Chin. Ann. of Math.
19B: 4(1998),409-414.

REFINED CONNECTIVITY PROPERTIES OF ABELIAN CAYLEY GRAPHS***

Li Qiaoliang* Li Qiao**

Abstract

Restricted edge connectivity λ' of a graph G is defined to be the minimum size $|U|$ of a set U of edges such that $G-U$ is disconnected and $G-U$ contains no trivial component K_1. The high order edge connectivity N_i, $i \geq 1$, is the number of edge cutsets of size i. To determine all N_i, $i \geq 1$, for a general graph is NP-hard. In this paper, the authors evaluated the restricted edge connectivity λ' and the high order edge connectivity N_i, $1 \leq i \leq \lambda' - 1$, for any connected Abelian Cayley graphs explicitly.

Keywords Abelian Cayley graph, Restricted edge connectivity, High order edge connectivity

1991 MR Subject Classification O5C40, O5C25

Chinese Library Classification O157.5

§1. Introduction

A graph $G = (V, E)$ means a finite graph without loops and multiple edges with vertex set V and edge set E, the classical edge connectivity $\lambda(G)$ of G is the minimum size of a set U of edges such that $G - U$ is disconnected, and such a set U is called a cutset of G. Note that in the above definition, absolutely no conditions or restrictions are imposed either on the components of $G - U$ or on the set U. Thus it would seem natural to generalize the concept of edge connectivity by introducing some conditions or restrictions on the components of $G - U$ and/or the set U.

As a generalization of classical edge connectivity, Esfahanian and Hakimi in [5] proposed the concept of restricted edge connectivity as follows:

Definiion 1.1. *A set U of edges of a connected graph G is called a restricted cutset (RC, in brevity) if $G - U$ is disconnected and $G - U$ contains no trivial component K_1. The restricted edge connectivity $\lambda'(G)$ is the minimum size of RCs in G.*

They also proved the following basic property:

Proposition 1.1. *If G is a connected graph with at least four vertices and it is not a star graph $K_{1,m}$ (from now on we make this assumption), then $\lambda'(G)$ is well defined and*

$$\lambda(G) \leq \lambda'(G) \leq \xi(G),$$

Manuscript received November 24, 1996.

*Department of Applied Mathematics, Shanghai Jiaotong University, Shanghai 200030, China.

Presend in Department of Mathematics, Hunan Normal University, Changsha 410082, China.

**Department of Applied Mathematics, Shanghai Jiaotong University, Shanghai 200030, China.

***Project supported by the National Natural Science Foundation of China (No. 19671057).

where $\xi(G) = \min\{d(x) + d(y) - 2 : \ \{x, y\} \in E(G)\}$.

Recently, motivated by the reliability analysis of communication networks, the following two concepts were proposed by Bauer, Boesch, Suffel, and Tinell[2].

Definition 1.2. *The high order edge connectivity* $N_i(G)$ *of a graph* G *is the number of cutsets of size* i.

Definition 1.3. *A connected graph is said to be super-*λ *if every cutset of size* λ *isolates a vertex with the minimum degree* δ *in* G.

It is easy to see that a graph G is super -λ iff $\lambda(G) = \delta(G)$ and $\lambda^{'}(G) > \lambda(G)$. Thus the qualitative property super-λ of G is quantified by $\lambda^{'}(G)$. By Proposition 1.1, we introduce the following concept:

Definition 1.4. *A super-*λ *graph* G *is said to be optimal super-*λ *if* $\lambda^{'}(G) = \xi(G)$.

Now we turn to a special class of regular graphs known as Cayley graphs.

Let S be a subset generating a finite group Γ. The pair (Γ, S) determines a connected Cayley digraph $\mathrm{Cay}(\Gamma, S)$, in which the vertices are the elements of Γ and the arcs are the pairs (g, gx) with $g \in \Gamma$ and $x \in S$. The arc (g, gx) is called an x-arc of the Cayley digraph.

If $1_\Gamma \notin S$ and $S = S^{-1}$ (ie., $x \in S \Leftrightarrow x^{-1} \in S$), then $\mathrm{Cay}(\Gamma, S)$ is a loopless symmetric digraph, which is considered as a graph, the edge $\{g, gx\} = \{g, gx^{-1}\}$ is called an x-edge or x^{-1}-edge.

