, we shall study harmonic Gauss maps of space-like submanifolds in $N_p^{n+p}(c)$ in a generalized sense. This is similar to the Riemannian case[9] and extends Theorem 1.2 of [3].

§2. Basic Formulas and Lemmas

Let M^n be an n-dimensional connected space-like submanifold isometrically immersed in an $(n+p)$-dimensional pseudo-Riemannian manifold N_p^{n+p} of index p. We choose a local field of pseudo-Riemannian orthonormal frames $e_1, \cdots, e_{n+p}$ in N_p^{n+p} such that, restricted to M^n, the vectors $e_1, \cdots, e_n$ are tangent to M^n. We shall make use of the following convention on the ranges of indices unless otherwise stated:

$$n+1 \le \alpha, \beta, \cdots \le n+p.$$

For each α, we denote by $A_\alpha : T_xM^n \to T_xM^n$ the Weingarten endomorphism with respect to the normal e_α at $x \in M^n$. The square S of the norm of the second fundamental form and the mean curvature vector $\mathfrak{h}$ for M^n are defined respectively by

$$S = \sum_\alpha \operatorname{tr}(A_\alpha^2), \qquad \mathfrak{h} = \frac{1}{n}\sum_\alpha (\operatorname{tr}A_\alpha)e_\alpha. \tag{2.1}$$

If $\mathfrak{h} = 0$ identically, then M^n ia said to be maximal in N_p^{n+p}. Instead of the maximal condition, a more general assumption is to require the submanifold to have parallel mean curvature vector, namely $\nabla^\perp \mathfrak{h} = 0$. This implies that the quantity

$$H^2 := |\mathfrak{h}|^2 = \sum_\alpha \left(\frac{1}{n}\operatorname{tr}A_\alpha\right)^2 \tag{2.2}$$

is constant on M^n, where H is called the mean curvature of M^n. In the case that $H \ne 0$, we can choose a local field of pseudo-Riemannian orthonormal frames in such a way that $e_{n+1} = \mathfrak{h}/H$. With this choice, we introduce linear maps $B_\alpha : T_xM^n \to T_xM^n$ given by

$$B_{n+1} = A_{n+1} - HI, \qquad B_\beta = A_\beta \quad (\beta > n+1), \tag{2.3}$$

where I denotes the identity. It is easy to check that each map B_α is traceless and that

$$\sigma := \sum_\alpha \operatorname{tr}(B_\alpha^2) = S - nH^2. \tag{2.4}$$

Clearly, σ is nonnegative and σ vanishes identically if and only if M^n is totally umbilical in N_p^{n+p}. Note that the following quantity

$$\sigma_{\mathfrak{h}} := \operatorname{tr}(B_{n+1}^2) = \operatorname{tr}(A_{n+1}^2) - nH^2 \tag{2.5}$$

is independent of the choice of the frame field and is a function globally defined on M^n.

Now assume that the ambient space is $N_p^{n+p}(c)$. Let $\triangle$ be the Laplacian on M^n. By using (2.1)–(2.5), a straightforward computation gives the following (see [2, 15])

Lemma 2.1. *Let M^n be a space-like n-submanifold in $N_p^{n+p}(c)$ with parallel mean curvature vector $\mathfrak{h}$. Then we have*

$$\frac{1}{2}\triangle\sigma_{\mathfrak{h}} = |\nabla B_{n+1}|^2 + \sigma_{\mathfrak{h}}(\sigma_{\mathfrak{h}} + nc - nH^2) - nH(trB_{n+1}^3) + \sum_{\beta>n+1}(tr(B_{n+1}B_\beta))^2, \tag{2.6}$$

$$\frac{1}{2}\triangle\sigma = \sum_\alpha |\nabla B_\alpha|^2 + n(c-H^2)\sigma - nH\sum_\alpha tr(B_{n+1}B_\alpha^2) + \sum_{\alpha,\beta}(tr(B_\alpha B_\beta))^2 - \sum_{\alpha,\beta>n+1} tr([B_\alpha, B_\beta])^2, \tag{2.7}$$

where $[B_\alpha, B_\beta] = B_\alpha B_\beta - B_\beta B_\alpha$.

In order to estimate the right hand sides of above formulas, we need the following algebraic lemma due to Santos.

