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RECOGNITION AND CLASSIFICATION FOR O(n)-EQUIVARIANT BIFURCATIONS WITH O(n)-CODIMENSION LESS THAN 5***

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Abstract

Bifurcation problems equivariant under the standard action of the orthogonal group $\mathbf{O}(n)$ up to $\mathbf{O}(n)$-codimension 4 are classified into 19 classes. For each class the normal form and one universal unfolding are calculated and the recognition problem is solved.

Keywords $\mathbf{O}(n)$-equivariant bifurcation, Normal form, Universal unfolding, Recognition problem

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§1. Introduction

In this paper we investigate local bifurcations equivariant under the standard action of the orthogonal group $\mathbf{O}(n)$ on $\mathbb{R}^n$. One of the motivations to study such problems is that many physical systems possess the spherical symmetry, for example, in the study of buckling of a planar disk of a spherical shell. Another motivation comes from some mathematical requirement, for example, the study of degenerate Hopf bifurcations (see [2, 6]). Most of these problems can be reduced to the study of the local bifurcation diagrams of $\mathbf{O}(n)$-equivariant bifurcation problems. A fundamental approach to the study of $\mathbf{O}(n)$-equivariant bifurcation problems is the equivariant singularity theory which was developed in [3, 4]. One of the goals of singularity theory is to classify and characterize equivalent classes. The other one is to study perturbation problems, which is related to universal unfoldings and then induces the notion of $\mathbf{O}(n)$-codimension.

There are some previous results for the special case $n = 1$ due to several authors. Golubitsky and Langford in [2] studied the $\mathbb{Z}_2$-equivariant bifurcation up to $\mathbb{Z}_2$-codimension three. They gave a complete discussion of classification, unfoldings and recognition. Shi in [6] generalized their result to the case of $\mathbb{Z}_2$-codimension four. In the present paper we show that for any positive integer n every $\mathbf{O}(n)$-equivariant bifurcation of $\mathbf{O}(n)$-codimension less than five is equivalent to one of the 19 normal forms listed in Table 3.1. For each class the

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normal form and one universal unfolding ard calculated, and by using the method proposed by Melbourne in [5] the recognition problem is solved.

The remainder of the paper is organized as follows. In Section 2 some notations and concepts are introduced. Then in Section 3 the classification and unfolding theorem (Theorem 3.1) are established. Finally in Section 4 the recognition conditions are derived (Theorem 4.1).

§2. Preliminaries

Let the n-dimensional orthogonal group $\mathbf{O}(n)$ act on $\mathbb{R}^n$ in the standard way. We first introduce some invariants related to $\mathbf{O}(n)$-equivariant bifurcation problems. Let $(x,\lambda) \in \mathbb{R}^n \times \mathbb{R}$. Denote by $\mathcal{E}_{x,\lambda}((n))$ the set of all $\mathbf{O}(n)$-invariant C^∞ germs, i.e., germs $f: (\mathbb{R}^n \times \mathbb{R}, 0) \to \mathbb{R}$ satifying

$$f(\gamma x,\lambda) = f(x,\lambda), \quad \forall \gamma \in \mathbf{O}(n), \quad x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}; \tag{2.1}$$

by $\overrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ the set of all $\mathbf{O}(n)$-equivariant C^∞ germs, i.e., germs $g: (\mathbb{R}^n \times \mathbb{R}, 0) \to \mathbb{R}^n$ satisfying

$$g(\gamma x,\lambda) = \gamma g(x,\lambda), \quad \forall \gamma \in \mathbf{O}(n), \quad x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}; \tag{2.2}$$

by $\overleftrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ the set of all matrix-valued $\mathbf{O}(n)$-equivariant C^∞ germs, i.e., germs g : $(\mathbb{R}^n \times \mathbb{R}, 0) \to \mathcal{L}(\mathbb{R}^n)$ satisfying

$$S(\gamma x,\lambda) = \gamma S(x,\lambda), \quad \forall \gamma \in \mathbf{O}(n), \quad x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}; \tag{2.3}$$

and finally, by $\mathcal{E}_\lambda$ the set of all C^∞ germs $(\mathbb{R},0) \to \mathbb{R}$.

The following proposition gives the $\mathbf{O}(n)$-invatriant theory.

Proposition 2.1. *Let* $u = |x|^2$. *Then*

(a) *Every* $f \in \mathcal{E}_{x.\lambda}(\mathbf{O}(n))$ *can be expressed as*

$$f(x,\lambda) = r(u,\lambda), \quad r \in \mathcal{E}_{u,\lambda}. \tag{2.4}$$

(b) *Every* $g \in \overrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ *can be expressed as*

$$g(x,\lambda) = r(u,\lambda)x, \quad r \in \mathcal{E}_{u,\lambda}. \tag{2.5}$$

(c) *Every* $S \in \overleftrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ *can be expressed as*

$$S(x,\lambda) = p(u,\lambda)I_n + q(u,\lambda)xx^T, \quad p,q \in \mathcal{E}_{u,\lambda}. \tag{2.6}$$

Proof. For the special case $n = 1, \mathbf{O}(n) = \mathbb{Z}_2$, the theorem is well-known[3]. We consider general case $n > 1$. Set the group

$$\mathbb{Z}_2 = \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & I_{n-1} \end{bmatrix} \right\}$$

that acts on the space $V := \{(x_1, 0, \cdots, 0) \in \mathbb{R}^n | x_1 \in \mathbb{R}\}$. Since $\mathbf{O}(n)$ acts transitively on the sphere $\mathbf{S}^{n-1}$, it follows that for any $x \in \mathbb{R}^n$ there exists an orthogonal transformation $\gamma_x \in \mathbf{O}(n)$ such that $\gamma_x x = (|x|,\ 0, \cdots, 0) \in V$.

(a) For $f \in \mathcal{E}_{x,\lambda}(\mathbf{O}(n))$, obviously, $f \mid V : (V \times \mathbb{R}, 0) \to \mathbb{R}$ is a C^∞ $\mathbb{Z}_2$-invariant germ. One has an $r \in \mathcal{E}_{u,\lambda}$ such that $f(y,\lambda) = r(|y|^2,\lambda)$ for all $(y,\lambda) \in (V \times \mathbb{R}, 0)$. So for $(x,\lambda) \in (\mathbb{R}^n \times \mathbb{R}, 0)$, $f(x,\lambda) = f(\gamma_x^{-1}(|x|,\ 0, \cdots, 0), \lambda) = f((|x|, 0, \cdots, 0), \lambda) = r(|x|^2, \lambda)$.

