Chin. Ann. of Math.
19B: 3(1998),381-390.

ON THE EXISTENCE OF ALMOST GLOBAL WEAK SOLUTION TO MULTIDIMENSIONAL VLASOV-POISSON EQUATION**

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Abstract

The authors prove the existence of almost global weak solution to multidimensional Vlasov-Poisson equation with a class of Randon measure as initial data.

Keywords Vlasov-Poisson equation, Almost global weak solution, Singular integral operator

1991 MR Subject Classification 35Q, 35Q72

Chinese Library Classification O175.24

§1. Introduction

Consider the following d-dimensional Vlasov-Poisson system, $d = 2, 3$,

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - E \cdot \nabla_v f = 0, \quad f(0, x, v) = f_0(x, v), \\ E(t, x) = c(d) \displaystyle\int \frac{x - y}{|x - y|^d} \rho(t, y)\, dy, \\ \rho(t, x) = \displaystyle\int f(t, x, v)\, dv. \end{cases} \tag{VP}$$

This system describes the problem of kinetic theory of galaxy in the growing process of a star, where the function $f(t, x, v)$ is denoted as the density of the star in phase space $R^d \times V$ at time t, $\rho(t, x)$ is total density at (t, x) and $E(t, x)$ is the gravitation derived from Newtonian potential associated with $\rho(t, x)$.

Many mathematicians have studied system (VP), and got a lot of interesting results. For example, in [1, 2, 8, 11], the authors studied their smooth solutions, and in [6, 7, 10, 13], the weak solutions. From those discussion, it should be pointed out that there exists a natural restriction for density distribution in phase space; that is, the density should be a bounded Randon measure. Under this restriction, Y. Zheng and A. Majda studied the existence of global weak solutions in their sense for 1-dimensional (VP) system in [10]. They assumed further that the initial data $f_0(x, v)$ is a probability measure satisfying $\iint e^{\alpha|v|} f_0(x, v) dv dx \le C_\alpha < \infty$ for any $\alpha \ge 0$, and utilized the special nonlinear structure of 1-dimensional (VP) system to obtain the existence mentioned as above. Now, the questions are: could the concept of weak solutions and the method provided by [10] be generalized

Manuscript received May 29, 1995.

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**Project supported by the National Natural Science Foundation of China.

to multidimensional (VP) system? What assumption posed to $f_0(x,v)$ is suitable for multidemensional case? (see [7]). We note that the assumption and method established in [10] are unsuitable for studying the corresponding problem of multidimensional case. In fact, when $d \geq 2$, we can not deduce $E(x,t) \in L^\infty([0,T],R^d)$ from $f \in L^\infty([0,T],L^1(R^d \times R^d))$; and since the operator $\partial_i \nabla \triangle^{-1}$ is only a Caldéron-Zygmund type singular integral operator which does not persist L^1-boundedness, we can not estimate the norm of $\partial_i E(t,x)$ in $L^1([0,T] \times R^d)$ (see the end of this section for details); finally, because of the appearance of the term $\sum\limits_{i=1}^{d} v_i \partial_i f$ in the first equation of (VP), we may not apply the proof argument of Theorem 2. 5 in [10] by use of the test function $v^\beta \varphi(t,x,v)$ with $\varphi(t,x,v) \in C_0^\infty(R_+ \times R^d \times R^d)$, where $\beta = (\beta_1, \cdots, \beta_\alpha)$ being multi-index. Therefore, in this paper, we have to impose a reasonable assumption on Randon measure $f_0(x,v)$, and then develop the concept of weak solution to multidimensional (VP) system and prove its existence.

For this end, we first state some background of the condition posed on the initial data here. For convenience, we assume that $f_0(x,v) \in L^1_{\text{comp}}$, and let $f_0^\varepsilon(x,v)$ be the Friedrichs' molifying of $f_0(x,v)$. So, there exists a unique global classical solution $f^\varepsilon(t,x,v), E^\varepsilon(t,x)$ to (VP) with the initial data $f_0^\varepsilon(x,v)$ for every $\varepsilon > 0$ (see [8, 11]), which satisfies

$$\iint f^\varepsilon(t,x,v)dxdv = \iint f_0^\varepsilon(x,v)dxdv = \iint f_0(x,v)dxdv.$$

Since $\text{div}_{x,v}(v, E(t,x)) = 0$, following the argument given in Theorem 1 in [12], we can extract a subsequence $\{f^{\varepsilon_j}(t,x,v)\}$ of $\{f^\varepsilon(t,x,v)\}$ such that

$$f^{\varepsilon_j}(t,x,v) \rightharpoonup f(t,x,v) \text{ in } L^\infty([0,\infty), L^1(\mathbb{R}^d \times \mathbb{R}^d)).$$