Let Z_n be the additive group of residue classes of numbers modulo n. By abuse of notation, we identify the elements of Z_n with the natural numbers $0, 1, 2, \cdots, n-1$. A Cayley graph of the form $\mathrm{Cay}(Z_n, S)$ with $0 \notin S$ and $S = -S$ is called a circulant (graph). In [3], Boesch and Wang showed that the only circulants that are not super-λ are the cycles C_n and $\mathrm{Cay}(Z_{2m}, \{2, 4, \cdots, m-1, m\})$ for some odd integer $m > 1$. They also proved:

Theorem A. *Let* $H = \mathrm{Cay}(Z_n, \{1, 2, \cdots, k, n-1, n-2, \cdots, n-k\})$, $k < n/2$, *and let* U *be a cutset of* H *of size* i, $2k \le i \le 4k-3$, *then* U *isolates exactly one vertex and*

$$N_i(H) = \binom{nk-2k}{i-2k} n.$$

In [10] Yang, Wang, Lee and Boesch proved that the generalized hypercube $\mathrm{Cay}(Z_r^n,$ $\{(1, 0, \cdots, 0), \ \cdots, \ (0, 0, \cdots, 1)\})$ is super-λ unless $n = r = 2$. Recently, Hamidoune and Tindell in [6] generalized both mentioned super-λ results as follows:

Theorem B. *A connected Abelian Cayley graph that is not super-* λ *is either a cycle* C_n *or is isomorphic to* $K_m \times K_2$ *for some* $m \ge 2$.

In the present work, we first prove a nice property of RCs of size $\lambda^{'}$ for any connected Abelian Cayley graphs (the key lemma). Then by the key lemma, we evaluate the restricted edge connectivity $\lambda^{'}$ and the high order edge connectivity N_i, $\lambda \le i \le \lambda^{'} - 1$, for any connected Abelian Cayley graph explicitly. In particular, we show that every connected Abelian Cayley graph is optimal super- λ unless S contains exactly one element x of order 2, such that $\langle \Gamma - x \rangle \ne \Gamma$ and $|\Gamma| < 4|S| - 4$. We also prove that the conclusion of Theorem A holds for every connected Abelian Cayley graphs $\mathrm{Cay}(\Gamma, S)$ whenever S contains no order 2 elements of Γ. As a corollary, the mesh $C_n \times C_k = \mathrm{Cay}(Z_n \times Z_k, \{(0, 1), (0, -1), (1, 0), (-1, 0)\}$ is optimal super-λ.

Throughout this paper, G denotes a (simple) graph, Γ denotes a finite group, and Cay (Γ, S) denotes the Cayley graph of a generating set S of Γ with $1_\Gamma \notin S$ and $S = S^{-1}$.

§2. Key Lemma

In order to state and prove the key lemma, we need some definitions and propositions.

Let $G=(V,E)$ be a graph. For $X\subseteq V$, $(X,V-X)$ denotes the set of edges with one end in X and the other not in X. $\delta(X)$ denotes the number of edges in $(X,V-X)$. X is called a super fragment of G if $(X,V-X)$ is a *RC* and $\delta(X)=\lambda^{'}$, and a minimal super fragment is called a super atom.

Proposition 2.1[9] *Every connected vertex–symmetric grpaph G has edge connectivity δ, where δ is the minimum vertex degree of G.*

Proposition 2.2.[8,Exercise 6.48] $\delta_G(X\cap Y)+\delta_G(X\cup Y)\leq\delta_G(X)+\delta_G(Y)$ *for any* $X,Y\subseteq V(G)$.

Now we consider properties of *RC*s of a connected Abelian Cayley graphs.

Proposition 2.3. *Let $G=\mathrm{Cay}(\Gamma,S)$ be a connected Abelian Cayley graph with $3\leq|S|\leq 4$. X is a super atom of G with $|X|=3$. If for any $x\in S$ there is at most one x-edge in the induced subgraph $G[X]$ of G on X, then $G[X]\neq K_3$.*

Proof (By contradiction). Suppose $G[X]=K_3$. Then there exists three distinct elements x_1,x_2,x_3 in S such that $gx_1x_2=gx_3$ for some $g\in X$. This implies

$$x_1x_2=x_3. \qquad (*)$$

There are two cases:

Case 1. $|S|=4$.