Lemma 2.2.[15] *Let A, $B : R^n \to R^n$ be symmetric linear maps such that $[A, B] = 0$ and* $\mathrm{tr}A = \mathrm{tr}B = 0$. *Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}(\mathrm{tr}A^2)(\mathrm{tr}B^2)^{1/2} \le \mathrm{tr}A^2B \le \frac{n-2}{\sqrt{n(n-1)}}(\mathrm{tr}A^2)(\mathrm{tr}B^2)^{1/2},$$

where the equality holds on the right (resp. left) hand side if and only if $(n-1)$ of the eigenvalues λ_i of A and the corresponding eigenvalues μ_i of B satisfy

$$|\lambda_i| = \frac{(\mathrm{tr}A^2)^{1/2}}{\sqrt{n(n-1)}}, \quad \lambda_i\lambda_j \ge 0, \quad \mu_i = \frac{(\mathrm{tr}B^2)^{1/2}}{\sqrt{n(n-1)}} \quad \left(resp. -\frac{(\mathrm{tr}B^2)^{1/2}}{\sqrt{n(n-1)}}\right).$$

We now want to establish the following analytic lemma used below.

Lemma 2.3. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below and f be a nonnegative C^2-function on M. If f satisfies*

$$\triangle f \ge a_0 f^{1+r} + \text{ finite terms as } \{a_i f^{r_i}\}, \tag{2.8}$$

where a_0 and r are positive real numbers, r_i's are nonnegative real numbers less than $1+r$, and a_i's are arbitrary real numbers, then $\sup_M f = f_0 < +\infty$ and f_0 satisfies

$$0 \ge a_0 f_0^{1+r} + \text{ finite terms as } \{a_i f_0^{r_i}\}.$$

Proof. Consider the function F on M defined by

$$F = (f+1)^{-r/2}, \tag{2.9}$$

which is bounded on M and, in fact, $0 \le F \le 1$. Since the Ricci curvature of M is bounded from below, we can apply the generalized maximal principle (see, e.g., [5]) to the function F bounded from below. Namely, for any given number $\varepsilon > 0$, there exists a point $x \in M$ such that

$$|\nabla F(x)| < \varepsilon, \qquad \triangle F(x) > -\varepsilon, \quad F(x) < \mathrm{inf}F + \varepsilon. \tag{2.10}$$

Consequently, by (2.9) and (2.10), it is easy to see that

$$r^2F^{2(1+r)/r}(x)\triangle f(x) < 2(2+r)\varepsilon^2 + 2rF(x)\varepsilon. \tag{2.11}$$

Thus, for any convergent sequence $\{\varepsilon_m > 0\}$ such that $\varepsilon_m \to 0$ $(m \to \infty)$, there exists a point sequence $\{x_m\}$ so that the sequence $\{F(x_m)\}$ satisfies (2.10) and converges to $F_0 = \inf F$ by taking a subsequence, if necessary. It implies that $f(x_m) \to f_0 = \sup f$ according to (2.9). Since F is bounded on M, (2.11) implies that for any positive number ε $(< a_0)$ there is a sufficiently large number m such that

$$F^{2(1+r)/r}(x_m)\triangle f(x_m) < \varepsilon. \tag{2.12}$$

This inequality and (2.8) yield

$$f^{1+r}(x_m)[a_0 - \varepsilon(1 + f^{-1}(x_m))^{1+r}] + \text{ finite terms as } \{a_i f^{r_i}(x_m)\} < 0,$$

which implies that the sequence $\{f(x_m)\}$ is bounded because $\varepsilon < a_0$ and $r_i < 1 + r$. Thus, $f_0 < +\infty$ and $F_0 \neq 0$. Consequently, it follows from (2.12) that

$$\lim_{m\to\infty} \triangle f(x_m) \le 0.$$

This and (2.8) complete the proof of the lemma.

Lemma 2.4. *Let M^n be an n-dimensional space-like submanifold in $N_p^{n+p}(c)$ with constant mean curvature H. Then the Ricci curvature of M^n satisfies* $\mathrm{Ric}(M^n) \ge (n-1)c - n^2H^2/4$, *and the equality holds everywhere if and only if either $n = 2$ and M^n is totally umbilical or $n \ge 3$ and M^n is totally geodesic.*

Proof. It follows directly from the Gauss equation of M^n in $N_p^{n+p}(c)$.

