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A THEOREM ON PLURICANONICAL MAPS OF NONSINGULAR MINIMAL THREEFOLDS OF GENERAL TYPE**

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Abstract

Let X be a complex nonsingular projective threefold of general type. It is shown that the dimension of the image of X through m-canonical maps is at least two for every $m \geq 3$.

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§1. Introduction and the Main Theorem

Let X be a complex nonsingular minimal projective threefold of general type. The nature of pluricanonical maps of X is very important to the classification theory. It is well-known from [10] that $\Phi_{|mK_X|}$ is a birational map for $m \geq 7$. In [3], it is proved that 6-canonical map of X is a birational map onto its image. In this paper, we mainly study the following problem:

Problem. *What is the greatest positive integer m_0 such that $|m_0K_X|$ is composed of a pencil of surfaces for some X, i.e.,* $\dim\Phi_{|m_0K_X|}(X) = 1$*?*

Benveniste[1] proved that $m_0 \leq 3$. We can easily see that $m_0 \geq 1$ (cf. [4]). Our result in this paper is that $m_0 \leq 2$.

Main Theorem *Let X be a nonsingular minimal projective threefold of general type. Then* $\dim\Phi_{|mK_X|}(X) \geq 2$ *for $m \geq 3$.*

Throughout this paper, all our arguments proceed over the the complex number field $\mathbb{C}$. Most terms and notations are standard except the following which we are in favor of:

$:=$ — definition;

$\sim_{\text{lin}}$ — linear equivalence;

$\sim_{\text{num}}$ — numerical equivalence.

§2. Preparation

We will use the vanishing theorem in the following form.

Proposition 2.1 (Theorem 1.2 of [6]). *Let X be a nonsingular complete variety, $D \in$ $\mathrm{Div}(X) \otimes \mathbb{Q}$. Assume the following two conditions:*

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(1) *D is nef and big;*

(2) *the fractional part of D has the support with only normal crossings.*

Then $H^i(X, \mathcal{O}_X(\lceil D \rceil + K_X) = 0$ for $i > 0$, where $\lceil D \rceil$ is the minimum integral divisor with $\lceil D \rceil - D \geq 0$.

Let X be a nonsingular projective threefold. For a divisor $D \in \mathrm{Div}(X)$, we have

$$\chi(\mathcal{O}_X(D)) = D^3/6 - K_X \cdot D^2/4 + D \cdot (K_X^2 + c_2)/12 + \chi(\mathcal{O}_X)$$

by Riemann-Roch theorem. The calculation shows that

$$\chi(\mathcal{O}_X(D)) + \chi(\mathcal{O}_X(-D)) = -K_X \cdot D^2/2 + 2\chi(\mathcal{O}_X) \in \mathbb{Z},$$

therefore $K_X \cdot D^2$ is an even integer, especially K_X^3 is even.

If K_X is nef and big, then we obtain by Kawamata-Viehweg's vanishing theorem

$$p(n) := h^0(X, \mathcal{O}_X(nK_X)) = (2n-1)[n(n-1)K_X^3/12 - \chi(\mathcal{O}_X)], \tag{2.1}$$

for $n \geq 2$.

Let X be a nonsingular projective threefold, $f : X \to C$ be a fibration onto a nonsingular curve C. From the spectral sequence:

$$E_2^{p,q} := H^p(C, R^q f_* \omega_X) \Longrightarrow E^n := H^n(X, \omega_X),$$

we get by direct calculation

$$h^2(\mathcal{O}_X) = h^1(C, f_*\omega_X) + h^0(C, R^1 f_* \omega_X), \tag{2.2}$$

$$q(X) := h^1(\mathcal{O}_X) = b + h^1(C, R^1 f_* \omega_X). \tag{2.3}$$

Therefore we obtain

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_F)\chi(\mathcal{O}_C) + \Delta_2 - \Delta_1, \tag{2.4}$$

where we set $\Delta_1 := \mathrm{deg} f_* \omega_{X/C}$ and $\Delta_2 := \mathrm{deg} R^1 f_* \omega_{X/C}$. We can also refer to Corollary 3.2 of [9] for the above formula.