(b) For $g \in \overrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ it is easy to see that $g(V \times \mathbb{R}) \subset V$ and $g \mid V : (V \times \mathbb{R}, 0) \to V$ is a C^∞ $\mathbb{Z}_2$-equivariant germ. Hence there exists a germ $r \in \mathcal{E}_{u,\lambda}$ such that $g(y, \lambda) = r(|y|^2, \lambda)y$ for all $(y, \lambda) \in (V \times \mathbb{R}, 0)$. So for $(x, \lambda) \in (\mathbb{R}^n \times \mathbb{R}, 0)$,

$$\begin{aligned} g(x, \lambda) &= g(\gamma_x^{-1}(|x|, 0, \cdots, o), \lambda) = \gamma_x^{-1} g((|x|, 0, \cdots, o), \lambda) \\ &= \gamma_x^{-1} r((|x|, \lambda)(|x|, 0, \cdots, 0) = r(|x|^2, \lambda)x. \end{aligned}$$

(c) can be proved by arguments similar to those for (b).

Remark 2.1. (i) We introduce an equivalence $\sim$ on $\mathcal{E}_{u,\lambda}$:

$$r_1 \sim r_2 \quad \text{if and only if } r_1(u, \lambda) = r_2(u, \lambda) \text{ for all } u > 0 \text{ and } \lambda \in (\mathbb{R}, 0).$$

Then (2.5) defines a 1-1 correspondence between germs in $\overrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ and equivalence classes in $\mathcal{E}_{u,\lambda}/\sim$. For this reasom we identify $g \in \overrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ with the corresponding equivalent class $[r] \in \mathcal{E}_{u,\lambda}/\sim$. It is easy to that $r_1 - r_2$ is a flat germ whenever $r_1 \sim r_2$. As we will see below, for germs of finite $\mathbf{O}(n)$-codimension the classfication and recognition problems depend only on their low order derivatives at the origin and then the solution of these problems do not depend on the particular choice of the representative in an equivalent class.

(ii) $\mathcal{E}_{u,\lambda}$ and $\mathcal{E}_\lambda$ are local rings whose maximal ideals are respectively

$$\mathcal{M}_{u,\lambda} = \{r \in \mathcal{E}_{u,\lambda} | r(0,0) = 0\} \quad \text{and} \quad \mathcal{M}_\lambda = \{r \in \mathcal{E}_\lambda | r(0) = 0\}.$$

By (b) and (c), $\overrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ and $\overleftrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ are $\mathcal{E}_{u,\lambda}$-modules generated by $\{x\}$ and $\{I_n, xx^T\}$ respectively. Let R be a ring and $\mathcal{I}$ be R or an ideal of R. Let M be an R-module and $\mathcal{S}$ a subset of M. We denote by

$$\mathcal{IS} := \{r_1 g_1 + \cdots + r_k g_k | r_i \in \mathcal{I},\ g_i \in \mathcal{S}, \quad i = 1, \cdots, k\}$$

the submodule of M generated by elements all like rg, where $r \in \mathcal{I}$ and $g \in \mathcal{S}$. Hence

$$\overrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n)) = \mathcal{E}_{u,\lambda}\{x\} \quad \text{and} \quad \overleftrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n)) = \mathcal{E}_{u,\lambda}\{I_n,\ xx^T\}.$$

Now we condider bifurcation problems. An $\mathbf{O}(n)$-equivariant bifurcation problem g is a germ $g \in \overrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ satisfying $g(0, 0) = 0$ and $\det(dg)_{0,0} = 0$, where dg is the derivative of g with respect to x. The (local) bifurcation diagram of g is the set

$$\{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}, 0) | g(x, \lambda) = 0\}.$$

A triple $(S, X, \Lambda) \in \overleftrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n)) \times \overrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n)) \times \mathcal{E}_\lambda$ is called an $\mathbf{O}(n)$-equivalence if it datisfies

$$X(0, 0) = 0, \quad \Lambda(0) = 0, \quad \det S(0, 0) > 0, \quad \det(dX)_{0,0} > 0, \quad \Lambda'(0) > 0; \tag{2.7}$$

and it is called a strong $\mathbf{O}(n)$-equivalence if furthermore $\Lambda(\lambda) \equiv \lambda$. Two germs $g, h \in \overrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ are said to be (strongly) equivalent if there is a (respectively, strong) $\mathbf{O}(n)$-equivalence (S, X, Λ) such that

$$h(x, \lambda) = S(x, \lambda) g(X(x, \lambda), \Lambda(\lambda)), \quad \forall (x, \lambda) \in (\mathbb{R}^n \times \mathbb{R}, 0), \tag{2.8}$$

Hence equivalent germs have diffeomorphic bifurcation diagrams. Let $\alpha \in \mathbb{R}^k$. The notations $\mathcal{E}_{\lambda,\alpha}, \mathcal{E}_{x,\lambda,\alpha}(\mathbf{O}(n)), \overrightarrow{\mathcal{E}}_{x,\lambda,\alpha}(\mathbf{O}(n))$ and $\overleftrightarrow{\mathcal{E}}_{x,\lambda,\alpha}(\mathbf{O}(n))$ have the similar meaning as their counterparts with a single paramter λ. A k-parameter $\mathbf{O}(n)$-unfolding of $g \in \overrightarrow{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ is a germ $G \in \overrightarrow{\mathcal{E}}_{x,\lambda,\alpha}(\mathbf{O}(n))$ satisfying $G(x, \lambda, 0) = g(x, \lambda)$ for all $(x, \lambda) \in (\mathbb{R}^n \times \mathbb{R}, 0)$, where

$\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{R}^k$ are called unfolding parameters. Equivalences on $\vec{\mathcal{E}}_{x,\lambda,\alpha}(\mathbf{O}(n))$ can be defined similarly as those on $\vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ (see [4]). An $\mathbf{O}(n)$-unfolding $G \in \vec{\mathcal{E}}_{x,\lambda,\alpha}(\mathbf{O}(n))$ of $g \in \vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ is said to be versal if any $\mathbf{O}(n)$-unfolding of g is equivalent to an unfolding induced from G. Universal $\mathbf{O}(n)$-unfoldings of g are the versal ones with the least number of unfolding parameters, which is called the $\mathbf{O}(n)$-codimension of g and denoted by $\mathrm{codim}_{\mathbf{O}(n)} g$.

Let $\mathcal{D}(\mathbf{O}(n))$ ($\mathcal{D}^s(\mathbf{O}(n))$) be set of all (respectively, strong) $\mathbf{O}(n)$-equivalences. Then $\mathcal{D}(\mathbf{O}(n))$ is a group with a suitably defined binary operation and $\mathcal{D}^s(\mathbf{O}(n))$ is a subgroup of $\mathcal{D}(\mathbf{O}(n))$ and their actions on $\vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ are defined as the right-hand side in (2.8) (see [5]). For $g \in \vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ denote by $T\mathcal{D}^{(s)}(g, \mathbf{O}(n))$ the tangent space to the group orbit $\mathcal{D}(\mathbf{O}(n)) \cdot g$ (respectively, $\mathcal{D}^s(\mathbf{O}(n)) \cdot g$). It is easy to see that

$$T\mathcal{D}^s([r], \mathbf{O}(n)) = \mathcal{E}_{u,\lambda}\{[r], [ur_u]\}, \tag{2.9}$$

$$T\mathcal{D}([r], \mathbf{O}(n)) = \mathcal{E}_{u,\lambda}\{[r], [ur_u]\} + \mathcal{E}_\lambda\{[\lambda r]\}. \tag{2.10}$$

They are respectively an $\mathcal{E}_{u,\lambda}$-submodule and an $\mathcal{E}_\lambda$-submodule of $\vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$.

