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ON THE COHOMOLOGY OF GENERALIZED RESTRICTED LIE ALGEBRAS**

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Abstract

In [21], generalized restricted Lie algebras, defined over a field F of positive characteristic p, were introduced. In this note their cohomology, especially the so-called generalized restricted cohomology is studies. Some reduction properties are obtained. For graded Cartan type Lie algebras the author determines the first Lie-cohomology groups and the first generalized restricted cohomology groups with the coefficients in the highest weight modules from which all irreducible generalized restricted modules are derived.

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§0. Introduction

This paper is aimed at developing the cohomology theory of modular Lie algebras and then determining the first cohomology groups of Cartan type Lie algebras.

As generalization of the concept of restricted Lie algebras, a generalized restricted Lie algebra (GR Lie algebra) was introduced in [21], which is associated with a basis and a mapping of the basis into the Lie algebra satisfying the generalized-restrictedness conditions. Generalized restricted representations (GR representations) were then introduced, which can be reduced to the representations of a finite-dimensional associative algebra whenever the Lie algebra is finite-dimensional. Furthermore any irreducible GR representation of a GR Lie algebra over an algebraically closed field coincides with a representation of the generalized restricted enveloping algebra or of the generalized reduced enveloping algebra which is finite-dimensional.

Any graded Cartan type Lie algebra is a GR Lie algebra associated with its standard basis (see [21, 2.13]). Hence we can employ the idea of generalized restrictedness to study its irreducible modules (in the situation of the graded modules, their determination was solved mainly by G. Shen, and N. Hu).

The cohomology theory of modular Lie algebras has received considerable attention during the past decade. Especially, for the cohomology of restricted Lie algebras, more precise information has been obtained (see [6–9, 12]). In this paper we extend some standard results

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of the restricted Lie algebra case to the situation of GR Lie algebras, and then apply them to determine two classes of the first cohomology groups of graded Cartan type Lie algebras. These results here are more extensive than the corresponding ones in [4, 5]. Our discussion appeals to the methods, developed in [8] and [9], of exploiting the theory of Frobenius extentions (in the sense of [15]). In addition, any GR Lie algebra L uniquely corresponds with a restricted Lie algebra $P(L)$, called the primitive p-envelope, such that the GR L-module category coincides with the restricted $P(L)$-module category (to see Proposition 2.5). Thus we establish the relationship between various cohomology groups and can transfer the results from one to another.

In the first two sections, we will introduce some basic results about GR Lie algebras and the Frobenius extension for GR Lie algebras. In the third section, we will generally discuss the ordinary and generalized restricted cohomology of GR Lie algebras and obtain the cohomology reduction theorems. Finally, as an application we will determine the first ordinary and generalized restricted cohomology groups respectively for graded Cartan type Lie algebras with the coefficients in the Verma modules $Z_{p^s}(\lambda)$ in the concluding section.

§1. Generalized Restricted Lie Algebras

Throughout this paper F always denotes a field of prime characteristic p.

1.1. Let L be a Lie algebra over F, $E=(e_i)_{i\in I}$ an ordered basis of L and $\mathbf{s}=(s_i)_{i\in I}$, $s_i\in\mathbb{N}$. If there is a mapping $\varphi_{\mathbf{s}}: E\to L$, $e_i\mapsto e_i^{\varphi_{\mathbf{s}}}$ such that

$$\mathrm{ad}e_i^{\varphi_{\mathbf{s}}}=(\mathrm{ad}e_i)^{p^{s_i}},\quad \forall i\in I,$$

then L is called a generalized restricted Lie algebra (GR Lie algebra) associated with E and $\varphi_{\mathbf{s}}$.

If L' is a subalgebra of L with a basis $E'\subset E$ and $e_i^{\varphi_{\mathbf{s}}}\in L'$, $\forall e_i\in E'$, then L' is said to be a GR subalgebra of L.

Let L be a GR Lie algebra associated with $\{E,\varphi_{\mathbf{s}}\}$ and $\rho: L\to\mathfrak{gl}(V)$ a representation of L. Call ρ a generalized restricted representation (GR representation) associated with $\{E,\varphi_{\mathbf{s}}\}$ if ρ satisfies the additional condition

$$\rho(e_i^{\varphi_{\mathbf{s}}})=(\rho(e_i))^{p^{s_i}},\quad \forall i\in I.$$

In this case, V is called a GR L-module.