For a nonsingular threefold X with nef and big canonical divisor K_X, Miyaoka[11] showed that $3c_2 - c_1^2$ is pseudo-effective, therefore we get $K_X^3 \leq -72\chi(\mathcal{O}_X)$ by the Riemann-Roch equality $\chi(\mathcal{O}_X) = -c_2 \cdot K_X/24$. In particular, $\chi(\mathcal{O}_X) < 0$.

Taking a combination of theorems of Kawamata and Kollár, we have the following Lemma.

Lemma 2.1. *Let X, C be nonsingular projective varieties and C be a curve, $f : X \to C$ be an algebraic fiber space. Then*

(1) *$f_*[\omega_{X/C}^{\otimes m}]$ is semi-positive for $m \geq 1$;* (2) *$R^i f_* \omega_{X/C}$ is semipositive for $i > 0$.*

Corollary 2.1. *Under the same condition as in Lemma 2.1, we have $\triangle_i \geq 0$ for $i = 1, 2$.*

§3. Proof of the Main Theorem

It has been shown in [10] that $\dim \Phi_{|mK_X|}(X) \geq 2$ for $m \geq 4$. Therefore we have to study the case when $m = 3$, i.e., the 3-canonical map.

Let X be a nonsingular projective threefold with nef and big canonical divisor K_X. From (2.1), we have

$$p(3) = h^0(X, 3K_X) = 5\Big[\frac{1}{2}K_X^3 - \chi(\mathcal{O}_X)\Big] \geq 10,$$

therefore $\dim \Phi_{|3K_X|}(X) \geq 1$.

Now suppose that $|3K_X|$ is composed of pencils, i.e., $\dim\Phi_{|3K_X|}(X) = 1$; we hope to derive a contradiction. Denote $\phi_3 := \Phi_{|3K_X|}$, $W_3 := \overline{\Phi_{|3K_X|}(X)} \subset \mathbb{P}^{p(3)-1}$. Take a succession of blow-ups with nonsingular centers $f_3 : X' \to X$ according to Hironaka such that $g_3 := \phi_3 \circ f_3$ is a morphism. Suppose $g_3 : X' \stackrel{h_3}{\longrightarrow} C \stackrel{s_3}{\longrightarrow} W_3$ is the Stein factorization. We have the following commutative diagram:

$$\begin{tikzcd}
X' \arrow[r, "h_3"] \arrow[d, equal] & C \arrow[d, "s_3"] \\
X' \arrow[r, "g_3"'] \arrow[d, "f_3"'] & W_3 \\
X &
\end{tikzcd}$$

where C is a nonsingular curve. $h_3 : X' \to C$ is a fibration with general fiber F_3 being a nonsingular projective surface of general type. Denote $b := g(C)$, the genus of C.

Proposition 3.1 (Theorem 7 of [10]). *Let X be a nonsingular projective threefold with nef and big canonical divisor K_X. Suppose that $|3K_X|$ is composed of pencils. Then $f_3^*(K_X)^2 \cdot F_3 = 1$. In this case, F_3 is a nonsingular projective surface of general type. Let $\pi_3 : F_3 \to F_{3,0}$ be the contraction onto the minimal model, $K_{3,0}$ a canonical divisor on $F_{3,0}$. Then $K_{3,0}^2 = 1$ and $\mathcal{O}_{F_3}(\pi_3^*(K_{3,0})|_{F_3}) \cong \mathcal{O}_{F_3}(f_3^*(K_X)|_{F_3})$.*

Theorem 3.1. *Let X be a nonsingular projective threefold with nef and big canonical divisor K_X. Suppose $p_g(X) \geq 2$, then* $\dim\phi_3(X) \geq 2$.

Proof. Suppose $|3K_X|$ be composed of pencils, we shall derive a contradiction.