Remark 2.2. Since (2.5), (2.9), and (2.10) have the same expressions as those in $\mathbb{Z}_2$ case, it is not surprising that some discussions in this paper parallel to that in [2].

A submodule M of $\vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ is said to be intrinsic if it consists of entire $\mathcal{D}^s(\mathbf{O}(n))$-orbits. For a subset $S \subset \vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ containing zero we denote by $\mathrm{It}r_{\mathcal{D}^s} S$ the maximal intrinsic submodule contained in S.

The following theorems are our main tools to classity $\mathbf{O}(n)$-equivariant germs and to calculate the normal forms and universal unfoldings for each class.

Theorem 2.1. *Let M be a submodule of $\vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ of finite codimension. Then M is intrisic if and only if $M = \langle u^{k_1}\lambda^{l_1}, u^{k_2}\lambda^{l_2}, \cdots, u^{k_s}\lambda^{l_s}\rangle\{x\}$ for some intrgers k_i, l_i ($i = 1, \cdots, s$) such that $k_1 > k_2 > \cdots > k_s = 0 = l_1 < l_2 < \cdots < l_s$. Here $\langle u^{k_1}\lambda^{l_1}, u^{k_2}\lambda^{l_2}, \cdots, u^{k_s}\lambda^{l_s}\rangle$ is the ideal of $\mathcal{E}_{u,\lambda}$ generated by $u^{k_i}\lambda^{l_i}$.*

Proof. This theorem can be proved as [3, Proposition VI.2.8].

Theorem 2.2. *Let $g, p \in \vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$.*

(a) (Homotopy, [1, Theorem 2.2], [4, Theorem XIV.1.3]) *If $T\mathcal{D}^{(s)}(g, \mathbf{O}(n)) = T\mathcal{D}^{(s)}(g + tp, \mathbf{O}(n))$, for all $t \in [0,1]$, then $g + tp$ is (strongly) $\mathbf{O}(n)$-equivalent to g for all $t \in [0,1]$.*

(b) ([4, Theorem XIV.7.2]) *If $p \in \mathrm{Itr}_{\mathcal{D}^s} \mathcal{M}_{u,\lambda} TD^s(g, \mathbf{O}(n))$, then $g + p$ is strongly $\mathbf{O}(n)$-equivalent to g.*

Theorem 2.3 (Versal Unfolding Theorem, [4,Theorem XV.2.1]). *Suppose $G \in \vec{\mathcal{E}}_{x,\lambda,\alpha}(\mathbf{O}(n))$ be a k-parameter $\mathbf{O}(n)$-unfolding of $g \in \vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$. Then G is versal if and only if*

$$\vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n)) = T(g, \mathbf{O}(n)) + \mathbb{R}\Big\{\frac{\partial G}{\partial \alpha_1}\Big|_{\alpha=0}, \cdots, \frac{\partial G}{\partial \alpha_k}\Big|_{\alpha=0}\Big\}, \tag{2.11}$$

where

$$T(g, \mathbf{O}(n)) = \mathcal{E}_{u,\lambda}\{g, dg \cdot x\} + \mathcal{E}_\lambda\{g_\lambda\} \tag{2.12}$$

is the tangent space. Moreover, if $\left\{\frac{\partial G}{\partial \alpha_1}(\cdot, 0), \cdots, \frac{\partial G}{\partial \alpha_k}(\cdot, 0)\right\}$ is a basis of a subspace of $\vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ complement to $T(g, \mathbf{O}(n))$, then G is a universal $\mathbf{O}(n)$-unfolding of g.

§3. Classification and Unfoldings

Throughout this paper we assume $\varepsilon, \delta, \sigma = \pm 1$ and denote

$$r_{i,j} = \frac{\partial^{i+j} r}{\partial u^i \partial \lambda^j}(0,0).$$

The following classsification and unfolding theorem is one of the two main results of this paper.

Theorem 3.1. *Let* $g \equiv [r] \in \vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$, *where* $r \in \mathcal{E}_{u,\lambda}$ *satisfying* $r_{0,0} = 0$. *If* $\mathrm{codim}_{\mathbf{O}(n)} g \le 4$, *then* g *is* $\mathbf{O}(n)$*-equivalent to one of the normal forms listed in* Table 3.1 *where the* $\mathbf{O}(n)$*-codimension and a universal unfolding for each normal form are also given.*