Remark. When $H = 0$, this lemma is due to Ishihara (see. [8, Proposition 2.1]).

§3. Proof of Theorem 1.1

If $H = 0$ or $p = 1$, then we have nothing to prove, because these cases reduce to the results of [8] and [10]. In sequence, we shall consider the only case that $H \neq 0$ and $p \ge 2$.

Firstly, under the hypothesis as in Theorem 1.1, the numbers $S_{H,1}$ and $S_{H,p}$ are well-defined by (1.3) and (1.4). Moreover, it is easy to see that

$$\begin{aligned} S_{H,1} - nH^2 &= \frac{n}{4(n-1)}\left[(n-2) \mid H \mid + \sqrt{n^2H^2 - 4(n-1)c}\right]^2 \\ &= \frac{1}{p}(S_{H,p} - nH^2). \end{aligned} \tag{3.1}$$

Next, applying Lemma 2.2 with $A = B$ to the estimate of $\mathrm{tr}(B_{n+1}^3)$, we have from (2.6)

$$\frac{1}{2}\triangle\sigma_{\mathfrak{h}} \ge \sigma_{\mathfrak{h}}\left(\sigma_{\mathfrak{h}} - \frac{n(n-2)}{\sqrt{n(n-1)}} \mid H \mid \sigma_{\mathfrak{h}}^{1/2} + nc - nH^2\right). \tag{3.2}$$

Since M^n is space-like, by Lemma 2.4 we can apply Lemma 2.3 to the nonnegative function $\sigma_{\mathfrak{h}}$. It follows from (3.2) that

$$(\sup\sigma_{\mathfrak{h}})\left\{(\sup\sigma_{\mathfrak{h}}) - \frac{n(n-2)}{\sqrt{n(n-1)}} \mid H \mid (\sup\sigma_{\mathfrak{h}})^{1/2} + nc - nH^2\right\} \le 0. \tag{3.3}$$

By considering the second factor of the left hand side of (3.3) as the quadratic function in $(\sup\sigma_{\mathfrak{h}})^{1/2}$, we can easily see that

$$\sigma_{\mathfrak{h}} \le \sup\sigma_{\mathfrak{h}} \le S_{H,1} - nH^2 \tag{3.4}$$

according to (3.1).

In order to estimate the right hand side of (2.7), we note the following facts:

$$-\sum_{\alpha,\beta}\mathrm{tr}([B_\alpha,B_\beta])^2\geq 0,\qquad \sum_\alpha(\mathrm{tr}B_\alpha^2)^2\geq\frac{1}{p}\sigma^2. \tag{3.5}$$

Since M^n has parallel mean curvature vector $\mathfrak{h}$, we have $[B_{n+1},B_\alpha]=0$ when $e_{n+1}=\mathfrak{h}/H$. Thus, we can apply Lemma 2.2 to estimate $\mathrm{tr}(B_{n+1}B_\alpha^2)$. By this and (3.5), it follows from (2.7) that

$$\frac{1}{2}\triangle\sigma\geq\sigma\left\{\frac{1}{p}\sigma-\frac{n(n-2)}{\sqrt{n(n-1)}}\mid H\mid\sigma_{\mathfrak{h}}^{1/2}+nc-nH^2\right\}. \tag{3.6}$$

On the other hand, by (3.1) and (3.4), the following relationship

$$\frac{n(n-2)}{\sqrt{n(n-1)}}\mid H\mid\sigma_{\mathfrak{h}}^{1/2}-nc+nH^2\leq S_{H,1}-nH^2=\frac{1}{p}(S_{H,p}-nH^2) \tag{3.7}$$

can be derived by a simple calculation. Combining (3.6) with (3.7) yields

$$\frac{1}{2}\triangle\sigma\geq\frac{1}{p}\sigma(\sigma+nH^2-S_{H,p}). \tag{3.8}$$

Applying Lemma 2.3 to the nonnegative function σ, we obtain immediately

$$\sigma\leq\sup\sigma\leq S_{H,p}-nH^2,$$

namely, $S\leq S_{H,p}$ by (2.4). This completes the proof of the first part of Theorem 1.1.

