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THE GROWTH THEOREM FOR STARLIKE MAPPINGS ON BOUNDED STARLIKE CIRCULAR DOMAINS**

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Abstract

The authors obtain the growth and covering theorem for the class of normalized biholomorphic starlike mappings on bounded starlike circular domains.

This type of domain discussed is rather general, since the domain must be starlike if there exists a normalized biholomorphic starlike mapping on it. In the unit disc, it is just the famous growth and covering theorem for univalent functions.

This theorem successfully realizes the initial idea of H. Cartan about how to extend geometric function theory from one variable to several complex variables.

Keywords Starlike mappings, Growth theorem, Starlike circular domains
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§1. Introduction

On the geometric function theory of one complex variable, the following growth and $\frac{1}{4}$-covering theorem is well known (see [2]).

Theorem A. *For each normalized univalent function f on the unit disc $D \subset \mathbf{C}$,*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}, \quad z \in D.$$

Especially, the left-hand side of the above inequality implies $f(D) \supseteq \frac{1}{4}D$. For each $z \in D$, $z \neq 0$, equality occurs in the above inequality if and only if f is Koebe function $K(z) = \frac{z}{(1-z)^2}$ or its rotation $e^{-i\theta}K(e^{i\theta}z)$.

It is natural to extend this and other results on the geometric function theory of one variables to several variables. But as early as fifty years ago, H. Cartan pointed out where the difficulty lies. For example, the following is a counter-example:

$$f(z) = \Big(z_1, \frac{z_2}{(1-z_1)^k}\Big), \quad k \in \mathbf{N}.$$

Obviously this is a normalized biholomorphic mapping on the unit ball B^2 in $\mathbf{C}^2$, which indicates that the corresponding result in several complex variables fails,

Then the question arises: to extend under some suitable restrict on biholomorphic mappings. Hence H. Cartan suggested the study of biholomorphic starlike mappings, convex

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mappings and some other important classes of mappings in several complex variables. Along this direction, in 1988, the first affirmative result about the growth and $\frac{1}{4}$-covering theorem in several complex variables was obtained by R. W. Barnard, C. H. Fitzgerald, S. Gong[3]. They gave the growth and $\frac{1}{4}$-covering theorem for the class of normalized biholomorphic starlike mappings in the unit ball $B^n \subset \mathbf{C}^n$. After that, for the class of normalized biholomorphic starlike mappings, several authors, Liu[9], Gong, Yu, Wang[5,17], Pfaltzgraff[12], Zhang, Dong[18], Chen[1] obtained respectively the generalization on other domains such as the bounded symmetric domains, egg domains, and the unit ball in finite dimensional Banach spaces. See [4] for more details.

In this paper, we obtain the growth and $\frac{1}{4}$-covering theorem on bounded starlike circular domains for the class of normalized biholomorphic starlike mappings, which extend all the corresponding results mentioned above. This paper was once reported by the first author at the Conference of Several Complex Variables of China held in Beijing in 1990.

§2. Main Theorem

Our main result is the following theorem.

Main Theorem. *Suppose Ω to be a bounded starlike circular domain in $\mathbf{C}^n$, its defining function $\rho(z)$ is a C^1 function except for a lower dimensional set. If $f(z)$ is a normalized biholomorphic starlike mapping on Ω, then*

$$\frac{\rho(z)}{(1+\rho(z))^2} \le \rho(f(z)) \le \frac{\rho(z)}{(1-\rho(z))^2}, \tag{2.1}$$

or equivalently

$$\frac{|z|}{(1+\rho(z))^2} \le |f(z)| \le \frac{|z|}{(1-\rho(z))^2}. \tag{2.2}$$

Especially

$$f(\Omega) \supset \frac{1}{4}\Omega. \tag{2.3}$$

Note 2.1. (i) This kind of domain on which we discuss is rather general, in the sense that the domain must be starlike, if there exists a normalized biholomorphic starlike mapping f on it, such that the "contractive" mappings $\frac{1}{r}f(rz)$ are also starlike. This is simply because the identity mapping then as the limit of $\frac{1}{r}f(rz)$ is also starlike. On the other hand it includes the following domains as examples.