Table 3.1 Normal forms, $\mathbf{O}(n)$-Codimensions and Universal Unfoldings

No.	Normal Form h	$\mathrm{codim}_{\mathbf{O}(n)} h$	Universal Unfolding of h
(1)	$(\delta u + \varepsilon\lambda)x$	0	h
(2)	$(\delta u + \varepsilon\lambda^2)x$	1	$h + \alpha_1 x$
(3)	$(\delta u^2 + \varepsilon\lambda)x$	1	$h + \alpha_1 ux$
(4)	$(\delta u + \varepsilon\lambda^3)x$	2	$h + (\alpha_1 + \alpha_2\lambda)x$
(5)	$(\delta u^3 + \varepsilon\lambda)x$	2	$h + (\alpha_1 u + \alpha_2 u^2)x$
(6)	$(\delta u^2 + \varepsilon\lambda^2)x$	3	$h + (\alpha_1 + \alpha_2 u + \alpha_3 u\lambda)x$
(7)	$(\delta u + \varepsilon\lambda^4)x$	3	$h + (\alpha_1 + \alpha_2\lambda + \alpha_3\lambda^2)x$
(8)	$(\delta u^2 + 2bu\lambda + \varepsilon\lambda^2)x$, $b \ne 0, b^2 \ne \varepsilon\delta$	3	$h + (\alpha_1 + \alpha_2\lambda + \alpha_3 u)x$
(9)	$(\delta u^2 + 2\sigma u(\lambda + \lambda^2) + \delta\lambda^2)x$	3	$h + (\alpha_1 + \alpha_2\lambda + \alpha_3\lambda^2)x$
(10)	$(\delta u^2 + \sigma u\lambda + \varepsilon\lambda^3)x$	3	$h + (\alpha_1 + \alpha_2\lambda + \alpha_3\lambda^2)x$
(11)	$(\delta u^3 + \sigma u\lambda + \varepsilon\lambda^2)x$	3	$h + (\alpha_1 + \alpha_2\lambda + \alpha_3 u^2)x$
(12)	$(\delta u^4 + \varepsilon\lambda)x$	3	$h + (\alpha_1 u + \alpha_2 u^2 + \alpha_3 u^3)x$
(13)	$(\delta u + \varepsilon\lambda^5)x$	4	$h + (\alpha_1 + \alpha_2\lambda + \alpha_3\lambda^2 + \alpha_3\lambda^3)x$
(14)	$(\delta u^2 + \sigma u\lambda + \varepsilon\lambda^4)x$	4	$h + (\alpha_1 + \alpha_2\lambda + \alpha_3\lambda^2 + \alpha_4 u)x$
(15)	$(\delta u^2 + \sigma u\lambda^2 + \varepsilon\lambda^3)x$	4	$h + (\alpha_1 + \alpha_2\lambda + \alpha_3 u + \alpha_4 u\lambda)x$
(16)	$(\delta u^3 + \sigma u\lambda + \varepsilon\lambda^3)x$	4	$h + (\alpha_1 + \alpha_2\lambda + \alpha_3\lambda^2 + \alpha_4 u^2)x$
(17)	$(\delta u^3 + \sigma u^2\lambda + \varepsilon\lambda^2)x$	4	$h + (\alpha_1 + \alpha_2\lambda + \alpha_3 u + \alpha_4 u\lambda)x$
(18)	$(\delta u^4 + \sigma u\lambda + \varepsilon\lambda^2)x$	4	$h + (\alpha_1 + \alpha_2 u + \alpha_3 u^2 + \alpha_4 u^3)x$
(19)	$(\delta u^5 + \varepsilon\lambda)x$	4	$h + (\alpha_1 u + \alpha_2 u^2 + \alpha_3 u^3 + \alpha_4 u^4)x$

Note. In this table $u = |x|^2$ and all the α_is are unfolding parameters.

The proof of Theorem 3.1 needs several lemmas.

Lemma 3.1. *Let* $r \in \mathcal{E}_{u,\lambda}$, $r(0,0) = 0$. *Suppose that* $[r]$ *has finite* $\mathbf{O}(n)$*-codimension. Then*

(a) $k \equiv \min\{i |\, r_{i,0} \ne 0\} < \infty$ *and* $l \equiv \min\{j |\, r_{0,j} \ne 0\} < \infty$.

(b) $T([r], \mathbf{O}(n)) \subset \mathcal{E}_{u,\lambda}\{u^k x, u\lambda x, \lambda^i x\} + \mathbb{R}\{[r_\lambda]\}$ *and hence* $\mathrm{codim}_{\mathbf{O}(n)}[r] \ge k + l - 2$.

(c) $[r]$ *is* $\mathbf{O}(n)$*-equivalent to*

$$(\varepsilon\lambda^l + a_1(\lambda)u + \cdots + a_{k-1}(\lambda)u^{k-1} + \delta u^k + a_{k+2}(u,\lambda)u^{k+2})x, \tag{3.1}$$

where $a_j(0) = 0,\ 1 \le j < k$.

Proof. Since $[r]$ is of finite $\mathbf{O}(n)$-codimension, it follows obviously that $k < \infty$ and $l < \infty$, so (a) holds. We may assume that

$$r(u,\lambda) = b_0(\lambda)\lambda^l + b_1(\lambda)u + \cdots + b_{k-1}(\lambda)u^{k-1} + b_k(\lambda)u^k + b_{k+1}(\lambda)u^{k+1} + b_{k+2}(u,\lambda)u^{k+2}, \tag{3.2}$$

where $b_0(0) \neq 0, b_k(0) \neq 0$ and $b_1(0) = \cdots = b_{k-1}(0) = 0$. It is easy to check that $[r], [ur_u]$ and $[r_\lambda]$ are in $\mathcal{E}_{u,\lambda}\{u^k x, u\lambda x, \lambda^l x\} + \mathbb{R}\{[r_\lambda]\}$ and hence by (2.12), (b) is valid. Note that for and $p, q, A \in \mathbb{R}$

$$pr(u + u^2 A, q\lambda) = c_0(A, p, q, \lambda)\lambda^l + \sum_{i=1}^{k+1} c_i(A, p, q, \lambda)u^i + c_{k+2}(A, p, q, u, \lambda)u^{k+2},$$

where

$$c_0(A, p, q, \lambda) = pq^l b_0(q\lambda),$$

$$c_j(A, p, q, \lambda) = p \sum_{i=0}^{[j/2]} \binom{j-i}{i} A^i b_{j-i}(q\lambda), \quad j = 1, 2, \cdots, k+1.$$

By the Implicit Function Theorem there exist smooth germs $A, p, q : (\mathbb{R}, 0) \to \mathbb{R}$ such that

$$c_0(A(\lambda), p(\lambda), q(\lambda), \lambda) = \varepsilon, \quad c_k(A(\lambda), p(\lambda), q(\lambda), \lambda) = \delta, \quad c_{k+1}(A(\lambda), p(\lambda), q(\lambda), \lambda) = 0,$$

where $\varepsilon = \operatorname{sgn} b_0(0)$ and $\delta = \operatorname{sgn} b_k(0)$. Denote $a_i(\lambda) = c_i(A(\lambda), p(\lambda), q(\lambda), \lambda)$ for $1 \le i < k$ and $a_{k+2}(u, \lambda) = c_{k+2}(A(\lambda), p(\lambda), q(\lambda), u, \lambda)$. Hence

$$p(\lambda)r(u + u^2 A(\lambda), q(\lambda)\lambda) = \varepsilon\lambda^l + \sum_{i=1}^{k-1} a_i(\lambda)u^i + \delta u^k + a_{k+2}(u, \lambda)u^{k+2}. \tag{3.3}$$

Lemma 3.2. *Let $r \in \mathcal{E}_{u,\lambda}$.*

(a) *If $r_{0,0} = \cdots r_{k-1,0} = 0$ and $r_{k,0} \cdot r_{0,1} \neq 0$ for some $k \ge 1$, then $[r]$ is $\mathbf{O}(n)$-equivalent to $(\varepsilon\lambda + \delta u^k)x$, where $\varepsilon = \operatorname{sgn} r_{0,1}$ and $\delta = \operatorname{sgn} r_{k,0}$.*

(b) *If $r_{0,0} = \cdots r_{0,l-1} = 0$ and $r_{1,0} \cdot r_{0,l} \neq 0$ for some $l \ge 1$, then $[r]$ is $\mathbf{O}(n)$-equivalent to $(\varepsilon\lambda^l + \delta u)x$, where $\varepsilon = \operatorname{sgn} r_{0,l}$ and $\delta = \operatorname{sgn} r_{1,0}$.*