In this case, we have $|K_X| \subset |3K_X|$. Therefore $\Phi_{|K_X|}$ generically factors through ϕ_3. Take a common modification X' of X such that both $g_1 := \Phi_{|K_X|} \circ f$ and $g_3 := \phi_3 \circ f$ are morphisms. We have the following commutative diagram:

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, equal] & C \arrow[d, "s_3"] \\
X' \arrow[r, "g_3"'] \arrow[d, equal] & W_3 \subset \mathbb{P}^{p(3)-1} \arrow[d, "s_0"] \\
X' \arrow[r, "g_1"'] \arrow[d, "f"'] & W_1 \subset \mathbb{P}^{p_g(X)-1} \\
X &
\end{tikzcd}$$

We see that $g_1 = s_0 \circ g_3$. Suppose $g_3 = s_3 \circ h$ is the Stein factorization, then we see that $g_1 = (s_0 \circ s_3) \circ h$ is just a Stein factorization of g_1. Therefore we see that both $\Phi_{|K_X|}$ and ϕ_3 derive the same fibration h. Take F be a general fiber of h and H_i be a general hyperplane section of W_i in $\mathbb{P}^{p(i)-1}$ for $i = 1, 3$.

$|3K_{X'}|$ is composed of pencils, so is $|K_{X'}|$. Set $K_{X'} \sim_{\text{lin}} g_1^*(H_1) + Z_1$, where Z_1 is the fixed part. We naturally have $g_1^*(H_1) \sim_{\text{num}} a_1 F$, where

$$a_1 = \deg W_1 \cdot \deg(s_0) \cdot \deg(s_3) \geq p_g(X) - 1.$$

Note that $g_1^*(H_1)$ can be a disjoint union of fibers at least over a nonempty open Zariski subset of C, we have the following exact sequence:

$$0 \to \mathcal{O}_{X'}(K_{X'} + f^*(K_X)) \to \mathcal{O}_{X'}(K_{X'} + f^*(K_X) + g_1^*(H_1))$$
$$\to \bigoplus_{i=1}^{a_1} \mathcal{O}_{F_i}(K_{F_i} + f^*(K_X)|_{F_i}) \to 0.$$

We see that each F_i is of the same kind as that in Proposition 3.1. Because $f^*(K_X)$ is nef and big, we get by Kawamata-Viehweg's vanishing theorem that $H^1(X', K_{X'}+f^*(K_X)) = 0$. We now study the divisor $G := K_{X'} + f^*(K_X) + g_1^*(H_1)$ and the system $|G|$. We obviously have $G \leq 3K_{X'}$. $\Phi_{|G|}|_{F_i}$ means $\Phi_{|K_{F_i}+f^*(K_X)|_{F_i}|}$, i.e., $\Phi_{|2K_{F_i}|}$ according to Prposition 3.1. From the exact sequence

$$0 \to H^0(X', K_{X'} - F) \xrightarrow{i} H^0(X', K_{X'}) \to H^0(K_F) \to \cdots$$

we see that i is an absolute inclusion and therefore $p_g(F) > 0$. Therefore by the results on surface, we see $\Phi_{|2K_F|}$ is generically finite. Therefore we see that $\dim\Phi_{|G|}(X) = 3$. Then $\Phi_{|3K_{X'}|}$ is generically finite, which is a contradiction. Thus the theorem is proved.

Proposition 3.2. *Let X be a nonsingular projective threefold with nef and big canonical divisor K_X. Suppose that $|3K_X|$ is composed of pencils and that $p_g(X) \leq 1$. Then $b = 1$, $p_g(X) = q(X) = 1$ and $h^2(\mathcal{O}_X) = 0$.*

Proof. We use the first commutative diagram of this section and keep the same notations there. Because

$$\chi(\mathcal{O}_X) = 1 - q(X) + h^2(\mathcal{O}_X) - p_g(X) \leq -1,$$

we have $q(X) \geq 1 + h^2(\mathcal{O}_X) + (1 - p_g(X)) \geq 1 > 0$. On the other hand, we have $q(X) = q(X') = b + h^1(R^1h_3 * \omega_{X'})$ by (2.3). From Proposition 3.1, we know that $K_{F_0}^2 = 1$, where F_0 is the minimal model of F. Thus we have $q(F) = 0$ by Bombieri's theorem in [2]. Therefore $R^1h_{3*}\omega_{X'} = 0$ and $b = q(X) > 0$, which means ϕ_3 is actually a morphism.