Subcase 1.1. S contains no element of order greater than 2. Then $G\cong Q_4$ (4-cube), which contains no induced K_3.

Subcase 1.2. S contains one element of order at least 3. As $S=S^{-1}$, let $S=\{a,b,c,c^{-1}\}$ where $o(c)$(the order of c in Γ) ≥ 3. By $(*)$ one of the following 10 equations holds:

(1) $ab=c$, (2) $ab=c^{-1}$, (3) $ac=b$, (4) $ac=c^{-1}$,
(5) $ac^{-1}=b$, (6) $ac^{-1}=c$, (7) $bc=a$, (7) $bc=c^{-1}$,
(9) $bc^{-1}=a$, (10) $bc^{-1}=c$.

And (1), (2), (5), (7), (9) each imply that $o(c)=2$ or $o(c^{-1})=2$, which contradicts the hypothesis. (4), (6), (8), (10) each imply that $o(c)=4$, thus Cay $(\Gamma,S)\cong K_4\times K_2$, and $\lambda^{'}(K_4\times K_2)=4$. But if $G[X]=K_3$, then $\delta(X)=(4-2)3=6>4$, which contradicts the definition of X.

Subcase 1.3. S contains two elements of order at least 3. Let $S=\{a,\ b,\ a^{-1},\ b^{-1}\}$ with $o(a)\geq 3$ and $o(b)\geq 3$. By (*), one of the following 8 equations holds:

(1) $ab=a^{-1}$, (2) $ab=b^{-1}$, (3) $ab^{-1}=a^{-1}$, (4) $ab^{-1}=b$,
(5) $a^{-1}b=a$, (6) $a^{-1}b=b^{-1}$, (7) $a^{-1}b^{-1}=a$, (7) $a^{-1}b^{-1}=b$.

Each of them implies $\mathrm{Cay}(\Gamma,S)\cong\mathrm{Cay}(Z_n,\{1,2,-1,-2\})$ with $n\geq 5$. If $G[X]=K_3$. Then $\lambda^{'}=(4-2)3=6$. But in this case every set of edges incident with two ends of some edges of K_3 is a *RC*, which contradicts the fact that X is minimal.

Case 2. $|S|=3$.

Subcase 2.1. S contains no element of order at least 3. Then $\mathrm{Cay}(\Gamma,S)\cong Q_3$, which contains no induced K_3.

Subcase 2.2. S contains one element of order at least 3. Since $S=S^{-1}$, let $S=\{a,b,b^{-1}\}$ with $o(b)\geq 3$. By $(*)$, we have $ab^{-1}=b$ or $ab=b^{-1}$, which implies $a^{-1}=a=b^2$ and

$G \cong K_4$, hence $G[X] \neq K_3$.

By Case 1 and Case 2, Proposition 2.3 is proved.

Proposition 2.4. *Let $G = \mathrm{Cay}(\Gamma, S)$ be a connected Abelian Cayley graph, and X a super atom of G. If $|X| \geq 3$, then there exists some $x \in S$ such that the induced subgraph $G[X]$ contains at least two x-edges.*

Proof (By contradiction). Assume that for any $x \in S$ there is at most an x-edge in $G[X]$. Then $G[X]$ has at most $|S|$ edges, thus

$$\lambda' = \delta(X) = |S||X| - 2|S| = (|X| - 2)|S|.$$

If $|X| \geq 4$, then $\delta(X) \geq 2|S|$.

If $|X| = 3$ and $3 \leq |S| \leq 4$, by Proposition 2.3, $\delta(X) \geq 2(|S| - 1) + |S| - 2 > 2|S| - 2$.

If $|X| = 3$ and $|S| > 4$, then $G[X]$ has at most 3 edges, thus

$$\delta_G(X) \geq 3(|S| - 2) > 2|S| - 2.$$

All contradict Proposition 1.1 which asserts $\lambda' \leq \xi = 2|S| - 2$.