(a) any bounded symmetric domain in $\mathbf{C}^n$ with its standard Harish-Chandra realization (See Lemma 3.4 in the next section for the proof).

(b) Egg domains

$$B_p = \{z = (z_1, \cdots, z_n) \in \mathbf{C}^n : |z_1|^{p_1} + \cdots + |z_n|^{p_n} < 1\}, \quad \forall p_1 > 0, \ \cdots, \ p_n > 0,$$

or more general domains

$$\left\{(z, \cdots, w) \in \mathbf{C}^{n_1+\cdots+n_k} : |A_1 z|^{p_1} + \cdots + |A_k w|^{p_k} < 1\right\},$$
$$\forall p_1 > 0, \cdots, p_k > 0, z \in \mathbf{C}^{n_1}, \cdots, w \in \mathbf{C}^{n_k},$$

where $A_1, \cdots, A_k$ are the non-singular linear transformations in $\mathbf{C}^{n_1}, \cdots, \mathbf{C}^{n_k}$ respectively.

(ii) Taking $\Omega = B^n; R_I, R_{II}, R_{III}, R_{IV}$ (the classical domains[6]); Bp; the unit ball in finite dimensional Banach spaces; and bounded strictly balanced domains respectively in the Main Theorem gives the corresponding growth and $\frac{1}{4}$-covering theorems in [1, 3, 9, 17, 18].

§3. Preliminaries

Definition 3.1. *A domain $\Omega \subseteq \mathbf{C}^n$ is said to be circular if $e^{i\theta}z \in \Omega$, whenever $z \in \Omega, \theta \in \mathbf{R}$. A domain $\Omega \subseteq \mathbf{C}^n$ is said to be starlike (with respect to the origin) if the line segment joining $0 \in \Omega$ and every other point in Ω lies entirely in Ω.*

Definition 3.2. *Let Ω be a domain containing the origin. We call a holomorphic mapping $f(z) = (f_1(z), \cdots, f_n(z))$ from Ω into $\mathbf{C}^n$ normalized if $f(0) = 0, J_f(0) = I$, where J_f is the Jacobian of f, I is the identity matrix. If $f(\Omega)$, the graph of f, is a starlike set with respect to the origin in $\mathbf{C}^n$, we call f a starlike mapping.*

The following lemmas are needed in the proof of main theorem and its note.

Lemma 3.1. *$\Omega \subseteq \mathbf{C}^n$ is a bounded starlike and circular domain if and only if there exists a unique real continuous function $\rho : \mathbf{C}^n \longrightarrow \mathbf{R}$, called the defining function of Ω, such that*

(i) $\rho(z) \geq 0,\ \forall z \in \mathbf{C}^n;\quad \rho(z) = 0 \Longleftrightarrow z = 0$,

(ii) $\rho(tz) = |t|\rho(z)$, *for any* $t \in \mathbf{C}$, $z \in \mathbf{C}^n$,

(iii) $\Omega = \{z \in \mathbf{C}^n : \rho(z) < 1\}$.

Note 3.1. The function $\rho(z)$ in the lemma is just the distance function in the book of [8], which plays an important role in the characterization of pseudoconvex domains.

Proof. If the continuous function $\rho(z)$ satisfies (i), (ii), and (iii), then clearly Ω is a starlike circular domain. Its boundedness is also easy to prove, since if there exists a ray coming from the origin which completely falls in the starlike domain Ω, then for any fixed point z_0 in this ray we have from (iii) $\rho(tz_0) < 1, \forall t \in [0, +\infty)$. But we then obtain $\rho(z_0) = 0$ by (ii). This contradicts (i).