Then $\int f(t,x,v)dv \in L^\infty([0,\infty], \quad L^1(\mathbb{R}^d))$. On the other hand, notice that $E(t,x) = \nabla \triangle^{-1} \rho(t,x)$, and

$$\partial_i \nabla \triangle^{-1} \rho(t,x) = \int \partial_i \left(\frac{x-y}{|x-y|^d} \right) \rho(t,y)dy = \int \frac{\Omega(x-y)}{|x-y|^d} \rho(t,y)dy,$$

where $\Omega(x)$ satisfies $\int_0^1 \frac{\omega(\delta)}{\delta} d\delta \leq 4$ as

$$\omega(\delta) = \sup_{\substack{|x'-x| \leq \delta \\ |x'|=|x|=1}} |\Omega(x) - \Omega(x')|.$$

Hence by Theorem 4 in [9, p. 42], $\partial_i E(t,x)$ is determined for x, a. e., and

$$m\left\{x \in \mathbb{R}^d \middle| |\partial_i E(t,x)| > \alpha\right\} \leq \frac{c}{\alpha} \int_{\mathbb{R}^d} \rho(t,y)dy, \text{ for } \alpha > 0, \tag{1.1}$$

where m denotes the Lebesgue measure on $\mathbb{R}^d$. Even if we strengthen (1.1) by assuming that $\partial_i E(t,x) \in L^\infty([0,\infty), L^1(\mathbb{R}^d))$, since $\int \varphi(t,x,v) f(t,x,v)dv$ can not be rewritten as a first derivative of a BV function here as the one-dimensional case does, for $\varphi(t,x.v) \in C_0^\infty(R_+ \times \mathbb{R}^d \times \mathbb{R}^d)$, it is still impossible for us to follow the definition of weak solution in [10] for multidimensional (VP) system with initial data in L^1 or Randon measure space.

Now, we impose the following condition on $f_0(x,v)$,

$$\int \psi(x,v) f_0(x,v)dxdv \leq \int \psi(x,v) h(|v|)dxdv, \tag{1.2}$$

where $\psi(x,v)$ is an arbitrary nonnegative function in $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, $h(|v|)$ is a non-increasing continuous function of $|v|$ with $\int_{\mathbb{R}^d} h(|v|)dv < \infty$, and $f_0(x,v)$ is a probability distribution whose support is in $\{(x,v)||x| \le R, |v| \le R\}$. The condition corresponds to the essential boundedness of the initial total density distribution. It is worth while noticing that here the hypothesis (1.2) generalizes the assumption given by many mathematicians when they studied system of Vlasov type. For instance, J. Cooper and A. Klimas posed a special similar restriction for $f_0(x,v) \in L^1(\mathbb{R} \times \mathbb{R})$ to prove the existence of generalized solution in their sense for one dimensional Vlasov-Maxwell system[1], and in [2, 11] for proving the existence of global classical solution to Vlasov-Fokker-Planck systems and 2-D Vlasov-Poisson systems, the authors assumed that $(1+|v|^2)^{\frac{r}{2}} f_0(x,v) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, and $(1+|x|)^{2\gamma}(1+|v|)^{2\gamma} f_0(x,v) \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ respectively for some $\gamma > d$.

Now, under the assumption (1.2), we develop the concept of almost global weak solution to (VP) system and prove its existence in the following theorem.

Theorem 1.1. *Let $f_0(x,v)$ be as above, and satisfy* (1.2)*, for $d = 2,3$. Then there exists an almost global weak solution $f(t,x,v) \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}^+, \mathrm{Prob}(\mathbb{R}^d \times \mathbb{R}^d)\big)$ and $E(t,x)$ to* (VP) *system in the following sense:*

(1) *$t \longmapsto f(t,\cdot,\cdot) \in \mathrm{Prob}(\mathbb{R}^d \times \mathbb{R}^d))$ is weakly continuous, that is, for any $\psi(x, \quad v) \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, we have*

$$\lim_{t \to t_0} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x,v) f(t,x,v)dxdv = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x,v) f(t_0,x,v)dxdv, 0 < t_0 < \infty.$$

(2) *$E(t,x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+, W^{1,p}_{\mathrm{loc}}) \cap C([0,\infty), W^{s,p}_{\mathrm{loc}})$, for all $0 < s < 1$, $1 < p < +\infty$.*

(3) *For every $t \in [0,T]$, $\rho(t,x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+, L^\infty) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^+, L^1)$.*

(4) *For every $\varphi(t,x,v) \in C_0^\infty|(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d)$, we have*

$$\int_0^\infty \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\partial_t \varphi(t,x,v) + v.\nabla_x \varphi(t,x,v) - E \cdot \nabla_v \varphi(t,x,v)) f(t,x,v) dtdxdv = 0,$$

$$E(t,x) = c(d) \int \frac{x-y}{|x-y|^d}\, \rho(t,y)dy.$$

Remark 1.1. As in [10], $\iint\limits_V f(t,x,v)dxdv$ is only regarded as the measure of $f(t,\cdot,\cdot)$ on a Borel subset V of $\mathbb{R}^{2d}$, so does $\int\limits_V f(t,x,v)dv$.