1.2. For a GR Lie algebra L with respect to $\{E,\varphi_{\mathbf{s}}\}$, let $U(L)$ be the universal enveloping algebra of L. Put $\mathfrak{A}_{p^s}(L,E)=U(L)/I(E)$, where $I(E)$ is the ideal of $U(L)$ generated by $e_i^{p^{s_i}}-e_i^{\varphi_{\mathbf{s}}}$, $i\in I$. Owing to Jacobson's refinement of the PBW theorem (see [17, p.58]), the monomials

$$e_{i_1}^{\alpha_{i_1}}\cdots e_{i_k}^{\alpha_{i_k}},\quad i_1<i_2<\cdots<i_k,\quad 0\leqslant\alpha_{i_k}<p^{s_{i_k}},$$

constitute a basis of $\mathfrak{A}_{p^s}(L,E)$. When L is of l dimension, $\mathfrak{A}_{p^s}(L,E)$ has a basis of a finite number of elements

$$e^{\alpha}=e^{\alpha_1}\cdots e^{\alpha_l},\ \ 0\leqslant\alpha\leqslant\tau:=\sum_{i=1}^{l}(p^{s_i}-1)\epsilon_i,$$

where $\epsilon_i=(\delta_{i1},\cdots,\delta_{il})$ and a partial ordering on $\mathbb{N}^l$ is defined via

$$\alpha\leqslant\beta:\Longleftrightarrow\alpha_i\leqslant\beta_i,\quad 1\leqslant i\leqslant l.$$

Put $|\alpha| := \sum_{1\leqslant i\leqslant l} \alpha_i$. Obviously, GR L-modules are precisely the unitary $\mathfrak{A}_{p^s}(L,E)$-modules.

1.3. If F is algebraically closed and L is a finite-dimensional GR Lie algebra over F, then any irreducible representation $\rho : L \to \mathfrak{gl}(V)$ is precisely an irreducible unitary representation of $\mathfrak{A}_{p^s}(L,E^\kappa)$ for some mapping κ of E to F (called a character of E) satisfying

$$\rho(e_i)^{p^{s_i}} - \rho(e_i^{\varphi_{\mathbf{s}}}) = \kappa(e_i), \quad \forall e_i \in E, \tag{1.3.1}$$

where $\mathfrak{A}_{p^s}(L,E^\kappa) := U(L)/I(E^\kappa)$, $I(E^\kappa)$ is the ideal of $U(L)$ generated by $\{e_i^{p^{s_i}} - e_i^{\varphi_{\mathbf{s}}} - \kappa(e_i)\cdot 1 \mid e_i \in E\}$.

An L-module V satisfying (1.3.1) is called a κ-reduced module, and $\mathfrak{A}_{p^s}(L,E^\kappa)$ is called a κ-reduced enveloping algebra. When κ is a zero mapping, ρ is exactly GR representation. In the following, $\mathfrak{A}_{p^s}(L,E)$ and $\mathfrak{A}_{p^s}(L,E^\kappa)$ will be often simply written as $\mathfrak{A}$ and $\mathfrak{A}^\kappa$ respectively if the context is clear.

1.4. Restricted Lie algebras are special examples of GR Lie algebras for which E may be an arbitrary basis and $\varphi_{\mathbf{s}}$ is taken to be the s_i-th powers (with a fixed order) of the basis elements of the p-mapping.

Conversely, by a Jacobson's result (see [17, (2.3), p.71]) a GR Lie algebra associated with E and $\varphi_{\mathbf{1}}$, $\mathbf{1} = (1,1,\cdots,1)$, is definitely restricted under the unique restricted mapping $[p] : L \to L$ with $[p]\big|_E = \varphi_{\mathbf{1}}$.