(c) *If $r = \varepsilon\lambda^l + \delta u^k$, where $\varepsilon, \delta = \pm 1$, then $\operatorname{codim}_{\mathbf{O}(n)}[r] = kl - 1$ and $[r]$ has a universal $\mathbf{O}(n)$-unfolding in one of the following forms:*

(c.1) $\big(\varepsilon\lambda^l + \delta u^k + \sum_{i=0}^{l-2} \alpha_i \lambda^i + \sum_{i=1}^{k-1}\sum_{j=0}^{l-1} \beta_{i,j} u^i \lambda^j\big)x \quad (k > 1,\ l > 1),$

(c.2) $\big(\varepsilon\lambda^l + \delta u + \sum_{i=0}^{l-2} \lambda^i\big)x \quad (k = 1,\ l > 1),$

(c.3) $\big(\varepsilon\lambda + \delta u^k + \sum_{i=1}^{k-1} \alpha_i u^i\big)x \quad (k > 1,\ l = 1),$

(c.4) $(\varepsilon\lambda + \delta u)x \quad (k = l = 1),$

where $\alpha_i, \beta_{i,j}$ are unfolding parameters.

Proof. By (2.9) and (2.10), (a) and (b) can be proved with an argument similar to that for $\mathbb{Z}_2$ cases in [2]. By (2.12) a simple calculation shows that

$$T((\varepsilon\lambda^l + \delta u^k)x, \mathbf{O}(n)) = \mathcal{E}_{u,\lambda}\{\lambda^l x, u^k x\} + \mathbb{R}\{\lambda^{l-1} x\}.$$

If $k > 1$ and $l > 1$, then $\{u^i\lambda^j x | 1 \le i < k,\ 0 \le j < l\} \cup \{\lambda^j x | 0 \le j < l-1\}$ form a basis for subspace complement to $T((\varepsilon\lambda^l + \delta u^k)x, \mathbf{O}(n))$, By the Versal Unfolding Theorem (Theorem 2.3), $(\varepsilon\lambda^l + \delta u^k)x$ is of $\mathbf{O}(n)$-codimension $kl - 1$ and (c.1) gives a universal unfolding. Other cases can be similarly proved.

Lemma 3.3. *If $r \in \mathcal{E}_{u,\lambda}$ satisfies $r_{0,0} = r_{1,0} = r_{0,1} = \cdots = r_{0,l-1} = 0$ and $r_{2,0} \cdot r_{0,l} \neq 0$ for some $l \ge 2$ and $\operatorname{codim}_{\mathbf{O}(n)}[r] \le 4$, then $[r]$ is $\mathbf{O}(n)$-equivalent to one of normal forms*

(6),(8),(9),(10),(14),(15) *in* Table 3.1

Proof. This lemma follows by an argument similar to that for $\mathbb{Z}_2$ cases in [2].

Lemma 3.4. *If* $r \in \mathcal{E}_{u,\lambda}$ *satisfies* $r_{0,0} = r_{1,0} = r_{2,0} = r_{0,1} = \cdots = r_{0,l-1} = 0$ *and* $r_{3,0} \cdot r_{0,l} \neq 0$ *for some* $l \geq 2$ *and* $\mathrm{codim}_{\mathbf{O}(n)}[r] \leq 4$, *then* $[r]$ *is* $\mathbf{O}(n)$*-equivalent to one of normal forms* (11),(16),(17) *in* Table 3.1

Proof. By Lemma 3.1, r can be written as

$$r = \varepsilon\lambda^l + a_1(\lambda)\lambda^p u + a_2(\lambda)\lambda^q u^2 + \delta u^3 + a_5(u,\lambda)u^5,$$

where $p, q \geq 1$, and $\varepsilon, \delta = \pm 1$. By the assumption that $\mathrm{codim}_{\mathbf{O}(n)}[r] \leq 4$ we have $l \leq 3$. If $p, q \geq l$, then

$$\lambda^p ux, \lambda^q u^2 x, u^5 x \in \mathcal{M}_{u,\lambda} T\mathcal{D}^s((\varepsilon\lambda^l + \delta u^3)x, \mathbf{O}(n))$$

and hence by Theorem 2.1 and Theorem 2.2 (b), $[r]$ is equivalent to $(\varepsilon\lambda^l + \delta u^3)x$. By Lemma 3.2, $\mathrm{codim}_{\mathbf{O}(n)}[r] = 3l - 1 \geq 5$, which contradicts the assumption that $\mathrm{codim}_{\mathbf{O}(n)}[r] \leq 4$. Therefore we have $p < l$ or $q < l$.

If $l = 2, p = 1, q \geq 1$, it can be proved similarly as its counterpart in [2] that $[r]$ is equivalent to $(\varepsilon\lambda^2 + \sigma\lambda u + \delta u^3)x$, which is of $\mathbf{O}(n)$-codimension three and has a universal unfolding of the form

$$(\varepsilon\lambda^2 + \sigma\lambda u + \delta u^3 + \alpha_1 + \alpha_2\lambda + \alpha_3 u^2)x.$$

If $l = 2, p > 1, q = 1$, let $h = (\varepsilon\lambda^2 + a_2(0)\lambda u^2 + \delta u^3)x$. Then

$$T\mathcal{D}^s(h, \mathbf{O}(n)) = \mathcal{E}_{u,\lambda}\{(\delta u^3 - 2\varepsilon\lambda^2)x, (3\varepsilon\lambda^2 + a_2(0)\lambda u)x\}$$

and

$$\frac{2\varepsilon}{\delta} u\lambda^2 x \equiv u^4 x \equiv -\frac{2a_2(0)}{3\delta} u^3\lambda x \equiv \frac{4(a_2(0))^2}{9} u^2\lambda^2 x, \quad \mathrm{mod}\, \mathcal{M}_{u,\lambda} T\mathcal{D}^s(h, \mathbf{O}(n)).$$

It follows that $u\lambda^2 x \in \mathcal{M}_{u,\lambda} T\mathcal{D}^s(h, \mathbf{O}(n))$ and hence $u^2\lambda^2 x, u^4 x \in \mathcal{M}_{u,\lambda} T\mathcal{D}^s(h, \mathbf{O}(n))$. Therefore $[r] - h \in \mathrm{Itr}_{\mathcal{D}^s} \mathcal{M}_{u,\lambda} T\mathcal{D}^s(h, \mathbf{O}(n))$ and by Theorem 2.2 $[r]$ is equivalent to h. Since $T\mathcal{D}^s(h, \mathbf{O}(n)) = \mathcal{E}_{u,\lambda}\{u^3 x, u^2\lambda x, \lambda^2 x\}$ is independent of $a_2(0)$ which is nonzero, by Theorem 2.2 (a), h is equivalent to $(\varepsilon\lambda^2 + \mathrm{sgn}\, a_2(0)\lambda u^2 + \delta u^3)x$. An easy calculation shows that $x, ux, \lambda x, u\lambda x$ span a complementary space to $T(h, \mathbf{O}(n))$ and this shows that $\mathrm{codim}_{\mathbf{O}(n)} h = 4$.