Remark 1.2. As in [3, 4, 6, 10], here we can not guarantee the uniqueness of the weak solution to (VP) in our sense because the operator $\partial_i \nabla \triangle^{-1}$ does not persist L^1 boundedness. On the other hand, by [8, 11], we know that for smooth initial data with compact support, there exists a unique classical solution to d-dimensional (VP) for $d = 2,3$. Hence, for convenience, we only prove Theorem 1.1 for $d = 2$; the method given here can be exactly extended to the 3-dimensional case.

§2. Proof of Theorem 1.1

Take $j(x) \in C_0^\infty(\mathbb{R}^2), j(x) \ge 0$, $\mathrm{supp}\, j(x) \subset \{x||x| \le 1\}$ and $\int j(x)dx = 1$, set

$$J_\varepsilon(x,v) = \varepsilon^{-4} j\Big(\frac{x}{\varepsilon}\Big) j\Big(\frac{v}{\varepsilon}\Big) \quad \text{for all} \quad \varepsilon > 0.$$

Let $f_0^\varepsilon(x,v) = f_0 * J_\varepsilon$. Then by the given conditions of $f_0(x,v)$ in Theorem 1.1, we have

$$\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} f_0^\varepsilon(x,v)dxdv = 1, \quad f_0^\varepsilon(x,v) \in C_0^\infty(\mathbb{R}^4),$$

and

$$\operatorname{supp} f_0^v(x,v) \subset \{(x,v)||x| \le R+1, |v| \le R+1\}, \quad \text{for} \quad 0 \le \varepsilon \le 1.$$

Now, we take $\varphi(y,v') = J_\varepsilon(x-y, v-v')$ in (1.2). We have

$$f_0^\varepsilon(x,v) \le \int \varepsilon^{-2} j\Big(\frac{v-v'}{\varepsilon}\Big) h_1(|v'|)dv' \le h_1(|v|), 0 \le \varepsilon \le 1,$$

where

$$h_1(|v|) = \begin{cases} h(0), & |v| \le 1, \\ h(|v|-1), & |v| \ge 1. \end{cases}$$

So, for convenience, we still assume $f_0^\varepsilon(x,v) \le h(|v|)$, with h satisfying the same conditions as the correspondence in (1.2). By [11], we know that there exists a unique smooth solution $f^\varepsilon(t,x,v) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2)$ and $E^\varepsilon(t,x) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$ to (VP) system with $f_0^\varepsilon(x,v)$ as initial data, and

$$\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} f^\varepsilon(t,x,v)\, dxdv = \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} f_0^\varepsilon(x,v)\, dxdv = 1, \tag{2.1}$$

$$f^\varepsilon(t,x,v) = f_0^\varepsilon(x_0,v_0), \tag{2.2}$$

where (x,v) and (x_0,v_0) satisfy

$$\begin{cases} \frac{dx}{dt} & = v, x|_{t=0} = x_0, \\ \frac{dv}{dt} & = -E^\varepsilon(t,x), v|_{t=0} = v_0. \end{cases} \tag{2.3}$$

Lemma 2.1. *For every $T > 0$, there exists $M_T > 0$ such that*

$$\|E^\varepsilon(t,x)\|_{L^\infty([0,T]\times\mathbb{R}^2)} \le M_T.$$

Proof. By $f_0^\varepsilon(x,v) \le h(|v|)$, (2.2) and (2.3), we find

$$f^\varepsilon(t,x,v) = f_0(x_0,v_0) \le h(|v_0|),$$

and

$$|v - v_0| \le t\|E^\varepsilon(t,x)\|_{L^\infty([0,t]\times\mathbb{R}^2)}.$$

So, by the noninreasing condition of $h(\cdot)$, we have

$$f^\varepsilon(t,x,v) \le h_{t\|E^\varepsilon(t,x)\|_{L^\infty([0,t]\times\mathbb{R}^2)}}(|v|),$$

where

$$h_r(|v|) = \begin{cases} h(0), & |v| \le r, \\ h(|v|-r), & |v| \le r. \end{cases}$$

On the other hand,

$$\|E^\varepsilon(t,\cdot)\|_{L^\infty} \le \int\limits_{|x-y|\le r} \frac{1}{|x-y|}\rho^\varepsilon(t,y)\, dy + \int\limits_{|x-y|\ge r} \frac{1}{|x-y|}\rho^\varepsilon(t,y)\, dy$$
$$\le 2\pi r\|\rho^\varepsilon(t,\cdot)\|_{L^\infty} + \frac{1}{r}\|\rho^\varepsilon(t,\cdot)\|_{L^1};$$

taking $r = \sqrt{\frac{\|\rho^\varepsilon(t,\cdot)\|_{L^1}}{2\pi\|\rho^\varepsilon(t,\cdot)\|_{L^\infty}}}$, we find

$$\|E^\varepsilon(t,\cdot)\|_{L^\infty} \le 2\pi\|\rho^\varepsilon(t,\cdot)\|_{L^1}^{\frac{1}{2}}\|\rho^\varepsilon(t,\cdot)\|_{L^\infty}^{\frac{1}{2}}, \tag{2.4}$$

while

$$\begin{aligned}\|\rho^\varepsilon(t,\cdot)\|_{L^1} &= \iint f^\varepsilon(t,x,v)\,dxdv = 1,\\ \|\rho^\varepsilon(t,\cdot)\|_{L^\infty} &\le \int h_{t\|E^\varepsilon(\cdot,\cdot)\|_{L^\infty([0,t]\times\mathbb{R}^2)}}(|v|)\,dv\\ &= \pi t^2\|E^\varepsilon(t,x)\|^2_{L^\infty([0,t]\times\mathbb{R}^2)}h(0) + \int h(|v|)\,dv.\end{aligned}$$