1.5. In the rest of the section, let L be one of graded Lie algebras of Cartan type $X(m:\mathbf{n})^{(2)}$, $X \in \{W,S,H,K\}$ (see [23]). Then the zero-grade part $L_{[0]}$ is isomorphic to $\mathfrak{gl}(m), \mathfrak{sl}(m), \mathfrak{sp}(m-1)$ or $\mathfrak{sp}(m-1)\oplus F\mathcal{D}_K(x^{\epsilon_m})$ for $X = W,S,H$ or K respectively (see [17, Chapter 4]). Let $\mathfrak{h}$ be the standard Cartan subalgebra of $\mathfrak{g} := L_{[0]}$, $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) the sum of positive (resp. negative) root spaces of $\mathfrak{g}$,

$$N^+ := \mathfrak{n}\bigoplus\sum_{i>0} L_{[i]}, \quad B := \mathfrak{h}\bigoplus N^+, \quad N^- := \mathfrak{n}^-\bigoplus\sum_{i<0} L_{[i]}, \quad B^- := \mathfrak{h}\oplus N^-.$$

A standard basis $E = E^-\cup E^0\cup E^+$ was naturally given in [21] where E^0 (resp. E^-, E^+) is a basis of $\mathfrak{h}$ (resp. N^-, N^+). The notations concerning graded Lie algebras of Cartan type will follow [23] or [17]. Note that L_0 is restricted under the p-mapping: $D \mapsto D^{[p]}$ (see [2]). We have the following result.

Proposition 1.5.1. *Let $\mathbf{s} = (s_1,\cdots,s_l)$ be an l-tuple of positive integers with $s_i \geqslant n_i$, $i = 1,2,\cdots,m$. Then L is a GR Lie algebra associated with a standard basis E and $\varphi_{\mathbf{s}}$ defined by*

$$e_i^{\varphi_{\mathbf{s}}} = \begin{cases} 0, & e_i \in E^-\cap L^-, \\ e_i, & e_i \in E^0, \\ e_i^{[p]^{s_i}}, & e_i \in E^+ \text{ or } E^-\cap L_{[0]}, \end{cases}$$

where $D^{[p]}$, for $D \in L_0 = \sum_{i\geqslant 0} L_{[i]}$, denotes the usual p-th power of the derivation D.

Remark 1.5.1. The minimal $\mathbf{s}$ such that $(L,E,\varphi_{\mathbf{s}})$ becomes a GR Lie algebra admits

$$s_i = \begin{cases} n_i, & 1 \leqslant i \leqslant m, \\ 1, & \text{otherwise.} \end{cases}$$

1.6. As $N^-, \mathfrak{h}, N^+$ and B are GR Lie algebras associated with $E^-, E^0, E^+, E^+\cup E^0$ respectively, by Jacobson's refinement of the PBW theorem

$$\mathfrak{A} := \mathfrak{A}_{p^s}(L,E) = \mathfrak{A}^-\mathfrak{A}^0\mathfrak{A}^+, \quad \mathfrak{B} := \mathfrak{A}_{p^s}(B, E^0\cup E^+) = \mathfrak{A}^0\mathfrak{A}^+,$$

where $\mathfrak{A}^0 := \mathfrak{A}_{p^s}(\mathfrak{h}, E^0)$ and $\mathfrak{A}^{\pm} := \mathfrak{A}_{p^s}(N^{\pm}, E^{\pm})$.

Let F be a perfect field. Denote by $\mathcal{X}_{p^s}(E^0)$ the set of all mappings of E^0 to F which send $e_i \in E^0$ into the finite field $F_{p^{s_i}}$. Then any $\lambda \in \mathcal{X}_{p^s}(E^0)$ can be uniquely extended to an algebra homomorphism $\mathfrak{A}^0 \to F$ (still denoted by λ). These λ constitute Alg-Hom$(\mathfrak{A}^0, F)$ and determine all irreducible $\mathfrak{A}^0$-modules which are 1-dimensional. Thus any $\lambda \in \mathcal{X}_{p^s}(E^0)$ can uniquely determine a 1-diminsional $\mathfrak{B}$-module F_λ on which the action of the augmentation ideal of $\mathfrak{A}^+$ is trivial. Set $Z_{p^s}(\lambda) = \mathfrak{A} \otimes_{\mathfrak{B}} F_\lambda$, which has a unique maximal proper $\mathfrak{A}$-submodule and a unique irreducible quotient $M_{p^s}(\lambda)$.