Proposition 2.5. *Let $G = \mathrm{Cay}(\Gamma, S)$ be a connected Abelian Cayley graph, X a super atom of* $\mathrm{Cay}(\Gamma, S)$ *and $|X| \leq n/2$. For any $x \in S$, if there are at least two x-edges in $G[X]$, then $Xx = X$.*

Proof. Let $Xx = X'$. Since G is vertex symmetric, X' is also a super atom and $|X'| = |X| \leq n/2$, $|V - (X \cap X')| \geq n/2 \geq 2$. Since there are at least two x-edges in $G[X]$, then

$$|X \cap X'| \geq 2; \quad 2 \leq |X \cup X'| = |X| + |X'| - |X \cap X'| \leq n - 2 \quad \text{and } |V - (X \cup X')| \geq 2.$$

Thus $\delta_G(X \cap X') \geq \lambda + \delta > \lambda'$ if $G[X \cap X']$ or $G[V - (X \cap X')]$ has a trivial component K_1; $\delta_G(X \cap X') \geq \lambda'$ otherwise.

Similarly, $\delta_G(X \cup X') \geq \delta + \lambda > \lambda'$ if $G[V - (X \cup X')]$ has a trivial component K_1; $\delta_G(X \cup X') \geq \lambda'$ otherwise.

Therefore, if $G[X \cap X']$ or $G[V - (X \cap X')]$ or $G[V - (X \cup X')]$ has a trivial component K_1, then

$$\delta_G(X \cap X') + \delta_G(X \cup X') > \lambda' + \lambda' = \delta_G(X) + \delta_G(X'),$$

which contradicts Proposition 2.2. So each of the three induced subgraphs $G[X \cap X']$ $G[V - (X \cap X')]$ and $G[V - (X \cup X')]$, has no trivial component K_1. Hence $\delta_G(X \cup X') \geq \lambda'$ and $\delta_G(X \cap X') \geq \lambda'$. Using Proposition 2.2 again, we have

$$\lambda' + \lambda' \leq \delta_G(X \cap X') + \delta_G(X \cup X') \leq \delta_G(X) + \delta_G(X') = \lambda' + \lambda'.$$

Thus $\delta_G(X \cap X') = \lambda'$ and $\delta_G(X \cup X') = \lambda'$. Since X is minimal, we have $X \cap X' = X$, i.e., $X' = X$, the proposition is proved.

Proposition 2.5 can be strengthened as follows:

Proposition 2.6. *Let $G = \mathrm{Cay}(\Gamma, S)$, and X be defined as in Proposition* 2.5*. If for some $x \in S$ there is an x-edge in $G[X]$, then $Xx = X$.*

Proof. By Propositions 2.4 and 2.5, there exists some $x' \in S$ such that $Xx' = X$. Suppose $\{g, gx\}$ is an x-edge of $G[X]$. Since both gx' and gxx' are in X, the x-edge $\{gx', gx'x\}$ is also in $G[X]$, and $\{gx', gx'x\} \neq \{g, gx\}$. By Proposition 2.3, $Xx = X$.

Key Lemma. *Let $G = \mathrm{Cay}(\Gamma, S)$ be a connected Abelian Cayley graph and X a super atom of G with $|X| \leq |\Gamma|/2$. If $|X| = 2$, then $G[X] \cong K_2$. If $|X| \geq 3$, there exists some subset*

$S^{'} \subseteq S$ such that the induced subgraph $G[X]$ is isomorphic to a connected Abelian Cayley graph of order $|X| = |\langle S^{'}\rangle|$ with generating set $S^{'} \subseteq S$, where $\langle S^{'}\rangle$ denotes the subgraph of Γ generated by $S^{'}$.

Proof. By the definition of RC, we know that $G[X]$ is connected and $|X| \geq 2$. If $|X| = 2$, then $G[X] \cong K_2$. Now assume that $|X| \geq 3$ and, without loss of generality $1 \in X$.

Let $S^{'} = \{x \in S$: there exists at least one x-edge in $G[X]\}$. Consider $S^{'}$ and X as subsets of Γ and let $\langle S^{'}\rangle$ be the subgraph of Γ generated by $S^{'}$. Since $1 \in X$, we have $\langle T\rangle \subseteq X$ by Proposition 2.5.