Conversely, if Ω is a bounded starlike and circular domain in $\mathbf{C}^n$, then we define

$$\rho(z) = \inf\{c > 0 : c^{-1}z \in \Omega\}.$$

Obviously $\rho(z)$ satisfies (ii), (iii) and $\rho(z) \geq 0,\ \forall z \in \mathbf{C}^n$. If there exists a point $z_0 \neq 0$ such that $\rho(z_0) = 0$, then by (ii) and (iii), Ω includes the whole ray which comes from the origin and through the point z_0, hence Ω is unbounded. Thus (i) holds. Finally we prove that $\rho(z)$ is continuous. Clearly $\{z \in \mathbf{C}^n : r < \rho(z) < R\} = R\Omega \setminus \overline{r\Omega}$ is an open set in $\mathbf{C}^n$, which implies the continuity of ρ. This completes the proof of Lemma 3.1.

Lemma 3.2. *If Ω is a Harish-Chandra realization of some bounded symmetric domain, then its defining function $\rho(z)$ is holomorphic about $z, \overline{z}$ (except for a lower dimensional set).*

Proof. From [15, Proposition 4.6], $\Omega = \{z \in \mathbf{C}^n : ||ad(\mathrm{Re}z)|| < 1\}$. Here $\mathbf{C}^n$ is regarded as a subset $\mathfrak{p}_+$ of complexification of Lie algebra of $\mathrm{Aut}(\Omega)$, $||\cdot||$ is the operator norm determined by the inner product of Killing form, $\mathrm{Re}z = \frac{1}{2}(z + \overline{z})$ for $z \in \mathfrak{p}_+ \equiv \mathbf{C}^n$. Note that $\overline{\mathfrak{p}_+} = \mathfrak{p}_-$, $\mathfrak{p}_+ \cap \mathfrak{p}_- = 0$.

Denote $\tilde{\rho}(z) = ||ad(\mathrm{Re}z)||$. From [15, Lemma 4.5], it is easy to check that $\tilde{\rho}(z)$ satisfies Lemma 3.1 (i), (ii) and (iii). Note that the function which satisfies the condition of Lemma 3.1 (ii) and (iii) is unique, hence $\rho(z) = \tilde{\rho}(z) = ||ad(\mathrm{Re}z)||$. Since the computation of the operator norm $||ad(\mathrm{Re}z)||$ only involves the computation of Lie algebra and of solving characteristic polynomial to get the maximum characteristic value, we see that $\rho(z)$ is holomorphic about $z, \overline{z}$ except for a lower dimensional set.

Lemma 3.3. *If $\Omega \subseteq \mathbf{C}^n$ is a bounded symmetric domains, and is convex and circular, then its defining function $\rho(z)$ is holomorphic about $z, \overline{z}$ (except for a lower dimensional set).*

Proof. Firstly suppose Ω is a irreducible bounded symmetric domain of rank≥ 2, and is convex and circular domain, then Ω and its Harish-Chandra imbedding are the same up to an affine linear transformation by Main Theorem in [10]. Thus the result follows from Lemma 3.2.

Secondly, if Ω is a irreducible bounded symmetric domain of rank 1, and is a convex and circular domain, then Ω is exactly the common ball. The result also holds.

Note that if $\rho_1(z), \rho_2(z)$ are the defining functions of Ω_1, Ω_2 respectively, then the defining function of $\Omega_1 \times \Omega_2$ is $\rho(z, w) = \max(\rho_1(z), \rho_2(w))$. Thus Lemma 3.3 holds for the reducible situation.

Finally, let us recall the well-known result in the theory of one complex variable.

Lemma 3.4. *Let g be a holomorphic function on the unit disc in $\mathbf{C}$. If $\operatorname{Re} g(0) = 1$, $\operatorname{Re} g(\varsigma) \geq 0$, $\forall \varsigma \in D$, then the following inequality holds:*

$$\frac{1-|\varsigma|}{1+|\varsigma|} \leq \operatorname{Re} g(\varsigma) \leq \frac{1+|\varsigma|}{1-|\varsigma|}, \quad \forall \varsigma \in D.$$

§4. Preparatory Theorem

Now, we provide a necessary condition of starlikeness for normalized biholomorphic mapping, which has its own interest.