Theorem 1.6.1.[21] *The assignment $\lambda \to M_{p^s}(\lambda)$ is a one-to-one correspondence between the elements of $\mathcal{X}_{p^s}(E^0)$ and the isomorphism classes of the irreducible $\mathfrak{A}$-modules.*

§2. Frobenius Extensions for GR Lie Algebras and Primitive *p*-Envelops

In this section, L is always assumed to be a finite-dimensional GR Lie algebra associated with $\{E, \varphi_{\mathbf{s}}\}$. For a GR subalgebra (L', E') of (L, E), by the argument of [7] we will know that $\mathfrak{A}_{p^s}(L, E) : \mathfrak{A}_{p^s}(L', E')$ is a Frobenius extension. At first, we give the definition of Frobenius extensions.

2.1. Definition 2.1.1. *Let A be an associative F-algebra and C a subalgebra of A. The extension $A : C$ is called a θ-Frobenius extension if the following two statements hold.*

(i) *A is a finitely generated projective (left) C-module.*

(ii) *There exists an (A, C)-bimodule isomorphism of A onto* $\mathrm{Hom}_C(A, {}_\theta C)$ *where for a (left) C-module V, ${}_\theta V$ denotes the new (left) C-module with induced action $c \cdot v = \theta(c)v$.*

Thus there is a natural isomorphism of (left) A-modules (see [15, (18), p.91]).

$$\theta_V : A \otimes_C V \cong \mathrm{Hom}_C(A, {}_\theta V) \tag{2.1.1}$$

for any (left) C-module V.

2.2. In the Hopf algebra $U(L)$, $I(E)$ is obviously a coideal. In addition, the supplementation mapping ϵ satisfies the condition $\epsilon(I(E)) = 0$, and the canonical antipode mapping $\mathcal{S}$ of $U(L)$ preserves $I(E)$ since

$$\mathcal{S}(e_i^{\varphi_{\mathbf{s}}} - e_i^{e_i^{s_i}}) = -e_i^{\varphi_{\mathbf{s}}} - (-1)^{p^{s_i}} e_i^{p^{s_i}} = -(e_i^{\varphi_{\mathbf{s}}} - p^{p^{s_i}}).$$

So the ideal $I(E)$ is a Hopf ideal of $U(L)$. According to [1, Theorem 4.21, p.174], $\mathfrak{A}_{p^s}(L, E) = U(L)/I(E)$ has a unique Hopf algebra structure such that the canonical projection $\pi : U \to \mathfrak{A}_{p^s}(L, E)$ is a Hopf algebra morphism.

For the basis $\{e^\alpha = e_1^{\alpha_1} e_2^{\alpha_2} \cdots e_l^{\alpha_l}, 0 \leqslant \alpha \leqslant \tau\}$ of $\mathfrak{A} := \mathfrak{A}_{p^s}(L, E)$, let χ be the projection $\mathfrak{A} \to F$ along with the line Fe^τ. Via the Hopf algebra homomorphism π, it is not hard to see that the comultiplication of $\mathfrak{A}$ satisfies the identies

$$\Delta(e^\alpha) = \sum_{0 \leqslant \alpha' \leqslant \alpha} \binom{\alpha}{\alpha'} e^\alpha \otimes e^{\alpha - \alpha'}, \ 0 \leqslant \alpha \leqslant \tau. \tag{2.2.1}$$

Thereby for any element f of the dual algebra $\mathfrak{A}^{\#}$, we have

$$\begin{aligned}(\chi\# f)(e^{\alpha}) &= \sum_{0\leqslant\alpha'\leqslant\alpha}\binom{\alpha}{\alpha'}\chi(e^{\alpha'})f(e^{\alpha-\alpha'})\\ &= \sum_{0\leqslant\alpha'\leqslant\alpha}\binom{\alpha}{\alpha'}\delta_{\alpha',\tau}f(e^{\alpha-\alpha'})\\ &= \delta_{\alpha,\tau}f(1) = \chi(e^{\alpha})f(1),\end{aligned}$$

where δ is the Kronecker symbol. Consequently, χ is a nontrivial right integral of $\mathfrak{A}^{\#}$ (see [1, §3.3, p.144]). According to the theorem in [13, p.79], the finite-dimensional Hopf algebra $\mathfrak{A}$ is a Frobenius algebra, i.e. with the nondegenerate associative bilinear form given by $(x, y) = \chi(xy)$.