Now we prove that $X \subseteq \langle S^{'}\rangle$. For any $g \in X$, as $G[X]$ is connected, there is a path joining e and g which consists of a sequence of x-edges for $x \in S^{'}$, thus $g \in \langle S^{'}\rangle$. Since $\langle S^{'}\rangle \subseteq X$, we have $\langle S^{'}\rangle = X$. Therefore X is a subgroup of Γ and thus $G[X]$ is isomorphic to a conected Abelian Cayley graph $\mathrm{Cay}(\langle S^{'}\rangle, S^{'})$ with $S^{'} \subseteq S$ and $|\langle S^{'}\rangle| = |X|$.

§3. Main results

Theorem 3.1. *Let $G = \mathrm{Cay}(\Gamma, S)$ be a connected Abelian Cayley graph. Then*

$$(1)\quad \lambda^{'}(G) = \begin{cases} |\Gamma|/2, & \text{if } S \text{ contains exactly one element } x \text{ of order } 2 \\ & \text{such that } \langle S - x\rangle \neq \Gamma \text{ and } |\Gamma| \leq 4|S| - 4, \\ 2|S| - 2, & \text{otherwise.} \end{cases}$$

(2) *If $G \neq K_m \times K_2$, then every cutset of size $i, |S| \leq i \leq \lambda^{'} - 1$, isolates exactly one vertex, and*

$$N_i = \binom{|E| - |S|}{i - |S|}|\Gamma|.$$

Proof. Let X be a super atom of G with $|X| \leq |\Gamma|/2$. By the key lemma, $G[X]$ is isomorphic to K_2 or a connected Abelian Cayley graph $\mathrm{Cay}(\langle S^{'}\rangle, S^{'})$ with $|\langle S^{'}\rangle| = |X|$ and $S^{'} \subseteq S$. If $G[X] \cong K_2$, then $\lambda^{'}(G) = \delta(X) = 2|S| - 2$. Now assume that $G[X]$ is isomorphic to a connected Abelian Cayley graph $\mathrm{Cay}(\langle S^{'}\rangle, S^{'})$ of order $|X|$. Since $G[X]$ is regular of degree $|S^{'}|$, we have

$$\lambda^{'} = \delta(X) = (|S| - |S^{'}|)|X| \geq (|S| - |S^{'}|(|S^{'}| + 1).$$

It is easy to verify that for $1 \leq |S^{'}| \leq |S| - 2$,

$$(|S| - |S^{'}|)(|S^{'}| + 1) \geq 2|S| - 2.$$

If every vertex of $G[X]$ has degree $|S| - 1$, then $G[X]$ is isomorphic to $\mathrm{Cay}(\langle S^{'}\rangle, S^{'})$ with $|S^{'}| = |S| - 1$. This implies that there is some $x \in S - S^{'}$ with $o(x) = 2$ and $\langle S - x\rangle \neq G$, in this case $\delta(X) = |\Gamma|/2$.

If there are at least two $x_1, x_2 \in S$ with $o(x_1) = o(x_2) = 2$ such that $\langle S - x_1\rangle \neq \Gamma$ and $\langle S - x_2\rangle \neq \Gamma$, then

$$|\langle S - x_1\rangle| = |\Gamma|/2, \quad |\langle S - x_2\rangle| = |\Gamma|/2 \ \text{ and } \ \langle S - x_1 - x_2\rangle \neq \langle S - x_1\rangle$$

(Otherwise x_2 can be expressed as the product of elements of $S - x_1 - x_2$, this implies that $\langle S - x_2\rangle = \Gamma$, contradicting the assumption). So $|\langle S - x_1\rangle| = 2|\langle S - x_1 - x_2\rangle|$. As $|\langle S - x_1 - x_2\rangle| \geq |S| - 1$, we have $|\Gamma|/2 \geq 2|S| - 2$.

The above discussion leads to $\lambda^{'}(G) \geq \min\{2|S| - 2, |\Gamma|/2\}$ and $\lambda^{'}(G) = |\Gamma|/2$ iff there is exactly one element $x \in S$ with $o(x) = 2$ and $\langle S - x\rangle \neq \Gamma$.

On the other hand, by Proposition 1.1 and the fact that if there is some $x \in S$ with $o(x) = 2$, and $\langle S - x \rangle \neq \Gamma$, then the set of the $|\Gamma|/2$ edges between $\langle S - x \rangle$ and $x\langle S - x \rangle$ is a RC, we have $\lambda^{'} \leq \min\ \{2|S| - 2, |\Gamma|/2\}$. Thus conclusion (1) is proved. The conclusion (2) follows immediately from (1) and the definition of $\lambda^{'}$.