Denoting $\int h(|v|)\,dv$ by C_0, substituting the above two formulas to (2.4), and taking $t_1 = \frac{1}{4\pi^2}\sqrt{\frac{2\pi}{h(0)}}$, we obtain

$$\|E(t,x)\|_{L^\infty([0,t_1]\times\mathbb{R}^2)} \le 2\pi\sqrt{2C_0}. \tag{2.5}$$

In what follows, we will inductively prove the following assertion:

For any $t \in [\frac{n-1}{4\pi^2}\sqrt{\frac{2\pi}{h(0)}}, \frac{n}{4\pi^2}\sqrt{\frac{2\pi}{h(0)}}]$, n being an arbitrary positive integer, we have

$$\|E(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \le 2\pi\sqrt{2C_{n-1}}, \tag{2.6}$$

where

$$C_{n-1} = 1 + (n-1)^2(n-2)^2 + \cdots + (n-1)^2(n-2)^2\cdots(n-k)^2 + \cdots + [(n-1)!]^2.$$

To this end, we assume the above assertion is true for $n = k$. By

$$\frac{k}{4\pi^2}\sqrt{\frac{2\pi}{h(0)}}\cdot 2\pi\sqrt{2C_{k-1}} = \frac{k}{\pi}\sqrt{\frac{\pi C_{k-1}}{k(0)}},$$

and (2.2), we get

$$f^\varepsilon(t,x,v) \le h_{\frac{k}{\pi}\sqrt{\frac{\pi C_{k-1}}{h(0)}}}(|v|), \quad t \in \Big[0, \frac{k}{4\pi^2}\sqrt{\frac{2\pi}{k(0)}}\Big].$$

Denote $h_{\frac{k}{\pi}\sqrt{\frac{\pi C_{k-1}}{h(0)}}}(|v|)$ by $h_k(|v|)$. Then $h_k(0) = h(0)$. Repeating the proof of (2.5) with $h_k(|v|)$ instead of $h(|v|)$, we find

$$\|E(t,\cdot)\|_{L^\infty} \le 2\pi\sqrt{2C_k}, \quad t \in \Big[\frac{k}{4\pi^2}\sqrt{\frac{2\pi}{h(0)}}, \frac{k+1}{4\pi^2}\sqrt{\frac{2\pi}{h(0)}}\Big],$$

where C_k is defined as above. This complets the proof of (2.6).

On the other hand, for every $T > 0$, there exists a positive integer k such that

$$\frac{k-1}{4\pi^2}\sqrt{\frac{2\pi}{h(0)}} \le T \le \frac{k}{4\pi^2}\sqrt{\frac{2\pi}{h(0)}}.$$

Thus by (2.6),

$$\|E(\cdot,\cdot)\|_{L^\infty([0,t]\times\mathbb{R}^2)} \le 2\pi\sqrt{2C_{k-1}}.$$

This concludes Lemma 2.1.

Noticing that $E(t,x) = \nabla\triangle^{-1}\rho(t,x)$, where ∇ and $\triangle^{-1}$ are the divengence and inverse operator of $\triangle$ with respect to x respectively, we further have the following Lemma 2.2.

Lemma 2.2. *For every $T>0$, there exists $K_T>0$ such that*

$$\|E^\varepsilon(t,x)\|_{L^\infty([0,T],W^{1,p}_{loc})}\le K_T,$$

for $1<p<\infty$.

Proof. By Lemma 2.1, we have

$$\|E^\varepsilon(t,x)\|_{L^\infty([0,T],L^p_{loc})}\le N_T. \tag{2.7}$$

$$\partial_{x_i}E^\varepsilon(t,x)=\partial_i\nabla\triangle^{-1}\rho^\varepsilon(t,\cdot),$$

and $\xi_i\xi_j/|\xi|^2$ is a multiplier which is homogeneous of degree zero and is infinitely differentiable on the sphere. Hence by Theorem 3.6 of [9], $\partial_i\nabla\triangle^{-1}$ persists L^p boundedness for $1<p<\infty$; that is,

$$\begin{aligned}\|\partial_i E^\varepsilon(t,\cdot)\|_{L^\infty([0,T],L^p(\mathbb{R}^2))}&\le C\|\rho^\varepsilon(t,\cdot)\|_{L^\infty([0,T],L^p(\mathbb{R}^2))}\\&\le C\|\rho^\varepsilon(t,\cdot)\|^{p-1}_{L^\infty([0,t]\times\mathbb{R}^2)}\|\rho^\varepsilon(t,\cdot)\|_{L^\infty([0,t],L^1)}\\&\le((TM_T)^2h(0)+c_0)^{p-1},\quad i=1,2,\end{aligned} \tag{2.8}$$

where M_T is the same as the correspondence in Lemma 2.1. Combining (2.7) and (2.8) we find that Lemma 2.2 holds.