Assume now that $S=S_{H,p}$ on M^n everywhere. Then, $\triangle\sigma=0$ and this implies that all estimetes used to obtain (3.2) and (3.8) are equalities. Thus, from these equalities and Lemma 2.2 we have

$$\mid\nabla B_\alpha\mid^2=0,$$

$$\mathrm{tr}(B_{n+1}B_\alpha^2)=\pm\frac{n-2}{\sqrt{n(n-1)}}(\mathrm{tr}B_\alpha^2)(\mathrm{tr}B_{n+1}^2)^{1/2}, \tag{3.9}$$

$$\mathrm{tr}B_\alpha^2=\mathrm{tr}B_{n+1}^2=\sigma_{\mathfrak{h}}=S_{H,1}-nH^2,$$

$$\mathrm{tr}([B_\alpha,B_\beta])^2=0. \tag{3.10}$$

Let (B_α) denote the matrices which define the maps B_α's. By the equality part of Lemma 2.2, (3.9) implies that there exists an orthonormal frame $e_1,\cdots,e_n$ of TM^n such that, in this frame, (B_α) has the following form:

$$(B_\alpha)=\lambda_\alpha\begin{pmatrix}1 & & & & 0 & \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & 0 & & 1 & & \\ & & & & -(n-1) & \end{pmatrix},\quad \lambda_\alpha=\pm\sqrt{\frac{S_{H,1}-nH^2}{n(n-1)}}. \tag{3.11}$$

This shows that the first normal space of M^n in $N_p^{n+p}(c)$, namely

$$T_1^\perp(x)=\mathrm{span}\Big\{\sum_\alpha\langle A_\alpha(X),Y\rangle e_\alpha,\ X,Y\in T_xM^n\Big\}$$

has constant dimension in M^n. It is easy to see that $\dim T_1^\perp(x)\leq 2$ for all $x\in M^n$. The formula (3.10) implies that the normal connection of M^n is flat, i.e., $R^\perp=0$. Since $\nabla^\perp\mathfrak{h}=0$ in M^n, the first normal bundle $T_1^\perp$ is a parallel normal subbundle (see [6, 15]). Let $T_2^\perp$ be the orthogonal complement of $T_1^\perp$ in the normal bundle of M^n, which is also a parallel normal subbundle. By the definition of $T_1^\perp$, M^n is totally geodesic with respect to the normal subbundle $T_2^\perp$. So, it is possible to reduce the codimension of M^n to two (see [18, Theorem 1]). Thus, we can regard M^n as a space-like submanifold in $N_2^{n+2}(c) \hookrightarrow N_p^{n+p}(c)$ with $S = S_{H,p}$.

We now will prove that M^n has a parallel umbilic direction in $N_2^{n+2}(c)$. In fact, if we choose a new pseudo-Riemannian orthonormal frame $\{e'_{n+1}, e'_{n+2}\}$ of $T^\perp M^n$ given by

$$e'_{n+k} = \frac{1}{\sqrt{\lambda_{n+1}^2 + \lambda_{n+2}^2}}\{\lambda_{n+2}e_{n+1} + (-1)^{k-1}\lambda_{n+1}e_{n+2}\},\ k = 1, 2, \tag{3.12}$$

then e'_{n+1}, e'_{n+2} are parallel in $T^\perp M^n$ and e'_{n+2} is an umbilic direction. Hence, there exists a totally umbilical hypersurface $N_1^{n+1}(\tilde{c})$ in $N_2^{n+2}(c)$ such that M^n lies in $N_1^{n+1}(\tilde{c})$ as a space-like hypersurface and M^n is not totally umbilical in $N_1^{n+1}(\tilde{c})$ because $S = S_{H,p}$. Namely, we have the following composition:

$$M^n \hookrightarrow N_1^{n+1}(\tilde{c}) \hookrightarrow N_2^{n+2}(c) \hookrightarrow N_p^{n+p}(c).$$