Preparatory Theorem. *Suppose Ω to be a bounded starlike circular domain in $\mathbf{C}^n$, its defining function $\rho(z)$ is a C^1 function except for a lower dimensional set. If $f(z)$ is a normalized biholomorphic starlike mapping, then we have*

$$\operatorname{Re}\frac{\partial \rho}{\partial z}(z) J_f^{-1}(z) f(z) \geq 0, \quad \forall z \in \Omega. \tag{4.1}$$

Moreover

$$\rho(z)\frac{1+\rho(z)}{1-\rho(z)} \geq 2\operatorname{Re}\frac{\partial \rho}{\partial z}(z) J_f^{-1}(z) f(z) \geq \rho(z)\frac{1-\rho(z)}{1+\rho(z)}, \quad \forall z \in \Omega, \tag{4.2}$$

where $f(z)$ is denoted by a column vector and $\frac{\partial \rho}{\partial z}(z) = \left(\frac{\partial \rho}{\partial z_1}(z), \cdots, \frac{\partial \rho}{\partial z_n}(z)\right)$.

Note 4.1. In the special cases, that is, in the bounded strictly balanced domains, this result was also obtained by Chen[1, Lemma 3.2].

Proof. Now, we list some simple properties of defining function derived directly from Lemma 3.2(ii), which will be used very often below.

$$2\operatorname{Re}\frac{\partial \rho}{\partial z}(z) z = \rho(z), \qquad \forall\ z \in \mathbf{C}^n, \tag{4.3}$$

$$2\operatorname{Re}\frac{\partial \rho}{\partial z}(z_0) z_0 = 1, \quad \forall\ z_0 \in \partial\Omega, \tag{4.4}$$

$$\frac{\partial \rho}{\partial z}(\lambda z) = \frac{\partial \rho}{\partial z}(z), \qquad \forall\ \lambda \in [0,\ \infty), \tag{4.5}$$

$$\frac{\partial \rho}{\partial z}(e^{i\theta} z) = e^{-i\theta}\frac{\partial \rho}{\partial z}(z), \quad \forall\ \theta \in \mathbf{R}. \tag{4.6}$$

The above equalities hold except for a lower dimensional set.

In the following, we fix $0 \neq z \in \Omega$ and denote $z_0 = \frac{z}{\rho(z)}$. Note that $z_0 \in \partial\Omega$.

First we begin to prove (4.1), which will be treated in two cases.

Case 1. z_0 be a non-essential boundary point[8, p.125] of Ω.

This means $f(\xi)$, $J_f^{-1}(\xi)$ can be holomorphically continued to some neighbourhood of z_0, so is the mapping $f^{-1}\left((1-r)f(\xi)\right)$ for any fixed $r \in (0,1)$. Note that under this homomorphic extension, $f(z_0)$ is defined. Clearly $f^{-1}\left((1-r)f(z_0)\right) \in \overline{\Omega}$, that is, $\rho\left(f^{-1}((1-r)f(z_0))\right) \leq 1,\ 0<r<1$. By expanding in Taylor series of r, the above inequality changes into

$$\rho\left\{z_0 - J_f^{-1}(z_0)f(z_0)r + O(r^2)\right\} \leq 1.$$

Furthermore

$$\rho(z_0) - 2\mathrm{Re}\frac{\partial\rho}{\partial z}(z_0)J_f^{-1}(z_0)f(z_0)r + O(r^2) \leq 1.$$

Recall that $\rho(z_0)=1$, thus

$$\mathrm{Re}\frac{\partial\rho}{\partial z}(z_0)J_f^{-1}(z_0)f(z_0) \geq 0. \tag{4.7}$$

Since Ω is a circular domain, for each $\theta \in \mathbf{R}$, $e^{i\theta}z_0$ is also a non-essential boundary point of Ω just as z_0, the same argument as (4.7) gives

$$\mathrm{Re}\frac{\partial\rho}{\partial z}(e^{i\theta}z_0)J_f^{-1}(e^{i\theta}z_0)f(e^{i\theta}z_0) \geq 0.. \tag{4.8}$$

Denote

$$h(\varsigma) = \mathrm{Re}\frac{\partial\rho}{\partial z}(z_0)\frac{J_f^{-1}(\varsigma z_0)f(\varsigma z_0)}{\varsigma}, \quad \varsigma \in \overline{D},$$

where D is the unit disc in $\mathbf{C}$.