By (1.3) and applying Theorem 1.2 in [5], we know that for all $t\in[0,T]$, there exists a subsequence $\{f^{\varepsilon_j}(t,x,v)\}$ of $\{f^\varepsilon(t,x,v)\}$ such that $\{f^{\varepsilon_j}(t,x,v)\}$ weakly converges to $f(t,\cdot,\cdot)$ in $\mathcal{M}^+(\mathbb{R}^2\times\mathbb{R}^2)$(the positive Radon measure space). And by (2.2) and Lemma 2.1, we find

$$\operatorname{supp} f^\varepsilon(t,\cdot,\cdot)\subset\{(x,v)||x|\le R+Rt+\frac{1}{2}M_tt^2,|v|\le R+M_tt\}.$$

Take $\chi_t(x,v)\in C_0^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ which is equal to 1 in $\operatorname{supp} f^\varepsilon(t,\cdot,\cdot)$. Then we have

$$\begin{aligned}1&=\lim_{\varepsilon_j\to0}\iint_{\mathbb{R}^2\times\mathbb{R}^2}f^{\varepsilon_j}(t,x,v)\chi_t(x,v)dxdv=\iint\chi_t(x,v)\,f(t,x,v)dxdv\\&=\iint_{\mathbb{R}^2\times\mathbb{R}^2}f(t,\cdot,\cdot)dxdv;\end{aligned}$$

that is, $f(t,\cdot,\cdot)\in$Prob$(\mathbb{R}^2\times\mathbb{R}^2)$.

Now, for every $T>0$, denote the rational number set of $[0,T]$ by $\{t_r|r\in\mathbb{N}\}$. Then by diagnonal process, we can take a subsequence $\{f^{\varepsilon_j}(t,x,v)\}$ of $\{f^\varepsilon(t,x,v)\}$ and a sequence of probability measure $f(t_\gamma,x,v)$ such that $\{f^{\varepsilon_j}(t_r,x,v)\}$ weakly converges to $f(t_r,x,v)$ in $\mathcal{M}^+(\mathbb{R}^2\times\mathbb{R}^2)$. Hence, similiar to the idea in [6], we have

Lemma 2.3. *There exists a probability measure $f(t,\cdot,\cdot)$ which is weakly continuous with respect to $t\in[0,T]$, that is, for every $\psi\in C_0^\infty(\mathbb{R}^4)$, the function $\iint_{\mathbb{R}^2\times\mathbb{R}^2}\psi(x,v)f(t,x,v)\,dxdv$ is continuous with respect to $t\in[0,T]$, and a subsequence $\{f^{\varepsilon_j}(t,x,v)\}$ of $\{f^\varepsilon(t,x,v)\}$ such that*

$$\lim_{j\to\infty}\iint\psi(x,v)f^{\varepsilon_j}(t,x,v)dxdv=\iint\psi(x,v)f(t,x,v)dxdv.$$

Proof. Firstly, we prove the following assertion:

for every function $\psi(x,v)$ in a enumerable dense subset of $C_0^\infty(\mathbb{R}^4)$, the function

$$[0,T]\ni t\longmapsto\iint_{\mathbb{R}^2\times\mathbb{R}^2}\psi(x,v)f^{\varepsilon_j}(t,x,v)\,dxdv \tag{2.9}$$

is equicontinuous.

Take arbitrary t_1, $t_2 \in [0,T]$, then

$$\begin{aligned}
&\Big|\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \psi(x,v)(f^{\varepsilon_j}(t_2,x,v)-f^{\varepsilon_j}(t_1,x,v))dxdv\Big| \\
&=\Big|\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \psi(x,v)\int_{t_1}^{t_2}\partial_s f^{\varepsilon_j}(s,x,v)dsdxdv\Big| \\
&=\int_{t_1}^{t_2}\Big|\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \psi(x,v)(-v\cdot\nabla_x f^{\varepsilon_j}(s,x,v)+E^{\varepsilon_j}\cdot\nabla_v f^{\varepsilon_j}(s,x,v)dxdv\Big|ds \\
&=\int_{t_1}^{t_2}\Big|\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} (v\cdot\nabla_x\psi(x,v)-E^{\varepsilon_j}\cdot\nabla_v\psi(x,v))f^{\varepsilon_j}(s,x,v)dxdv\Big|ds.
\end{aligned}$$

Then, by Lemma 2.1,

$$\Big|\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \psi(x,v)(f^{\varepsilon_j}(t_2,x,v)-f^{\varepsilon_j}(t_1,x,v))dxdv\Big|\le C_{T,\varphi}|t_1-t_2|.$$