2.3. Definition 2.3.1. *Let A be a Frobenius algebra over F with nondegenerate associative bilinear form $(\quad,\quad) : A \times A \to F$. An automorphism μ of A is called a Nakayama automorphism if*

$$(x, y) = (y, \mu(x)), \quad \forall x, y \in A.$$

Any derivation of L can be uniquely extended to a derivation of $U(L)$. By

$$\mathrm{ad}x(e_i^{\varphi_{\mathbf{s}}} - e_i^{p^{s_i}}) = 0, \quad \forall x \in L,$$

we have $\mathrm{ad}x(I(E)) \subset I(E)$. Thus from [17, (4.1), p.215], it follows that in $\mathfrak{A}$,

$$[x, e^{\alpha}] \equiv \Big(\sum_{1\leqslant i\leqslant l}\alpha_i a_{ii}(x)\Big)e^{\alpha} + \sum_{\substack{|\alpha'|=|\alpha|\\ \alpha'\neq\alpha}} b(\alpha')e^{\alpha'}\mathrm{mod}(\mathfrak{A}_{(|\alpha|-1)}),$$

where $[x, e_i] = \sum\limits_{i=1}^{l} a_{ij}(x)e_i \in L$, and

$$\mathfrak{A}_{(|\alpha|-1)} = \{e^{\beta} \mid |\beta| \leqslant |\alpha| - 1\}.$$

It is readily seen that

$$\chi([x, e^{\alpha}]) = -\mathrm{tr}(\mathrm{ad}_L x)\chi(e^{\alpha}), \quad \forall x \in L.$$

Thus

$$\chi(xe^{\alpha}) = \chi([x, e^{\alpha}]) + \chi(e^{\alpha}x) = \chi(e^{\alpha}(x - \mathrm{tr}(\mathrm{ad}_L x)\cdot 1)).$$

Let $\nu'(x) := x - \mathrm{tr}(\mathrm{ad}_L x)\cdot 1$. Then ν' uniquely induces an automorphism ν of $\mathfrak{A}$. From the above equalities we have

Proposition 2.3.1. *For a finite-dimensional GR Lie algebra L associated with $E, \varphi_{\mathbf{s}}$, $\mathfrak{A}_{p^s}(L, E)$ is a Frobenius algebra with Nakayama automorphism satisfying*

$$\nu(x) = x - \mathrm{tr}(\mathrm{ad}_L x)\cdot 1, \quad \forall x \in L.$$

2.4. Let L' be a GR subalgebra of L with a basis $E' \subset E$. Then the subalgebra $\mathfrak{A}' := \mathfrak{A}_{p^s}(L', E')$ of $\mathfrak{A}$ is generated by primitive elements as $P(\mathfrak{A}') \supset \pi(P(U(L')))$, where $P(A)$ denotes the set of primitive elements for a Hopf algebra A. According to Proposition 2.3.1 and [7, (3.3), p.139], $\mathfrak{A} : \mathfrak{A}'$ is a free θ-Frobenius extension, where

$$\theta(x) = \nu(x) + \mathrm{tr}(\mathrm{ad}_{P(\mathfrak{A}')}x)\cdot 1, \quad \forall x \in P(\mathfrak{A}').$$

Combining with (2.1.1), we have

Proposition 2.4.1. Let L' be a GR subalgebra of L and V a GR L'-module. Then there is a GR L-module isomorphism

$$\mathfrak{A} \otimes_{\mathfrak{A}'} V \cong \mathrm{Hom}_{\mathfrak{A}'}(\mathfrak{A}, {}_\theta V).$$