Let h_k, $k = 1, 2$, be the mean curvature of M^n in $N_2^{n+2}(c)$ with respect to the normal e'_{n+k}. By (3.11) and (3.12), it is easy to see that

$$h_1^2 = h_2^2 = \frac{1}{2}H^2. \tag{3.13}$$

By the Gauss equation of $N_1^{n+1}(\tilde{c})$ in $N_2^{n+2}(c)$, we then have

$$\tilde{c} = c - h_2^2 = c - \frac{1}{2}H^2. \tag{3.14}$$

Since M^n is not totally umbilical in $N_1^{n+1}(\tilde{c})$, then, by [1] or [14], it would be satisfied that

$$\begin{aligned} h_1^2 &> \tilde{c} \quad \text{for} \quad n = 2, \\ n^2 h_1^2 &\geq 4(n-1)\tilde{c} \quad \text{for} \quad n \geq 3. \end{aligned} \tag{3.15}$$

By (3.13) and (3.14), it follows from (3.15) that

$$n^2H^2 - \frac{1}{2}(n-2)^2H^2 \geq 4(n-1)c. \tag{3.16}$$

On the other hand, let S' and $S_{h_1,1}$ denote respectively the square of the norm of the second fundamental form of M^n in $N_1^{n+1}(\tilde{c})$ and the constant defined as in (1.2) where H is replaced by h_1. By [10], we would have

$$S' \leq S_{h_1,1}. \tag{3.17}$$

Let S'' denote the square of the norm of the second fundamental form of M^n in $N_2^{n+2}(c)$ with respect to the normal e'_{n+2}. Since e'_{n+2} is an umbilic direction, we have $S'' = nh_2^2 = nh_1^2$. Thus, by (3.13) and (3.1), we have

$$S' = S - S'' = S_{H,p} - \frac{1}{2}nH^2 = p(S_{H,1} - nH^2) + \frac{1}{2}nH^2.$$

This together with (3.17) yields

$$2(S_{H,1} - nH^2) \leq p(S_{H,1} - nH^2) \leq S_{h_1,1} - nh_1^2.$$

By the formulas (3.1) and (3.13), the inequality above can be reduced to

$$(n-2) \mid H \mid + 2\sqrt{n^2H^2 - 4(n-1)c} \leq \sqrt{n^2H^2 + 4(n-1)(H^2 - 2c)},$$

namely,

$$(n-2)\mid H\mid\sqrt{n^2H^2-4(n-1)c}\le 2(n-1)c-(n^2-2n+2)H^2. \tag{3.18}$$

By combining (3.18) with (3.16), we obtain a contradiction when $H\neq 0$ and $n\neq 2$. If $n=2$, then (3.18) becomes $c\ge H^2$, i.e., $\tilde{c}\ge h_1^2$ according to (3.13) and (3.14). This contradicts $(3.15)_1$.

Hence, to sum up, there is no any complete space-like submanifold M^n in $N_p^{n+p}(c)$ with nonzero parallel mean curvature vector such that $p\ge 2$ and $S=S_{H,p}$ everywhere. This proves Theorem 1.1 completely.

Remark. A submanifold is said to be pseudo-umbilical if its mean curvature vector is an umbilic direction everywhere. From the proof of the first part of Theorem 1.1 we have the following: Let M^n be a complete space-like n-submanifold in $N_p^{n+p}(c)$ with parallel mean curvature vector $\mathfrak{h}$, $\mid\mathfrak{h}\mid^2=H^2$. If M^n is pseudo-umbilical, then M^n is a maximal space-like submanifold in a totally umbilical hypersurface $N_{p-1}^{n+p-1}(c')$ of $N_p^{n+p}(c)$ with $c'=c-H^2$, so that either M^n is totally umbilical in $N_p^{n+p}(c)$ for $c\ge H^2$, or the square S of the norm of the second fundamental form of M^n in $N_p^{n+p}(c)$ satisfies $S\le n(p+1)H^2-npc$ for $c<H^2$.

§4. A Generalization to Lorentz Manifolds

In this section we prove the following

Theorem 4.1. *Let N_1^{n+1} be a locally symmetric Lorentz manifold whose sectional curvature K_N is pinched by $c_2\le K_N\le c_1$ for some two real numbers c_1 and c_2. Let M^n be a complete space-like hypersurface in N_1^{n+1} with constant mean curvature H. Put $c=(5c_2-3c_1)/2$ and $\sigma=S-nH^2$ where S stands for the square of the norm of the second fundamental form of M^n. Then, we have the following estimates for σ:*