For the simplicity, we have $X' = X$ and a fibration $h_3 : X \to C$. Denote the relative dualizing sheaf by $\omega_{X/C} = \omega_X \otimes h_3^*\omega_C^{-1}$. Also $h_{3*}[\omega_{X/C}^{\otimes 3}] = h_{3*}\omega_X^{\otimes 3} \otimes \omega_C^{\otimes -3}$. Let $\mathcal{E}_0$ be the saturated subbundle of $h_{3*}[\omega_X^{\otimes 3}]$ which is generated by all those global sections in $H^0(h_{3*}[\omega_X^{\otimes 3}])$. Because $|3K_X|$ is composed of pencils and ϕ_3 factors through h_3, we see that $\mathcal{E}_0$ is a rank one vector bundle. We have the following exact sequence

$$0 \to \mathcal{E}_0 \to h_{3*}[\omega_X^{\otimes 3}] \to \mathcal{E}_1 \to 0.$$

Note that $r = rk(h_{3*}[\omega_X^{\otimes 3}]) = h^0(3K_F) \geq 4$. We have

$$h^1(\mathcal{E}_0) \geq h^0(\mathcal{E}_1) \geq \deg\mathcal{E}_1 + (r-1)(1-b). \tag{3.1}$$

We also have the following sequence $h_{3*}[\omega_{X/C}^{\otimes 3}] \to \mathcal{E}_1 \otimes \omega_C^{\otimes -3} \to 0$. Because $h_{3*}[\omega_{X/C}^{\otimes 3}]$ is semipositive by Lemma 2.1, we have $\deg\mathcal{E}_1 \geq 6(r-1)(b-1)$. Thus, according to (3.1), we have

$$h^1(\mathcal{E}_0) \geq h^0(\mathcal{E}_1) \geq 5(r-1)(b-1). \tag{3.2}$$

If $h^1(\mathcal{E}_0) > 0$, then, by Clifford's theorem,

$$\deg\mathcal{E}_0 \geq 2h^0(\mathcal{E}_0) - 2 > h^0(\mathcal{E}_0) \tag{3.3}$$

because $\deg\mathcal{E}_0 > 0$. Because $h^0(\mathcal{E}_0) - h^1(\mathcal{E}_0) = \deg\mathcal{E}_0 - b + 1$, we get $h^1(\mathcal{E}_0) < b - 1$ by (3.3). Combining the above formulae, we obtain

$$5(r-1)(b-1) \le h^0(\mathcal{E}_1) \le h^1(\mathcal{E}_0) < b - 1,$$

which is impossible. Thus $h^1(\mathcal{E}_0) = h^0(\mathcal{E}_1) = 0$ and $b = 1$ by (3.2). Therefore we have $q(X) = p_g(X) = 1$, $h^2(\mathcal{O}_X) = 0$ and $\chi(\mathcal{O}_X) = -1$.

Theorem 3.2. *Let X be a nonsingular projective threefold with nef and big canonical divisor K_X. Suppose $p_g(X) \le 1$. Then $|3K_X|$ can not be composed of pencils.*

Proof. Suppose $|3K_X|$ is composed of pencils, we should derive a contracdiction.

From Proposition 3.1, we know that the minimal model F_0 of F is a surface of general type with $c_1^2 = 1$. On the other hand, $p_g(F) > 0$. We easily obtain two cases: (1) $p_g(F) = 2$ and (2) $p_g(F) = 1$.

Case 1. $p_g(F) = 2$. We first study $|2K_X|$. According to (2.1), we have $p(2) = \frac{1}{2}K_X^3 - 3\chi(\mathcal{O}_X) \ge 4$. Therefore $\phi_2 := \Phi_{|2K_X|}$ is a well-defined rational map. We have $|2K_X| \subset |3K_X|$, then $|2K_X|$ is also composed of pencils and ϕ_2 generically factors through ϕ_3. We have the following commutative diagram:

$$\begin{array}{ccc}
X & \xrightarrow{h_3} & C \\
\| & & \downarrow s_3 \\
X & \xrightarrow[\phi_3]{} & W_3 \\
\| & & \downarrow s_{3,2} \\
X & & W_2 = \overline{\phi_2(X)} \subset \mathbb{P}^{p(2)-1}
\end{array}$$