Recall that all ςz_0 ($\forall\varsigma \in \partial D$) are nonessential boundary points of Ω, thus $h(\varsigma)$ is the real part of a homomorphic function, or a harmonic function on $\overline{D}$. By (4.6) and (4.8), $h(\varsigma) \geq 0$ on ∂D. Using the minimal value principle of harmonic function, we have $h(\varsigma) \geq 0, \quad \varsigma \in D$. Particularly taking $\varsigma = \rho(z)$, by (4.5) and (4.6), we obtain $\mathrm{Re}\frac{\partial\rho}{\partial z}(z)J_f^{-1}(z)f(z) \geq 0$.

Case 2. z_0 be a essential boundary point of Ω.

From the Theorem 3.4.5 in [8], there exists a neighbourhood of z_0 in $\partial\Omega$ which consists of essential boundary points of Ω. Now Choosing some sufficient small neighborhood of z_0 on $\partial\Omega$ and joining every point in this neighbourhood with the origin, we obtain a cone-type domain $\tilde{\Omega}$ in $\mathbf{C}^n$, which is a domain of holomorphy.

Consider the closed analytic disc in $\mathbf{C}^n$:

$$\varphi_{r,t}(\varsigma) = \frac{f^{-1}\left((1-r)f(t\varsigma z_0)\right)}{\varsigma}, \quad \varsigma \in \overline{D},$$

for any fixed $t \in (0,1), r \in [0,1]$. It is easy to observe that $\varphi_{r,t}(\partial D) \subset\subset \Omega$.

Moreover, note that for the sufficient small r,

$$\varphi_{r,t}(\varsigma) = \frac{f^{-1}\left((1-r)f(t\varsigma z_0)\right)}{\varsigma} = tz_0 + O(r)$$

is very closed to the ray tz_0, the center ray of the cone-type domain $\tilde{\Omega}$. Hence

$$\varphi_{r,t}(\partial D) \subset\subset \tilde{\Omega}, \quad \forall t \in (0,1),\ r \in [0,\delta], \tag{4.9}$$

where δ is a sufficient small positive number.

Now we claim that

$$\varphi_{r,t}(\overline{D}) \subseteq \tilde{\Omega}, \quad \forall t \in (0,1),\ r \in [0,\delta]. \tag{4.10}$$

If not, we can assume that there exists a positive constant $s_0 < 1$ such that $s\varphi_{r,t}(\overline{D}) \subseteq \tilde{\Omega}$ for any $\frac{1}{2}s_0 < s < s_0$, while for $s_0 < s < 1$, $s\varphi_{r,t}(\overline{D}) \nsubseteq \tilde{\Omega}$.

This means $\bigcup\limits_{\frac{1}{2}s_0<s<s_0} \varphi_{s,r,t}(\overline{D})$ is not relative compact in $\tilde{\Omega}$, where $\varphi_{s,r,t}(z) = s\varphi_{r,t}(z)$. At the same time recalling that $\tilde{\Omega}$ is a cone-type domain, by (4.9) we have $\varphi_{s,r,t}(\partial D) = s\varphi_{r,t}(\partial D) \subset\subset \tilde{\Omega}$ for $\frac{1}{2}s_0 < s < 1$. Furthermore $\bigcup\limits_{\frac{1}{2}s_0<s<1} \varphi_{s,r,t}(\partial D) \subset\subset \tilde{\Omega}$.