Corollary 3.1. *A connected Abelian Cayley graph* $G = \mathrm{Cay}(\Gamma, S)$ *is optimal super-*λ *unless it is a cycle* C_n *or* $K_m \times K_2$ *for some* $m \geq 2$ *or there is exactly one element* $x \in S$ *with* $o(x) = 2$ *such that* $\langle S - x \rangle \neq \Gamma$ *and* $|\Gamma| \leq 4|S| - 4$.

Now we consider the case when $\mathrm{Cay}(\Gamma, S)$ is a circulant, i.e., $\Gamma = Z_n$ and $0 \notin S$, $S = -S$. From Theorem 3.1 we have

Corollary 3.2. *Let* $H = \mathrm{Cay}(Z_n, \{a_1, a_2, \cdots, a_k, n - a_k, \cdots, n - a_2, n - a_1\})$, *with* $k \geq 2$ *and* $a_k < n/2$, *and let* U *be a cutset of size* i *with* $2k \leq i \leq 4k - 3$. *Then* U *isolates exactly one vertex and*

$$N_i(H) = \binom{nk - 2k}{i - 2k} n.$$

A detailed report for related results of circulants see our recent work [7].

Notice that a connected Abelian Cayley graph $\mathrm{Cay}(\Gamma, S)$ is not super-λ iff $\lambda^{'} = |S|$ and there exists at least one RC, say U, with $|U| = \lambda^{'}$ such that $\mathrm{Cay}(\Gamma, S) - U$ is disconnected. So if $\lambda^{'} = |S| = 2|S| - 2$ then $|S| = 2$, which implies $\mathrm{Cay}(\Gamma, S) \cong C_n$. $\lambda^{'} = |S| = |\Gamma|/2$ iff there exists exactly one element x of order 2 such that $\langle S - x \rangle \neq \Gamma$ and $|\Gamma| \leq 4|S| - 4$. $|\Gamma|/2 = |S|$ iff $\mathrm{Cay}(\Gamma, S) \cong K_m \times K_2$. Thus we have:

Corollary 3.3.[6] *A connected Abelian Cayley graph that is not super-*λ *is either a cycle* C_n *or is isomorphic to* $K_m \times K_2$ *for some* $m \geq 2$.

Corollary 3.4.[10] *The generalized hypercube* $\mathrm{Cay}(Z_r^n, \{(1, 0, \cdots, 0), \cdots, (0, 0, \cdots, 1)\}$ *is optimal super-*λ *unless* $n = r = 2$.

Corollary 3.5. *The mesh* $C_n \times C_k$ *is optimal super-*λ *unless* $n = k = 2$.

References

[1] Ball, M. O., Complexity of network reliability computation, *Networks,* 10(1980), 153–165.
[2] Bauer, D., Boesch, F., Suffel, C. & Tindell, R., Connectivity extremal problems and the design of reliable probabilistic networks, in The Theory and Application of Graphs, Wiley, New York, 1981, 45–54.
[3] Boesch, F. & Wang, J., Super line connectivity properties of circulant graphs, *SIAM J. Alg. Disc. Meth.,* 7(1986), 89–98.
[4] Esfahanian, A. H., Generalized measure of fault tolerance with application to n-cube, *IEEE Trans. Comput.,* **38**(1989), 1586–1590.
[5] Esfahanian, A. H. & Hakimi, S. L., On computing a conditional edge connectivity of a graph, *Inform. Process. Lett.,* **27**(1988), 195–199.
[6] Hamidoune, Y. & Tindell, R., Vertex transitive and super line connectedness, *SIAM J. Disc. Math.,* **3**(1990), 524–530.
[7] Li, Q. L. & Li, Q., Reliability analysis of circulant graphs, *Networks,* **31**(1998), 61–65.
[8] Lovasz, L., Combinatorial problems and exercises, Northholland Publishing Company, 1979.
[9] Mader, W., Minimale n-fach kantenzusammenhangende graphen, *Math Ann.,* **191**(1971), 21–28.
[10] Yang, C. S., Wang, J. F., Lee, J. & Boesch, F.T., Reliability properties of hypercube networks, *Comput. Math. Appl.,* **14**(1987), 541–548.