where we note that $s_{3,2}$ is a well-defined map except finite points on W_3. We can choose a general hyperplane section H_2 of W_2, then $M_2 := h_3^* s_3^* s_{3,2}^*(H_2)$ is just the moving part of $|2K_X|$. Because h_3 is a fibration, we see that $|M_2|$ has no base points. Therefore ϕ_2 is actually a morphism. Note that $\phi_2 = (s_{3,2} \circ s_3) \circ h_3$ is just a Stein factorization of ϕ_2. We set

$$2K_X \sim_{\text{lin}} \phi_2^*(H_2) + Z_2 \sim_{\text{lin}} \sum_{i=1}^{a_2} F_i + Z_2,$$

where $F_i's$ are fibers of h_3. Because $\phi_2^*(H_2)$ can be a disjoint union of fibers at least over a nonempty Zariski open subset of W_3, we can have the following exact sequence:

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + \phi_2^*(H_2)) \to \bigoplus_{i=1}^{a_2} \mathcal{O}_{F_i}(K_{F_i}) \to 0.$$

Because $b = 1$, we easily see $a_2 = p(2)$ by Riemann-Roch equality. Because $h^2(\mathcal{O}_X) = 0$, we have the following surjective map:

$$H^0(X, K_X + \phi_2^*(H_2)) \to \bigoplus_{i=1}^{a_2} H^0(F_i, K_{F_i}) \to 0.$$

This means that $\Phi_{|K_X + \phi_2^*(H_2)|}$ can generically separate distinct fibers of h_3 and that

$$\Phi_{|K_X + \phi_2^*(H_2)|}|_{F_i} = \Phi_{|K_{F_i}|}$$

which is exactly a canonical map of pencils. Therefore $\dim\Phi_{|K_X+\phi_2^*(H_2)|}(X) = 2$. And then $\dim\phi_2(X) \geq 2$, which is a contradiction.

Case 2. $p_g(F) = 1$. In this case, the above method fails to be effective. From the exact sequence in Case 1, we have $h^0(X, K_X + \phi_2^*(H_2)) = 1 + a_2 = 1 + p(2)$. Denote $G_1 := K_X + \phi_2^*(H_2)$. We obviously have $|G_1| \subset |3K_X|$. Therefore $|G_1|$ is also composed of pencils. Using the same argument as in Case 1, we can see that $\Phi_{|G_1|}$ is also a morphism and that $G_1 = \sum_{i=1}^{1+a_2} F_i + \overline{Z}$, where $\overline{Z}$ is the fixed part of $|G_1|$. Thus, from the following relation

$$K_X + \phi_2^*(H_2) = G_1 = \sum_{i=1}^{1+a_2} F_i + \overline{Z}$$

we should have $K_X \sim_{\rm lin} F_k + \overline{Z}$ for some fiber F_k of h_3. Now let $G_2 := K_X + F_k$ and $G_3 := 2K_X + F_k$. We have $|G_2| \subset |G_3| \subset |3K_X|$. Therefore both $|G_2|$ and $|G_3|$ are also composed of pencils. On the other hand, we can write $G_2 = \sum_{i=1}^{a} F_i + Z'$ by a similar argument, where F_i is a general fiber of h_3 for each i. Using the following exact sequence

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X\Big(K_X + \sum_{i=1}^{a} F_i\Big) \to \bigoplus_{i=1}^{a} \mathcal{O}_{F_i}(K_{F_i}) \to 0,$$

because $h^2(\mathcal{O}_X) = 0$, we see that $\Phi_{|K_X+G_2-Z'|}$ can generically separate distinct fibers of h_3, so is $\Phi_{|G_3|}$. In what follows, we study $\Phi_{|G_3|}|_F$ for a general fiber F of h_3.

We have the exact sequence

$$0 \to \mathcal{O}_X(2K_X + F_k - F) \to \mathcal{O}_X(G_3) \to \mathcal{O}_F(2K_F) \to 0.$$

Because $K_X + F_k - F$ is nef and big, we see by Kawamata-Viehweg's vanishing theorem that $H^1(X, 2K_X + F_k - F) = 0$. Thus $\Phi_{|G_3|}|_F = \Phi_{|2K_F|}$, which is a generically finite map onto its image. Therefore we see that $\dim\Phi_{|G_3|}(X) = 3$ and then $\dim\phi_3(X) = 3$, which leads to a contradiction.

Theorem 3.1 and Theorem 3.2 imply the main theorem.

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