But on the other hand, from the property of analytic disc of holomorphic domain (see [8, Theorem 3.3.5 (3.3.5.1)]), we immediately obtain $\bigcup\limits_{\frac{1}{2}s_0<s<s_0} \varphi_{s,r,t}(\overline{D}) \subset\subset \tilde{\Omega}$, which is a contradiction. This proves the desired claim.

By Lemma 3.1, we know that (4.10) implies

$$\rho\left(\frac{f^{-1}\left((1-r)f(t\varsigma z_0)\right)}{\varsigma}\right) \le 1, \quad \forall |\varsigma| < 1.$$

Letting $t \longrightarrow 1$, we have $\rho\left(f^{-1}\left((1-r)f(\varsigma z_0)\right)\right) \le |\varsigma|, \quad \forall |\varsigma| < 1$. Then take $\zeta = \rho(z)$ to acquire $\rho\left(f^{-1}\left((1-r)f(z)\right)\right) \le \rho(z)$. Namely, $\rho(z) - 2\mathrm{Re}\frac{\partial\rho}{\partial z}(z)J_f^{-1}(z)f(z)r + o(r^2) \le \rho(z)$. Therefore $\mathrm{Re}\frac{\partial\rho}{\partial z}(z)\rho(z)J_f^{-1}(z)f(z) \ge 0$. This establishes the inequality (4.1).

Secondly, we prove the furthermore inequality (4.2).

Denote

$$g(\varsigma) = 2\frac{\partial\rho}{\partial z}(z_0)\frac{J_f^{-1}(\varsigma z_0)f(\varsigma z_0)}{\varsigma}, \quad \forall \varsigma \in D.$$

From (4.5), (4.6), we know that if let $\varsigma = |\varsigma|e^{i\theta}$, then $\frac{\partial\rho}{\partial z}(\varsigma z_0) = \frac{\partial\rho}{\partial z}(z_0)e^{-i\theta}$. Therefore

$$g(\varsigma) = \frac{2\frac{\partial\rho}{\partial z}(\varsigma z_0)J_f^{-1}(\varsigma z_0)f(\varsigma z_0)}{|\varsigma|}.$$

Clearly from the normalized condition of f, (4.4), and (4.1), we know that $g(\varsigma)$ satisfies the conditions of Lemma 3.4. This implies

$$|\varsigma|\frac{1+|\varsigma|}{1-|\varsigma|} \ge 2\mathrm{Re}\frac{\partial\rho}{\partial z}(\varsigma z_0)J_f^{-1}(\varsigma z_0)f(\varsigma z_0) \ge |\varsigma|\frac{1-|\varsigma|}{1+|\varsigma|}, \quad \forall \varsigma \in D.$$

Thus, set $\varsigma = \rho(z)$ in the above equality to obtain the desired equality (4.2).

This completes the proof of the preparatory theorem.

§5. The proof of Main Theorem

Now we can give the proof of the Main Theorem.

Proof of Main Theorem. Denote $z(t) = f^{-1}(tf(z)),\ t \in (0,1)$. Geometrically this represents a curve in the domain Ω, obtained by pulling back the straight line segment linking the point $f(z)$ and the origin in $f(\Omega)$ through the mapping f^{-1}. Then it is easy to see that

$$f(z(t)) = tf(z), \tag{5.1}$$

$$J_{f^{-1}}(tf(z)) = J_f^{-1}(z(t)), \tag{5.2}$$

$$\frac{dz(t)}{dt} = \frac{1}{t} J_f^{-1}(z(t))f(z(t)), \tag{5.3}$$

$$z(t) = tf(z) + O(t^2). \tag{5.4}$$

Obviously,

$$\frac{d\rho(z(t))}{dt} = 2\mathrm{Re}\left(\frac{\partial\rho}{\partial z}(z(t))\frac{dz(t)}{dt}\right). \tag{5.5}$$