Secondly, by the discussion after Lemma 2.2, we know that $\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2}\psi(x,v)f^{\varepsilon_j}(t,x,v)dxdv$ is pointwise convergent on a dense subset of [0, T]. Combining the above facts with the following Lemma 2.4, we find that $\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2}\psi(x,v)f^{\varepsilon_j}(t,x,v)dxdv$ is uniformly convergent in [0,T]. Furthermore, following the discussion after Lemma 2.2, we know that for every $t\in[0,T]$, there exists a probability measure $f(t,\cdot,\cdot)$ and a subsequence $\{f^{\varepsilon_{j_n}}(t,x,v)\}$ of $\{f^{\varepsilon_j}(t,x,v)\}$ such that

$$\begin{aligned}
\lim_{j\to\infty}\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} f^{\varepsilon_j}(t,x,v)\psi(x,v)dxdv&=\lim_{n\to\infty}\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} f^{\varepsilon_{j_n}}(t,x,v)\psi(x,v)dxdv\\
&=\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2}\psi(x,v)f(t,x,v)dxdv.
\end{aligned}$$

Then, by diagnoal process and standard dense argument, there exists a common subsequence of $\{f^{\varepsilon_j}(t,x,v)\}$ for every $\psi\in C_0^\infty$, without arousing ambiguity, still denoted by $\{f^{\varepsilon_j}(t,x,v)\}$, such that

$$\lim_{j\to\infty}\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} f^{\varepsilon_j}(t,x,v)\psi(x,v)dxdv=\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2}\psi(x,v)f(t,x,v)dxdv.$$

And by the fact that the uniform limit of a sequence of continuous function is still a continuous function, the map $[0,T]\ni t\mapsto f(t,\cdot,\cdot)\in\mathrm{Prob}(\mathbb{R}^2\times\mathbb{R}^2)$ is weakly continuous.

Lemma 2.4. *Assume that $\{G^n(z)\}$ is a equicontinuous sequence of function defined on a compact subset K of $\mathbb{R}^n$ and pointwise convergent on a dense subset $\mathcal{A}$ of K. Then $\{G^n(z)\}$ is uniformly convergent on K.*

Lemma 2.5. *There exists a subsequence of $\{E^{\varepsilon_j}(t,x)\}$, without arousing ambiguity, still denoted by $\{E^{\varepsilon_j}(t,x)\}$, and $E(t,x)\in L^\infty([0,T],W^{1p}_{\mathrm{loc}})\cap C([0,T],W^{sp}_{\mathrm{loc}})$, for all $T>0, 0<s<1, 1<p<+\infty$, such that*

$$\lim_{j\to+\infty}E^{\varepsilon_j}(t,x)=E(t,x)\quad in\quad C([0,T],W^{sp}_{\mathrm{loc}}).$$

Proof. First, we prove that for every $t_1, t_2 \in [0,T]$, there exist $L > 0, C_T > 0$, such that

$$\|\rho^{\varepsilon_j}(t_1,\cdot) - \rho^{\varepsilon_j}(t_2,\cdot)\|_{H^{-L}} \le C_T|t_1 - t_2|. \tag{2.10}$$

Now, we take an arbitrary function $\psi(x) \in C_0^\infty(\mathbb{R}^2)$, multiplying this function by the first equation in (VP),

$$\begin{aligned}\Big|\int \psi(x)(\rho^{\varepsilon_j}(t_1,x) - \rho^{\varepsilon_j}(t_2,x))dx\Big| &\le \int_{t_1}^{t_2} \Big|\iint_{\mathbb{R}^2\times\mathbb{R}^2} \psi(x)\partial_s f^{\varepsilon_j}(s,x,v)dvdx\Big|ds \\ &= \int_{t_1}^{t_2} \Big|\iint_{\mathbb{R}^2\times\mathbb{R}^2} \psi(x)v\cdot\nabla_x f^{\varepsilon_j}(t,x,v)dxdv\Big|dt \\ &\le (M_T T + R)\|\nabla_x\psi\|_{L^\infty}|t_1 - t_2| \\ &\le (M_T T + R)\|\psi\|_{H^L}|t_1 - t_2|;\end{aligned}$$

for every $L > \frac{2}{2} + 1 = 2$ by Sobolev interpolation inequality, (2.10) holds.

$$\begin{aligned}&\Big|\int \psi(x)(E^{\varepsilon_j}(t_1,x) - E^{\varepsilon_j}(t_2,x))\,dx\Big| \\ &= \Big|\int \psi(x)(\nabla\triangle^{-1}\rho^{\varepsilon_j}(t_1,x) - \nabla\triangle^{-1}\rho^{\varepsilon_j}(t_2,x))\,dx\Big| \\ &= \Big|\int_{|x|\le R+RT+\frac{1}{2}MT^2} \nabla\triangle^{-1}\psi(x)(\rho(t_1,x) - \rho(t_2,x))\,dx\Big| \\ &\le C_T\|\nabla\triangle^{-1}\psi(x)\|_{H^L\left\{x\middle||x|\le R+T+\frac{1}{2}M_T T^2\right\}}\cdot|t_1 - t_2|.\end{aligned} \tag{2.11}$$