$$\sigma\le\frac{2n(n-1)(c_1-c_2)H^2}{4(n-1)c-n^2H^2}\quad \textit{for}\quad n^2H^2<4(n-1)c; \tag{4.1}$$

$$\sigma\le\left(\sqrt{a}+\frac{n-2}{2}\sqrt{c/n}\right)^2\qquad \textit{for}\quad n^2H^2=4(n-1)c, \tag{4.2}$$

where

$$a=(n-2)^2\frac{c}{4n}+\sqrt{2(n-1)(c_1-c_2)c};$$

$$\sigma\le\frac{1}{4}\left(a+\sqrt{a^2+2n\mid H\mid\sqrt{2(c_1-c_2)}}\right)^2\quad \textit{for}\quad n^2H^2>4(n-1)c, \tag{4.3}$$

where

$$a=\frac{1}{2}\sqrt{\frac{n}{n-1}}\left\{(n-2)\mid H\mid+\sqrt{n^2H^2-4(n-1)c}\right\}.$$

Proof. We choose a local field of Lorentzian orthonormal frames $e_1,\cdots,e_n,e_{n+1}$ in N_1^{n+1} such that, restricted to M^n, the vector e_{n+1} is time-like so that e_{n+1} is normal to M^n. Furthermore, the tangent vectors $e_1,\cdots,e_n$ can be chosen in such a way that

$$A(e_i)=\lambda_i e_i\quad(1\le i,j,\cdots\le n),$$

where A is the Weingarten endomorphism of M^n. Denote by $K(e_i\wedge e_a)$ the sectional curvature of N_1^{n+1} with respect to the nondegenerate 2-plane spanned by vectors e_i and

e_a in TN_1^{n+1} for $1 \leq a \leq n+1$. Without loss of generality, we may assume that the mean curvature $H = (\sum\limits_i \lambda_i)/n$ is nonnegative. Let $\triangle$ be the Laplacian on M^n. By the computation as in [11], it is not hard to see that

$$\frac{1}{2}\triangle S = |\nabla A|^2 + nH\sum_i \lambda_i K(e_i \wedge e_{n+1}) - S\sum_i K(e_i \wedge e_{n+1}) + \sum_{i,j}(\lambda_i - \lambda_j)^2 K(e_i \wedge e_j) - nH(\mathrm{tr}A^3) + S^2, \tag{4.4}$$

where $S = \mathrm{tr}A^2$.

Since the sectional curvature of N_1^{n+1} is pinched, we have the following estimates:

$$\begin{aligned} & nH\sum_i \lambda_i K(e_i \wedge e_{n+1}) - S\sum_i K(e_i \wedge e_{n+1}) \\ = & -\frac{1}{2}\sum_{i,j}(\lambda_i - \lambda_j)^2 K(e_i \wedge e_{n+1}) - \frac{1}{2}S\sum_i K(e_i \wedge e_{n+1}) + \frac{n}{2}\sum \lambda_i^2 K(e_i \wedge e_{n+1}) \\ \geq & -nc_1(S - nH^2) - \frac{n}{2}(c_1 - c_2)S, \end{aligned} \tag{4.5}$$

$$\sum_{i,j}(\lambda_i - \lambda_j)^2 K(e_i \wedge e_j) \geq 2nc_2(S - nH^2). \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4) yields

$$\frac{1}{2}\triangle S \geq n(2c_2 - c_1)(S - nH^2) - \frac{n}{2}(c_1 - c_2)S - nH(\mathrm{tr}A^3) + S^2. \tag{4.7}$$

By introducing the linear map $B = A - HI$ and putting $f^2 = \mathrm{tr}B^2 = \sigma$, we have from (4.7)

$$\frac{1}{2}\triangle f^2 \geq f^4 + n(c - H^2)f^2 - nH(\mathrm{tr}B^3) - \frac{1}{2}(c_1 - c_2)n^2H^2, \tag{4.8}$$

where $c = (5c_2 - 3c_1)/2$. Since the map B is traceless, we can apply Lemma 2.2 to estimate $\mathrm{tr}B^3$ in (4.8) and obtain

$$\frac{1}{2}\triangle f^2 \geq f^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}Hf^3 + n(c - H)f^2 - \frac{1}{2}(c_1 - c_2)n^2H^2. \tag{4.9}$$