2.5. For a GR Lie algebra $(L, E, \varphi_{\mathbf{s}})$, set $P(L) := P(\mathfrak{A}_{p^s}(L, E))$. It is not difficult to verify that

$$P(L) = \sum_{\substack{0 \leqslant t_i < s_i \\ e_i \in E}} F e_i^{p^{t_i}}$$

in view of the equalities (2.2.1). Note that in $\mathfrak{A}_{p^s}(L, E)$, L (more precisely, the image of the imbedding of L into $U(L)/I(E)$) is an ideal of $P(L)$. Obviously, $P(L)$ is a p-envelope of L, which will be called *the primitive p-envelope of L*, and is of dimension $\sum_{i=1}^{l} s_i$ if $\dim L = l$.

Recall that the primitive p-envelope of a restricted Lie algebra is the Lie algebra itself. More generally, for any GR Lie algebra, we have the following

Proposition 2.5.1. *Let L be a GR Lie algebra associated with $\{E, \varphi_{\mathbf{s}}\}$. Then*

(i) *$P(L)$ is the unique restricted Lie algebras (up to isomorphism) such that its universal restricted enveloping algebra is isomorphic to $\mathfrak{A}_{p^s}(L, E)$.*

(ii) *The GR L-module category coincides with the restricted $P(L)$-module category.*

Proof. As $P(\mathfrak{A}_{p^s}(L, E)) \supset \pi(P(U(L)))$, $\mathfrak{A}_{p^s}(L, E)$ is a primitively generated Hopf algebra. According to [14, (6.11), p.246], $\mathfrak{A}_{p^s}(L, E) \cong V(P(L))$, here and further $V(\mathfrak{L})$ denotes the universal restricted envelope of a restricted Lie algebra $\mathfrak{L}$. The uniqueness also follows from [14, (6.11)]. The second assertion follows directly from the first.

Thus we establish the relationship between GR Lie algebras and restricted Lie algebras. And we especially obtain another description of θ via the restricted Lie algebra $P(L)$ (see [8, (1.5), p.155] or [9, (1.1), p.2868]), i.e. the Frobenius twist θ associated with the free Frobenius extension $V(P(L))(=\mathfrak{A}) : V(P(L'))(=\mathfrak{A}')$ is induced by

$$\theta(x) = x - \mathrm{tr}(\mathrm{ad}_{P(L)/P(L')})1_{\mathfrak{A}'}, \quad \forall x \in P(L').$$

§3. Cohomology of GR Lie Algebras

3.1. Definition 3.1. *Let L be a GR Lie algebra associated with $\{E, \varphi_{\mathbf{s}}\}$, V a GR L-module. Define the generalized restricted cohomology (GR cohomology) group of L with coefficients in V by*

$$H^n_{p^s}(L, V) := \mathrm{Ext}^n_{\mathfrak{A}_{p^s}(L,E)}(F, V), \ \forall n \in \mathbb{N}.$$

Owing to Proposition 2.5.1, we have the following fundamental lemma.

Lemma 3.1.1. *Let L be a GR Lie algebra associated with $\{E, \varphi_{\mathbf{s}}\}$. Then*

$$H^*_{p^s}(L, V) = H^*_p(P(L), V),$$

where $H^*_p(\mathfrak{L}, -)$ denotes the restricted cohomology functor of a restricted Lie algebra $\mathfrak{L}$, introduced first by Hochschild in [10].

3.2. According to [3, §XIII.8, p.282], the ordinary Lie cohomology groups of L with coefficients in an L-module V coincide with $\mathrm{Ext}^*_{U(L)}(F, V)$, the corresponding cohomology groups of the supplemented algebra $(U(L), \epsilon)$. ϵ induces a supplemention ϵ' of $\mathfrak{A}$ to F whose kernel is the image $\mathfrak{A}^+$ of $U(L)^+ := \ker\epsilon$.

The canonical projection π allows us to regard any $\mathfrak{A}$-module as a $U(L)$-module. As argumented in [10, p.561], there is a canonical imbedding homomorphism of $H^*_{p^s}(L,V)$ into $H^*(L,V)$ for a GR L-module V. Furthermore, in the following there is a description of the first GR cohomology groups, whose proof is merely a modification of that of [10, (2.1), p.563].