If $l = 3, p = 1, q \geq 1$, it can be proved similarly that $[r]$ is equivalent to $(\varepsilon\lambda^3 + \sigma u\lambda + \delta u^3)x$, which is of $\mathbf{O}(n)$-codimension four and has a universal unfolding of the form

$$(\varepsilon\lambda^3 + \sigma\lambda u + \delta u^3 + \alpha_1 + \alpha_2\lambda + \alpha_3\lambda^2 + \alpha_4 u^2)x.$$

If $l = 3, p > 1, q \geq 1$, similar argument as in the proof of Lemma 3.1(b) shows

$$T([r], \mathbf{O}(n)) \subset \mathcal{E}_{u,\lambda}\{u^3 x, u^2\lambda x, u\lambda^2 x, \lambda^3 x\} + \mathbb{R}\{r_\lambda x\}$$

and hence $\mathrm{codim}_{\mathbf{O}(n)}[r] \geq 5$.

Lemma 3.5. *If* $r \in \mathcal{E}_{u,\lambda}$ *satifies* $r_{0,0} = r_{1,0} = r_{2,0} = r_{3,0} = r_{0,1} = \cdots = r_{0,l-1} = 0$ *and* $r_{4,0} \cdot r_{0,l} \neq 0$ *for some* $l \geq 2$ *and* $\mathrm{codim}_{\mathbf{O}(n)}[r] \leq 4$, *then* $[r]$ *is of* $\mathbf{O}(n)$*-codimension four and* $\mathbf{O}(n)$*-equivalent to* $(\varepsilon\lambda^2 + \sigma\lambda u + \delta u^4)x$ *which has a universal unfolding as*

$$(\varepsilon\lambda^2 + \sigma\lambda u + \delta u^4 + \alpha_1 + \alpha_2 u + \alpha_3 u^2 + \alpha_4 u^3)x.$$

Proof. As in Lemma 3.1 r can be written as

$$r = \varepsilon\lambda^l + a_1(\lambda)\lambda^p u + a_2(\lambda)\lambda^q u^2 + a_3(\lambda)\lambda^m u^3 + \delta u^4 + a_5(u,\lambda)u^5,$$

where $\varepsilon, \delta = \pm 1$ and $p, q, m \geq 1$. By the restriction on the $\mathbf{O}(n)$-codimension of $[r]$ it follows that $l = 2$. If $p \geq 2$, then

$$T([r], \mathbf{O}(n)) \subset \mathcal{E}_{u,\lambda}\{u^4 x, u^2\lambda x, \lambda^2 x\} + \mathbb{R}\{r_\lambda x\}$$

and hence $\operatorname{codim}_{\mathbf{O}(n)}[r] \geq 5$, which is a contradiction to the assumption that $\operatorname{codim}_{\mathbf{O}(n)}[r] \leq 4$. Therefore $p = 1$ and a similar argument as in the proof of Lemma 3.4 shows that $[r]$ is equivalent to $(\varepsilon\lambda^2 + \operatorname{sgn} a_1(0)\lambda u + \delta u^4)x$, which is of $\mathbf{O}(n)$-codimension four and has a universal unfolding of the form $h + (\alpha_1 + \alpha_2 u + \alpha_3 u^2 + \alpha_4 u^3)x$.

Proof of Theorem 3.1. By Lemma 3.1, $[r]$ can be assumed to take the form

$$r = \varepsilon\lambda^l + a_1(\lambda)u + \cdots + a_{k-1}(\lambda)u^{k-1} + \delta u^k + a_{k+2}(u,\lambda)u^{k+2},$$

where $a_1(0) = \cdots = a_{k-1}(0) = 0,\ \varepsilon, \delta = \pm 1$ and $k + l \leq 6$.

If $k = 1$ or $l = 1$, then $[r]$ satisfies the condition in Lemma 3.2 and hence is equivalent to one of the normal forms (1),(2),(3),(4),(5),(7),(12),(13),(19).

If $k = 2$ and $1 < l \leq 4$, then $[r]$ satisfies the condition in Lemma 3.3 and hence $[r]$ is equivalent to one of the normal forms (6),(8),(9),(10),(14),(15).

If $k = 3$ and $1 < l \leq 3$, then $[r]$ satisfies the condition Lemma 3.4 and $[r]$ is equivalent to one of the normal forms (11),(16),(17).

If $k = 4$ and $l = 2$, then $[r]$ satisfies the condition in Lemma 3.5 and $[r]$ is equivalent to nomal form (18).

§4. Recognition Problem

In this section we focus on characterizing the group orbit $\mathcal{D}(\mathbf{O}(n)) \cdot g$ for a given $\mathbf{O}(n)$-equivariant germ g, i.e., solving the recognition problem for g. We prove

Theorem 4.1. *Let $g \equiv [r] \in \vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$, where $r \in \mathcal{E}_{u,\lambda}$ satisfies $r(0,0) = 0$. Then g is $\mathbf{O}(n)$-equivalent to one of the normal forms listed in Theorem* 3.1 *if and only if r satisfies the corresponding conditions in* Table 4.1

According to [5], $\mathcal{D}(\mathbf{O}(n))$ can be expressed as a product of the unipotent subgroup $\mathcal{U}(\mathbf{O}(n))$ and the scaling subgroup $\mathcal{S}(\mathbf{O}(n))$ (see [5] for their exact definitions):

$$\mathcal{D}(\mathbf{O}(n)) = \mathcal{U}(\mathbf{O}(n)) \cdot \mathcal{S}(\mathbf{O}(n)).$$

$\mathcal{S}(\mathbf{O}(n))$ consists of $\mathbf{O}(n)$-equivalences (S, X, Λ), where S, X, Λ are scalar maps and hence $\mathcal{S}(\mathbf{O}(n))$-orbits can be easily characterized and hence recognition problem concentrates on discribing $\mathcal{U}(\mathbf{O}(n))$-orbits. Let

$$M(g,\mathcal{U}) = \{h \in \vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n)) | g + h \in \mathcal{U}(\mathbf{O}(n)) \cdot g\}.$$

Then $\mathcal{U}(\mathbf{O}(n)) \cdot g = g + M(g,\mathcal{U})$. Then main result in [5] can be stated as

Theorem 4.2.[5] Let $g \in \vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ be of finite $\mathbf{O}(n)$-codimension. Then

(a) $\mathrm{Itr}_{\mathcal{U}} M(g,\mathcal{U}) = \mathrm{Itr}_{\mathcal{U}} T\mathcal{U}(g, \mathbf{O}(n))$;

(b) $M(g,\mathcal{U})$ is $\mathcal{U}$-intrinsic if and only if $T\mathcal{U}(g, \mathbf{O}(n))$ is.