Since $|\nabla\triangle^{-1}\psi(x)| \le \int \frac{1}{|x-y|}|\varphi(y)|dy$, by Riesz potential theory[9],

$$\|\nabla\triangle^{-1}\psi(x)\|_{L^2} \le \|\psi\|_{L^1}.$$

Hence, by Poincare's inequality[14]

$$\begin{aligned}\|\nabla\triangle^{-1}\varphi(x)\|_{L^2(x||x|\le R+RT+\frac{1}{2}MT^2)} &\le g_T(\|\partial_1\nabla\triangle^{-1}\psi(x)\|_{L^2} + \|\partial_2\nabla\triangle^{-1}\psi(x)\|_{L^2(R^2)}) \\ &\le g_T\|\psi(x)\|_{L^2}.\end{aligned}$$

Noticing that $\partial_i\nabla\triangle^{-1}$ is an operator of Calderon-Zygmund type, and ∂^α commutes with $\partial_i\nabla\triangle^{-1}$, where $\alpha = (\alpha_1,\alpha_2), \alpha_1 + \alpha_2 \le L-1$, we have

$$\|\partial^\alpha\partial_i\nabla\triangle^{-1}\psi(x)\|_{L^2} \le C\|\partial^\alpha\psi(x)\|_{L^2} \le C\|\psi\|_{H^{L-1}}.$$

Combining the above and (2.11), we know that

$$\|E^{\varepsilon_j}(t_1,\cdot) - E^{\varepsilon_j}(t_2,.)\|_{H^{-L+1}} \le C_T|t_1 - t_2|. \tag{2.12}$$

The mapping $W^{1p}_{\rm loc} \hookrightarrow W^{sp}_{\rm loc}$, for $0 < s < 1$, is a compact imbeding. Hence, by (2.12) and using Lions-Aubin Lemma, we know that, for every $\varepsilon > 0$, the following inequality holds:

$$\|E^{\varepsilon_j}(t_1,.) - E^{\varepsilon_j}(t_2,.)\|_{W^{sp}_{\rm loc}} \le \varepsilon\|E^{\varepsilon_j}(t,s)\|_{L^\infty([0,T],W^{1,p}_{\rm loc})} + C_{\varepsilon,T}|t_1 - t_2| \tag{2.13}$$

for $t_1, t_2 \in [0,T], 0 < s < 1, 1 < p < \infty$. And by diagonal precess, we can extract a subsequence of $\{E^{\varepsilon_j}(t,x)\}$, still denoted by $\{E^{\varepsilon_j}(t,x)\}$, which converges in $W^{sp}_{\rm loc}$ for t in the rational number subset $\{t_\gamma|\gamma \in N\}$ of [0, T]. For every fixed $t \in [0,T]$ and $\delta > 0$ there exists some $t_\gamma \in \{t_\gamma|\gamma \in N\}$ such that $|t - t_\gamma| < \delta$. Then

$$\begin{aligned}\|E^{\varepsilon_j}(t,\cdot) - E^{\varepsilon_k}(t,\cdot)\|_{W^{sp}_{\rm loc}} \le{}& \|E^{\varepsilon_j}(t,\cdot) - E^{\varepsilon_j}(t_\gamma,\cdot)\|_{W^{sp}_{\rm loc}} + \|E^{\varepsilon_j}(t_\gamma,\cdot) - E^{\varepsilon_k}(t_\gamma,\cdot)\|_{W^{sp}_{\rm loc}} \\ &+ \|E^{\varepsilon_k}(t_\gamma,\cdot) - E^{\varepsilon_k}(t,\cdot)\|_{W^{sp}_{\rm loc}},\end{aligned}$$

and

$$\|E^{\varepsilon_j}(t_1,\cdot)-E^{\varepsilon_k}(t_2,\cdot)\|_{W^{sp}_{\rm loc}}\le\|E^{\varepsilon_j}(t_1,\cdot)-E^{\varepsilon_j}(t_2,\cdot)\|_{W^{sp}_{\rm loc}}+\|E^{\varepsilon_j}(t_2,\cdot)-E^{\varepsilon_k}(t_2,\cdot)\|_{W^{sp}_{\rm loc}}.$$

Hence by (2.13), we conclud that $\{E^{\varepsilon_j}(t,\cdot)\}$ converges to some $E(t,x)$ in $C([0,T],W^{sp}_{\rm loc})$. Therefore by Lemma 2.2, we know that $E(t,x)\in L^\infty([0,T],W^{1.p}_{\rm loc})\cap C([0,T],W^{s.p}_{\rm loc})$, for every $0<s<1$ and $1<p<+\infty$.

Since Lemma 2.3 and Lemma 2.5 hold for every $T>0$, there exist two subsequences, $\{f^{\varepsilon_j}(t,x,v)\}$, $\{E^{\varepsilon_j}(t,x)\}$ of $\{f^{\varepsilon}(t,x,v)\}$ and $\{E^{\varepsilon}(t,x)\}$ respectively, such that Lemma 2.3 and Lemma 2.5 hold for arbitrary $0<T<+\infty$ by diagonal process.