Proposition 3.2.1. *Let V be any GR L-module. Then the canonical homomorphism of $H^*_{p^s}(L,V)$ into $H^*(L,V)$ maps $H^1_{p^s}(L,V)$ isomorphically onto that subspace of $H^1(L,V)$ whose elements are represented by the Lie 1-cocycle f satisfying the relation*

$$e_i^{p^{s_i}-1}\cdot f(e_i)=f(e_i^{\varphi_{\mathbf{s}}}),\quad \forall e_i\in E. \tag{3.2.1}$$

Remark 3.2.1. With the aid of $P(L)$, we give a more natural explanation of the above proposition.

(i) As L is an ideal of $P(L)$, we have the canonical restriction transformation Res : $H^*(P(L),V)\to H^*(L,V)$. By Proposition 2.5.1, there is obviously a commutative diagram:

$$\begin{array}{ccc} H^*_p(P(L),V) & \longrightarrow & H^*(P(L),V) \\ \Big\| & & \Big\downarrow \text{Res} \\ H^*_{p^s}(L,V) & \longrightarrow & H^*(L,V), \end{array} \tag{3.2.2}$$

where $H^*_p(P(L),V)\to H^*(P(L),V)$ is the canonical imbedding homomorphism defined in [10, p.561].

(ii) By Lemma 3.1.1, $H^1_{p^s}(L,V)=H^1_p(P(L),V)$. The latter, under the canonical imbedding homomorphism of $H^*_p(P(L),V)$ into $H^*(P(L),V)$, is isomorphic to the subspace of $H^1(P(L),V)$ whose elements are represented by the Lie 1-cocycle g satisfying

$$x^{p-1}\cdot g(x)=g(x^{\{p\}}),\quad \forall x\in P(L)\ \text{ (see [10, (2.1), p.563])}. \tag{3.2.3}$$

Here we denote by $x^{\{p\}}$ the p-th power of x in the associative algebra $\mathfrak{A}_{p^s}(L,E)$ only on the purpose of stressing its p-mapping property, but in other places, e.g. §2.5, we do not distinguish between $x^{\{p\}}$ and x^p whenever the context is clear. Note that g is linear, and satisfies the relation

$$g([x,y])=xg(y)-yg(x),\quad \forall x,y\in P(L).$$

The condition (3.2.3) is equivalent to the following

$$e_i^{p^{t_i-1}(p-1)}\cdot g(e_i^{p^{t_i-1}})=g(e_i^{p^{t_i}}),\quad \forall e_i\in E,\ t_i\in\{1,\cdots,s_i\}. \tag{3.2.3$'$}$$

(iii) We will see that under the transformation Res, the subspace of $H^1(P(L),V)$ with elements represented by the Lie 1-cocycle g satisfying the relation (3.2.3) is exactly isomorphic to the subspace of $H^1(L,V)$ with elements represented by the Lie 1-cocycle f satisfying (1). In fact, fixed $e_i\in E$, by induction we have

$$g(e_i^{p^{s_i}})=e_i^{(p-1)(p^{s_i-1}+\cdots+1)}\cdot g(e_i)=e_i^{p^{s_i}-1}\cdot g(e_i),$$

i.e. $g(e_i^{\varphi_{\mathbf{s}}})=e_i^{p^{s_i}-1}\cdot g(e_i)$. Thereby

$$g\big|_L(e_i^{\varphi_{\mathbf{s}}})=e_i^{p^{s_i}-1}\cdot g\big|_L(e_i),\quad \forall e_i\in E.$$

Conversely, suppose the Lie 1-cocycle f satisfies the condition (3.2.1). Then it may be uniquely extended to a Lie 1-cocycle g whose cohomology class belongs to $H^1(P(L),V)$, such that

$$g(e_i^{p^{t_i}}) = e_i^{p^{t_i}-1} \cdot f(e_i), \quad \forall e_i \in E,\ t_i = 1, 2, \cdots, s_i.$$

Obviously, g satisfies (3.2.3′) as well as (3.2.3). Note that under Res, the cohomology class of g is mapped into that of $g\,\big|_L$. Thereby, Proposition 3.2.1 precisely shows the commutativity of the first cohomological diagram in (3.2.2).