Table 4.1. Defining Conditions and Non-Degeneracy Conditions

No.	Defining Conditions	Non-Degeneracy Conditions
(1)	$r_{0,0}=0$	$\operatorname{sgn} r_{1,0}=\delta$, $\operatorname{sgn} r_{0,1}=\varepsilon$
(2)	$r_{0,0}=r_{0,1}=0$	$\operatorname{sgn} r_{1,0}=\delta$, $\operatorname{sgn} r_{0,2}=\varepsilon$
(3)	$r_{0,0}=r_{1,0}=0$	$\operatorname{sgn} r_{2,0}=\delta$, $\operatorname{sgn} r_{0,1}=\varepsilon$
(4)	$r_{0,0}=r_{0,1}=r_{0,2}=0$	$\operatorname{sgn} r_{1,0}=\delta$, $\operatorname{sgn} r_{0,3}=\varepsilon$
(5)	$r_{0,0}=r_{1,0}=r_{2,0}=0$	$\operatorname{sgn} r_{3,0}=\delta$, $\operatorname{sgn} r_{1,0}=\varepsilon$
(6)	$r_{0,0}=r_{1,0}=r_{0,1}=r_{1,1}=0$	$\operatorname{sgn} r_{2,0}=\delta$, $\operatorname{sgn} r_{0,2}=\varepsilon$
(7)	$r_{0,0}=r_{0,1}=r_{0,2}=r_{0,3}=0$	$\operatorname{sgn} r_{1,0}=\delta$, $\operatorname{sgn} r_{0,4}=\varepsilon$
(8)	$r_{0,0}=r_{1,0}=r_{0,1}=0$	$\operatorname{sgn} r_{2,0}=\delta$, $\operatorname{sgn} r_{0,2}=\varepsilon$, $r_{1,1}[r_{1,1}^2-r_{2,0}r_{0,2}]\neq 0.$
(9)	$r_{0,0}=r_{1,0}=r_{0,1}=0$ $r_{1,1}^2-r_{2,0}r_{0,2}=0$	$\operatorname{sgn} r_{2,0}=\delta$, $\operatorname{sgn} r_{0,2}=\delta$ $\operatorname{sgn}\left(r_{3,0}-\frac{3r_{2,1}r_{1,1}}{r_{0,2}}+\frac{3r_{1,2}r_{1,1}^2}{r_{0,2}^2}-\frac{r_{0,3}r_{1,1}^3}{r_{0,2}^3}\right)=\delta\sigma$
(10)	$r_{0,0}=r_{1,0}=r_{0,1}=r_{0,2}=0$	$\operatorname{sgn} r_{2,0}=\delta$, $\operatorname{sgn} r_{1,1}=\sigma$, $\operatorname{sgn} r_{0,3}=\varepsilon$
(11)	$r_{0,0}=r_{1,0}=r_{2,0}=r_{0,1}=0$	$\operatorname{sgn} r_{3,0}=\delta$, $\operatorname{sgn} r_{1,1}=\sigma$, $\operatorname{sgn} r_{0,2}=\varepsilon$
(12)	$r_{0,0}=r_{1,0}=r_{2,0}=r_{3,0}=0$	$\operatorname{sgn} r_{4,0}=\delta$, $\operatorname{sgn} r_{0,1}=\varepsilon$
(13)	$r_{0,0}=r_{0,1}=r_{0,2}=r_{0,3}=r_{0,4}=0$	$\operatorname{sgn} r_{1,0}=\delta$, $\operatorname{sgn} r_{0,5}=\varepsilon$
(14)	$r_{0,0}=r_{1,0}=r_{0,1}=r_{0,2}=r_{0,3}=0$	$\operatorname{sgn} r_{2,0}=\delta$, $\operatorname{sgn} r_{1,1}=\sigma$ $\operatorname{sgn} r_{0,4}=\varepsilon$
(15)	$r_{0,0}=r_{1,0}=r_{0,1}=r_{0,2}=r_{1,1}=0$	$\operatorname{sgn} r_{2,0}=\delta$, $\operatorname{sgn} r_{1,2}=\sigma$ $\operatorname{sgn} r_{0,3}=\varepsilon$
(16)	$r_{0,0}=r_{1,0}=r_{2,0}=r_{0,1}=r_{0,2}=0$	$\operatorname{sgn} r_{3,0}=\delta$, $\operatorname{sgn} r_{1,1}=\sigma$ $\operatorname{sgn} r_{0,3}=\varepsilon$
(17)	$r_{0,0}=r_{1,0}=r_{2,0}=r_{0,1}=r_{1,1}=0$	$\operatorname{sgn} r_{3,0}=\delta$, $\operatorname{sgn} r_{2,1}=\sigma$ $\operatorname{sgn} r_{0,2}=\varepsilon$
(18)	$r_{0,0}=r_{1,0}=r_{2,0}=r_{3,0}=r_{0,1}=0$	$\operatorname{sgn} r_{4,0}=\delta$, $\operatorname{sgn} r_{1,1}=\sigma$ $\operatorname{sgn} r_{0,2}=\varepsilon$
(19)	$r_{0,0}=r_{1,0}=r_{2,0}=r_{3,0}=r_{4,0}=0$	$\operatorname{sgn} r_{5,0}=\delta$, $\operatorname{sgn} r_{0,1}=\varepsilon$

Here $T\mathcal{U}(g,\mathbf{O}(n))$ is the tangent space to the group orbit $\mathcal{U}(\mathbf{O}(n))\cdot g$ and a subspace V of $\vec{\mathcal{E}}_{x,\lambda}(\mathbf{O}(n))$ is said to be $\mathcal{U}$-intrinsic if it coincides with $\mathrm{Itr}_{\mathcal{U}}V$ by which we denote the maximal subsqace of V consisting of entire $\mathcal{U}(\mathbf{O}(n))$-orbits. By Theorem 4.2 the key steps in solving the recognition problem are to calculate the tangent space $T\mathcal{U}(g,\mathbf{O}(n))$ and to check whether it is $\mathcal{U}$-intrinsic.