Since the sectional curvature of N_1^{n+1} is bounded, the Ricci curvature of M^n is bounded from below according to the Gauss equation of M^n. Thus, applying Lemma 2.3 to the function f^2, we have from (4.9)

$$\begin{aligned} 0 \geq & f_0^4 - \frac{n(n-2)}{\sqrt{n(n-1)}H}f_0^3 + n(c - H^2)f_0^2 - \frac{1}{2}(c_1 - c_2)n^2H^2 \\ = & f_0^2\left(f_0 - \frac{n-2}{2}\sqrt{\frac{n}{n-1}}H\right)^2 + \frac{n}{4(n-1)}\left\{4(n-1)c - n^2H^2\right\}f_0^2 \\ & - \frac{1}{2}(c_1 - c_2)n^2H^2, \end{aligned} \tag{4.10}$$

where $f_0 = \sup_M f < +\infty$.

We now consider three cases separately.

Case (i) $n^2H^2 < 4(n-1)c$. In such a case we have from the second part of (4.10)

$$0 \geq \frac{1}{2(n-1)}\{4(n-1)c - n^2H^2\}f_0^2 - (c_1 - c_2)nH^2,$$

from which (4.1) follows directly.

Case (ii) $n^2H^2 = 4(n-1)c$. It follows from the second part of (4.10) that

$$f_0^2\left(f_0 - \frac{n-2}{2}\sqrt{\frac{n}{n-1}}H\right)^2 \le \frac{1}{2}(c_1 - c_2)n^2H^2. \tag{4.11}$$

If $f_0 \ge \frac{n-2}{2}\sqrt{\frac{n}{n-1}}H = (n-2)\sqrt{c/n}$, then (4.11) impies that

$$f_0^2 - (n-2)\sqrt{c/n}f_0 - \sqrt{2(n-1)(c_1-c_2)c} \le 0,$$

from which (4.2) follows. If $f_0 < (n-2)\sqrt{c/n}$, then (4.2) holds naturally.

Case (iii) $n^2H^2 > 4(n-1)c$. In such a case the first part of (4.10) can be reduced as

$$0 \ge f_0^2(f_0 - a)^2 + f_0^2(f_0 - a)\sqrt{\frac{n}{n-1}\{n^2H^2 - 4(n-1)c\}} - \frac{1}{2}(c_1 - c_2)n^2H^2, \tag{4.12}$$

where a is given as in (4.3). If $f_0 < a$, then (4.3) is trivial. If $f_0 \ge a$, then it follows from (4.12) that

$$f_0^2(f_0 - a)^2 \le \frac{1}{2}(c_1 - c_2)n^2H^2.$$

This impies that

$$f_0^2 - af_0 - \frac{n}{\sqrt{2}}\sqrt{c_1 - c_2}H \le 0,$$

from which (4.3) follows immediately.

Remark. If N_1^{n+1} is a Lorentz space form, i.e., $c_1 = c_2$, then (4.1) with (4.2) implies the result of [1] or [14], and (4.3) with (4.2) implies Theorem 1 of [10]. On the other hand, for complete maximal space-like hypersurfaces M^n in N_1^{n+1}, it is easy to see from (4.1) and (4.2) that M^n is totally geodesic if $5c_2 - 3c_1 \ge 0$. This has been shown in [11].

§5. Harmonic Gauss Maps

In this section, we assume that $N_p^{n+p}(c)$ is simply connected and $p \ge 1$. Denote by $O_p(m)$ the pseudo-orthogonal group which is the set of all matrices $A \in GL(m, R)$ that preserve the pseudo-Euclidean inner product of R_p^m (see [12]).

The bundle $F(N_p^{n+p}(c))$ of the pseudo-orthonormal frames on $N_p^{n+p}(c)$ can be identified with the group $G(n+p)$ which is one of the following:

(i) $O_p(n+p+1)$ for $c > 0$;

(ii) $O_p(n+p)$ for $c = 0$;

(iii) $O_{p+1}(n+p+1)$ for $c < 0$.