Having the above preparation, now we can prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.3, Lemma 2.5, it is easy to know that

$$\begin{aligned}&\lim_{j\to\infty}\int E^{\varepsilon_j}(t,x)\cdot\nabla_v\varphi(t,x,v)f^{\varepsilon_j}(t,x,v)dtdxdv\\&=\int E(t,x)\cdot\nabla_v\varphi(t,x,v)f(t,x,v)dtdxdv\end{aligned}\tag{2.14}$$

for every $\varphi(t,x,v)\in C_0^\infty((0,\infty)\times\mathbb{R}^4)$. It implies that first equality of (4) in Theorem 1.1 holds.

Up to now, to verify Theorem 1.1, we only need to prove (3) and the second equality of (4) in Theorem 1.1. It will be realized when we have the following Lemma 2.6.

Lemma 2.6. *Let $\rho(t,x)=\int_{\mathbb{R}^2}f(t,x,v)dv$, where $f(t,x,v)$ is obtained as above. Then $\rho(t,.)$ is absolutely continuous with respect to Lebesgue measure, and by redefining $\rho(t,\cdot)$ on a set of zero measure, $\rho(\cdot,\cdot)\in L^\infty_{\rm loc}(R^+,L^\infty(\mathbb{R}^2))\cap L^\infty_{\rm loc}(\mathbb{R}^+,L^1(\mathbb{R}^2))$, and*

$$E(t,x)=C(2)\int\frac{x-y}{|x-y|^2}\,\rho(t,y)\,dy,$$

where $t\in(0,\infty)$.

Proof. Take any function $\psi(x)\in C_0^\infty(\mathbb{R}^2)$ and $\chi(v)\in C_0^\infty(\mathbb{R}^2)$, such that $\chi(v)$ is equal to 1 when $|v|\le R+M_TT$, for some $T>t$. Then

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}\psi(x)\rho(t,x)dx\Big|&=\Big|\lim_{j\to+\infty}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\psi(x)\chi(v)f^{\varepsilon_j}(t,x,v)dxdv\Big|\\&\le\lim_{j\to+\infty}\int_{\mathbb{R}^2}|\psi(x)|\rho^{\varepsilon_j}(t,x)dx\\&\le((TM_T)^2h(0)+C_0)\|\psi\|_{L^1}\doteq L_T\|\psi\|_{L^1}.\end{aligned}\tag{2.15}$$

Notice that $C_0^\infty(\mathbb{R}^2)$ is dense in $L^1(\mathbb{R}^2)$ and L^1 is the dual of L^∞. Hence by (2.15) there exists $g(t,x)\in L^\infty_{\rm loc}([0,\infty),L^\infty(\mathbb{R}^2))$ such that

$$\int\psi(x)\rho(t,x)dx=\int\psi(x)g(t,x)dx.\tag{2.16}$$

Following this equality, we find that $\rho(t,\cdot)$ is absolutely continuous with respect to Lebesgue measure, and $g(t,\cdot)=\frac{d\rho(t,\cdot)}{dm}$, which is the Randon-Nikoydim derivetive of $\rho(t,\cdot)$ with respect to Lebesgue measure m, and by (2.16) $\rho(t,\cdot)\overset{\rm a.\ e.}{=}g(t,\cdot)$.

Next, we prove the last assertion of Lemma 2.6.

Take an arbitrary $\delta>0$,

$$\Big|\int_{|x-y|\le\delta}\frac{x-y}{|x-y|^2}\rho^{\varepsilon_j}(t,y)\,dy\Big|\le\Big(2\pi\int h_{tM_t}(|v|)\,dv\Big)\delta,$$

while

$$\Big|\int_{|x-y|\le\delta}\frac{x-y}{|x-y|^2}\rho(t,y)dy\Big|\le 2\pi\|\rho(t,.)\|_{L^\infty}\delta.$$

For every fixed $x\in\{\mathbb{R}^2\big||x-y|\ge\delta\}$, $\frac{x-y}{|x-y|^2}$ is a smooth function of y which tends to zero when $|y|$ tends to infinity, so we have

$$\lim_{\varepsilon_j\to 0}\Big|\int_{|x-y|\ge\delta}\frac{x-y}{|x-y|^2}\rho^{\varepsilon_j}(t,x)\,dx-\int_{|x-y|\ge\delta}\frac{x-y}{|x-y|^2}\rho(t,y)dy\Big|$$
$$=\lim_{\varepsilon_j\to 0}\iint_{|x-y|\ge\delta}\frac{x-y}{|x-y|^2}\chi(v)(f^{\varepsilon_j}(t,y,v)-f(t,y,v))\,dydv.$$

Hence from Lemma 2.3, the last conclusion of Lemma 2.6 holds.

Acknowledgement. The first author would thank professor Zheng Yuxi for sending him the papers [7] and [10].

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