Proof of Theorem 4.1. Let $\mathcal{U}^s(\mathbf{O}(n))=\mathcal{D}^s(\mathbf{O}(n))\cap\mathcal{U}(\mathbf{O}(n))$ and let $T\mathcal{U}^s(g,\mathbf{O}(n))$ be the tangent spaces to the group orbits $\mathcal{U}^s(\mathbf{O}(n))\cdot g$. By the results in [5] we have

$$T\mathcal{U}^s([r],\mathbf{O}(n))=\mathcal{E}_{u,\lambda}\{[ur],[\lambda r],[u^2r_u],[u\lambda r_u]\}, \tag{4.1}$$

$$T\mathcal{U}([r],\mathbf{O}(n))=\mathcal{E}_{u,\lambda}\{[ur],[\lambda r],[u^2r_u],[u\lambda r_u]\}+\mathcal{E}_{\lambda}\{[\lambda^2r_\lambda\}, \tag{4.2}$$

Table 4.2. Tangent Spaces to $\mathcal{U}(\mathbf{O}(n))$-Orbits

Normal Form h	$T\mathcal{U}(h,\mathbf{O}(n))$	No. in Table 3.1 and Table 4.1
$(\delta u^k+\varepsilon\lambda^l)x$	$\mathcal{E}_{u,\lambda}\{u^{k+1}x,u^k\lambda x,u\lambda^lx,\lambda^{l+1}x\}$	(1–7), (12), (13), (19)
$(\delta u^2+2bu\lambda+\varepsilon\lambda^2)x$ $b\neq 0, b^2\neq\varepsilon\delta$	$\mathcal{M}^3_{u,\lambda}\{x\}$	(8)
$(\delta u^2+2\sigma u(\lambda+\lambda^2)+\delta\lambda^2)x$	$\mathcal{M}^4_{u,\lambda}\{x\}+\mathbb{R}\{u^2(\delta u+\sigma\lambda)x,$ $u\lambda(\delta u+\sigma\lambda)x,\lambda^2(\delta u+\sigma\lambda)x\}$	(9)
$(\delta u^k+\sigma u\lambda++\varepsilon\lambda^l)x,k,l\geq 2$	$\mathcal{E}_{u,\lambda}\{u^{k+1}x,u^2\lambda x,u\lambda^2x,\lambda^{l+1}x\}$	(10), (11), (14), (16), (18)
$(\delta u^3+\sigma u^2\lambda++\varepsilon\lambda^2)x$	$\mathcal{M}^4_{u,\lambda}\{x\}+\mathbb{R}\{u\lambda^2x,\lambda^3x\}$	(17)
$(\delta u^2+\sigma u\lambda^2++\varepsilon\lambda^3)x$	$\mathcal{M}^4_{u,\lambda}\{x\}+\mathbb{R}\{u^3x,u^2\lambda x\}$	(15)

We list in Table 4.2 the tangent spaces to the $\mathcal{U}(\mathbf{O}(n))$-orbits of the normal forms as the main data for solving recognition problems and prove two cases as examples. By [5, Theorem 5.5] it can be checked that all these tangent spaces are $\mathcal{U}$-intrinsic.

(i) $(\delta u^k + \varepsilon\lambda^l)x$

$f \in \mathcal{S}(\mathbf{O}(n)) \cdot (\delta u^k + \varepsilon\lambda^l)x$ iff $f(x,\lambda) = (\delta A^{2k+1}u^k + \varepsilon AB^l\lambda^l)x$ for some $A, B > 0$. Then from (4.1) and (4.2)

$$T\mathcal{U}(f, \mathbf{O}(n)) = T\mathcal{U}^s(f, \mathbf{O}(n)) = \mathcal{E}_{u,\lambda}\{u^{k+1}x, u^k\lambda x, u\lambda^l x, \lambda^{l+1}x\}.$$

By Theorem 2.1 it is $\mathcal{U}$-intrinsic and hence

$$M(f, \mathbf{O}(n)) = T\mathcal{U}(f, \mathbf{O}(n)),$$
$$g \in \mathcal{U}(\mathbf{O}(n)) \cdot f \text{ iff } g \in f + \mathcal{E}_{u,\lambda}\{u^{k+1}x, u^k\lambda x, u\lambda^l x, \lambda^{l+1}x\}.$$

The necessary and sufficient condition for $g = [r]$ being equivalent to $(\delta u^k + \varepsilon\lambda^l)x$ is

$$r_{i,j} = 0, \quad 0 \le i < k, \quad 0 \le j < l; \quad \operatorname{sgn} r_{k,0} = \delta; \quad \operatorname{sgn} r_{0,l} = \varepsilon. \tag{4.3}$$

(ii) $(\delta u^k + \sigma u\lambda + \varepsilon\lambda^l)x, \quad k, l \ge 2$

$f \in \mathcal{S}(\mathbf{O}(n)) \cdot (\delta u^k + \sigma u\lambda + \varepsilon\lambda^l)x$ iff $f(x,\lambda) = (\delta A^{2k+1}u^4 + \sigma A^3Bu\lambda + \varepsilon AB^l\lambda^2)x$ for some $A, B > 0$. Then

$$T\mathcal{U}^s(f, \mathbf{O}(n)) = \mathcal{E}_{u,\lambda}\{P_1, P_2, P_3, P_4\},$$

where

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = \begin{pmatrix} k\delta A^{2k+1} & \sigma A^3B & 0 & 0 \\ 0 & k\delta A^{2k+1}u^{k-2} & \sigma A^3B & 0 \\ (k-1)\delta A^{2k+1} & 0 & -\varepsilon AB^l\lambda^{l-2} & 0 \\ 0 & (k-1)\delta A^{2k+1}u^{k-2} & 0 & -\varepsilon AB^l \end{pmatrix} \begin{pmatrix} u^{k+1}x \\ u^2\lambda x \\ u\lambda^2 x \\ \lambda^{l+1}x \end{pmatrix}.$$

Since the matrix in the above equation is invertible, we get

$$T\mathcal{U}(f, \mathbf{O}(n)) = T\mathcal{U}^s(f, \mathbf{O}(n)) = \mathcal{E}_{u,\lambda}\{u^{k+1}x, u^2\lambda x, u\lambda^2 x, \lambda^{l+1}x\}.$$

By Theorem 2.1 it is $\mathcal{U}$-intrinsic. Hence $M(f, \mathbf{O}(n)) = T\mathcal{U}(f, \mathbf{O}(n))$ and

$$g \in \mathcal{U}(\mathbf{O}(n)) \cdot f \text{ iff } g \in f + \mathcal{E}_{u,\lambda}\{u^{k+1}x, u^2\lambda x, u\lambda^2 x, \lambda^{l+1}x\}.$$

The necessary and sufficient condition for $g = [r]$ being equivalent to $(\delta u^k + \sigma u\lambda + \varepsilon\lambda^l)x, \ k, l \ge 2$, is

$$r_{0,0} = r_{1,0} = \cdots = r_{k-1,0} = r_{0,1} = \cdots = r_{0,l-1} = 0;$$
$$\operatorname{sgn} r_{k,0} = \delta; \quad \operatorname{sgn} r_{1,1} = \sigma; \quad \operatorname{sgn} r_{0,1} = \varepsilon. \tag{4.4}$$

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