Let $\theta_{A'B'}$ be the Maurer-Cartan forms of $G(n+p)$, where from now on we agree with the following ranges of indeces:

$$\begin{array}{ll} 0 \le A', B', \cdots \le n+p, & 1 \le A, B, \cdots \le n+p, \\ 1 \le i, j, \cdots \le n, & n+1 \le \alpha, \beta, \cdots \le n+p. \end{array}$$

Then $\theta_{A'B'}$ satisfy the structure equations:

$$d\theta_{A'B'} = \sum_{C'} \varepsilon_{C'}\theta_{A'C'} \wedge \theta_{C'B'}, \qquad \theta_{A'B'} + \theta_{B'A'} = 0, \tag{5.1}$$

where $\varepsilon_i = 1$, $\varepsilon_\alpha = -1$ and

$$\varepsilon_0 = \begin{cases} 1 & \text{for} \quad c > 0, \\ 0 & \text{for} \quad c = 0, \\ -1 & \text{for} \quad c < 0. \end{cases}$$

On putting

$$\theta_A = \frac{1}{\sqrt{\varepsilon_0 c}}\theta_{0A} \qquad \text{for} \quad c \neq 0, \tag{5.2}$$

the pseudo-Riemannian metric of $N_p^{n+p}(c)$ is given by

$$ds_N^2 = \sum_A \varepsilon_A \theta_A^2$$

and (5.1) becomes

$$\begin{aligned} d\theta_A &= \sum_B \varepsilon_B \theta_{AB} \wedge \theta_B, \\ d\theta_{AB} &= \sum_C \varepsilon_C \theta_{AC} \wedge \theta_{CB} - c\theta_A \wedge \theta_B, \end{aligned} \tag{5.3}$$

which is the structure equation of $N_p^{n+p}(c)$.

Let M^n be a space-like n-submanifold in $N_p^{n+p}(c)$ and $F(M^n)$ the bundle of orthonormal frames on M^n. If P is the set of elements $\mathfrak{p} = (x, e_1, \cdots, e_{n+p}) \in F(N_p^{n+p}(c))$ such that $x \in M^n$ and $(x, e_1, \cdots, e_n) \in F(M^n)$, then $\bar{\pi} : P \to M^n$ can be viewed as a principal bundle with the fibre $G(n) \times O_p(p)$ and $\mathfrak{i} : P \to F(N_p^{n+p}(c))$ is the natural identification.

Let Q be the set of all totally geodesic space-like n-spaces in $N_p^{n+p}(c)$, which is identified with a pseudo-Grassmannian $G(n+p)/G(n) \times O_p(p)$. By means of forms $\theta_{A'B'}$ of $G(n+p)$ we can introduce a pseudo-Riemannian metric on Q:

$$ds_Q^2 = \varepsilon_0 \sum_\alpha \varepsilon_\alpha (\theta_{0\alpha})^2 + \sum_{i,\alpha} \varepsilon_i \varepsilon_\alpha (\theta_{i\alpha})^2, \tag{5.4}$$

which is invariant under the action of $G(n+p)$.

As a natural generalization of the Gauss maps of Riemannian submanifolds described as in [13], the Gauss map for M^n of $N_p^{n+p}(c)$ in a generalized sense may be defined as the map $\mathfrak{g} : M^n \to Q$ such that $\mathfrak{g}(x)$, $x \in M^n$, is a totally geodesic space-like n-space in $N_p^{n+p}(c)$ which is tangent to M^n at x. Then, we have the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\mathfrak{i}} & F(N_p^{n+p}(c)) = G(n+p) \\ {\scriptstyle\bar{\pi}}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ M^n & \xrightarrow[\mathfrak{g}]{} & Q = G(n+p)/G(n) \times O_p(p) \end{array} \tag{5.5}$$

where π is the natural projection.

By using the same method as in [9] or [16], from (5.1)–(5.5) we can give easily the following theorem whose proof is omitted here.

Theorem 5.1. *Let M^n be a space-like n-submanifold in $N_p^{n+p}(c)$. Then, the Gauss map $\mathfrak{g} : M^n \to G(n+p)/G(n) \times O_p(p)$ is harmonic if and only if*

(i) *M^n has parallel mean curvature vector when $c = 0$; or*

(ii) *M^n is a maximal submanifold when $c \neq 0$.*

Remark. When $c = 0$ and $p = 1$, the proof of the theorem was given in [3]. This theorem provides a class of harmonic maps from Riemannian manifolds to pseudo-Riemannian manifolds.

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