Thus, by (4.2), (5.3) and (5.5), we obtain at once

$$\frac{1}{t}\rho(z(t))\frac{1+\rho(z(t))}{1-\rho(z(t))} \geq \frac{d\rho(z(t))}{dt} \geq \frac{1}{t}\rho(z(t))\frac{1-\rho(z(t))}{1+\rho(z(t))}. \tag{5.6}$$

Integrate the both sides of the right inequality above to yield

$$\int_\epsilon^1 \frac{(1+\rho(z(t)))\, d\rho(z(t))}{\rho(z(t))\,(1-\rho(z(t)))} \geq \int_\epsilon^1 \frac{dt}{t}.$$

An easy computation gives

$$\log\frac{\rho(z)}{(1-\rho(z))^2} - \log\frac{\rho(z(\epsilon))}{(1-\rho(z(\epsilon)))^2} \geq \log\frac{1}{\epsilon},$$

that is,

$$\frac{\rho(z)}{(1-\rho(z))^2} \geq \frac{\rho(z(\epsilon))}{(1-\rho(z(\epsilon)))^2\,\epsilon}.$$

If we set $\epsilon \longrightarrow 0$ in the above equality, then by Lemma 3.1(ii) and (5.4) we obtain the following $\frac{\rho(z)}{(1-\rho(z))^2} \geq \rho(f(z))$. Similarly, integrating the both sides of the left inequality in (5.6) gives $\rho(f(z)) \geq \frac{\rho(z)}{(1+\rho(z))^2}$.

Now, we turn to the equivalence of the inequalities (2.1) and (2.2). The method is the same as [5].

Denote $r = |z|$. Obviously

$$dr = \frac{1}{2r}\sum_{i=1}^n(\overline{z_i}dz_i + z_i d\overline{z_i}), \quad d\rho = \sum_{i=1}^n\left(\frac{\partial\rho}{\partial z_i}dz_i + \frac{\partial\rho}{\partial\overline{z_i}}d\overline{z_i}\right).$$

In the standard inner product

$$\langle dz_i,\ dz_i\rangle = \langle d\overline{z_i};\ d\overline{z_i}\rangle = 2, \qquad \langle dz_i,\ d\overline{z_j}\rangle = 0;$$

$$\langle dz_i,\ dz_j\rangle = \langle d\overline{z}_i,\ d\overline{z}_j\rangle = 0 \quad (i \neq j),$$

by (4.3), we have

$$\langle d\rho, dr\rangle = \frac{1}{r}2\mathrm{Re}\frac{\partial\rho}{\partial z}z = \frac{\rho}{r}. \tag{5.7}$$

On the other hand, it is easy to check that

$$\langle d\rho, dr\rangle = \langle\frac{\partial\rho}{\partial r}dr, dr\rangle = \frac{\partial\rho}{\partial r}. \tag{5.8}$$

Combining (5.7) and (5.8) gives

$$\frac{\partial\rho}{\partial r} = \frac{\rho}{r}. \tag{5.9}$$

Then integrating the both sides of the equality from $f(z)$ to $f(\epsilon z)$ for any small positive number ϵ, we obtain

$$\frac{\rho(f(z))}{\rho(f(\epsilon z))}=\frac{|f(z)|}{|f(\epsilon z)|}. \tag{5.10}$$

We will soon see that this equality gives the equivalence of (2.1) and (2.2). In fact, assume (2.1), then combining it with (5.10) gives

$$|f(\epsilon z)|\frac{(1-\epsilon\rho(z))^2}{\epsilon(1+\rho(z))^2}\le|f(z)|\le|f(\epsilon z)|\frac{(1+\epsilon\rho(z))^2}{\epsilon(1-\rho(z))^2}.$$

Then, letting $\epsilon\longrightarrow 0$, we obtain (2.2) since $\lim\limits_{\epsilon\to 0}\frac{|f(\epsilon z)|}{\epsilon}=|z|$ by the normalized condition of f.

The same argument shows that (2.2) implies (2.1). This completes the proof of Main Theorem.

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