3.3. Denote by $\mathrm{Der}(L, V)$ the set of derivations from L to V. Let

$$\mathrm{Der}_{p^s}(L, V) := \{d \in \mathrm{Der}(L, V) \mid d(e_i^{\varphi_s}) = e_i^{p^{s_i}-1} d(e_i), e_i \in E\}$$

and

$$\mathrm{Ider}(L, V) := \{d \in \mathrm{Der}(L, V) \mid \exists v \in V, \text{ s.t. } d(x) = x \cdot v, \quad \forall x \in L\}.$$

The latter is a subspace of the former as V is a GR module.

The above proposition shows that $H^1_{p^s}(L, V) = \mathrm{Der}_{p^s}(L, V)/\mathrm{Ider}(L, V)$. Especially if V is a trivial L-module, $H^1_{p^s}(L, V)$ may be described as follows.

Corollary 3.3.1. *For every trivial GR L-module V, there is a natural isomorphism:*

$$H^1_{p^s}(L, V) \cong \mathrm{Hom}_F(L/(L^{(1)} + \langle E^{\varphi_s}\rangle), V),$$

where $\langle E^{\varphi_s}\rangle$ is the F-span of all $e_i^{\varphi_s}$, $\forall e_i \in E$.

Proof. The condition that V is a trivial module implies that $\mathrm{Ider}(L, V) = 0$. Thus $H^1_{p^s}(L, V) \cong \mathrm{Der}_{p^s}(L, V)$. Furthermore, $d(L^{(1)} + \langle E^{\varphi_s}\rangle) = 0$ for any $d \in \mathrm{Der}_{p^s}(L, V)$; thereby there is a linear mapping χ_V from $\mathrm{Der}_{p^s}(L, V)$ to $\mathrm{Hom}_F(L/(L^{(1)} + \langle E^{\varphi_s}\rangle), V)$ which maps d into $\chi_V(d)$ such that

$$\chi_V(d)\bar{x} := d(x), \quad \forall x \in L.$$

Conversely, for any $f \in \mathrm{Hom}_F(L/(L^{(1)} + \langle E^{\varphi_s}\rangle), V)$, let $\chi'_V(f)x = f(\bar{x})$ for any $x \in L$. The triviality of V implies that $\chi'_V(f)$ belongs to $\mathrm{Der}_{p^s}(L, V)$. The fact that χ'_V is the inverse of χ_V shows that χ_V is bijective. The naturality is easily checked.

The above corollory is an extension of [9, (2.7)], which is useful in §4.

3.4. In the sequel, L is assumed to be finite-dimensional. Let (L', E') be a GR subalgebra of (L, E), V a GR L'-module, $\mathfrak{A}' := \mathfrak{A}_{p^s}(L', E')$, and θ as in §2. Note that $\mathfrak{A}$ is a free (left) $\mathfrak{A}'$-module. By [11, (12.3), p.164] and Proposition 2.4.1, we have the following important reduction theorem similar to Shapiro's Lemma for finite groups, whose variation in the restricted Lie algebra case was obtained in [8].

Theorem 3.4.1. $H^n_{p^s}(L, \mathfrak{A} \otimes_{\mathfrak{A}'} V) \cong H^n_{p^s}(L', {}_\theta V)$, $n \geqslant 0$.

3.5. We turn to a reduction result on the ordinary Lie cohomology with coefficients in a GR module. Let $(L', E') \subset (L, E)$ be a GR subalgebra with finite cobasis $\{e_{i_1}, \cdots, e_{i_r}\}$, the notations $\theta, \mathfrak{A}$ and $\mathfrak{A}'$ are as in §2. Therefore, $L = L' + \langle e_{i_1}, \cdots, e_{i_r}\rangle$, implied by the definition of GR subalgebras. Let

$$z_j := e_{i_j}^{p^{s_{i_j}}} - e_{i_j}^{\varphi_s} \subset C(U(L)), \quad j = 1, 2, \cdots, r,$$

and

$$\mathcal{O}(L, L') := \mathrm{alg}_F(L' \cup \{z_1, \cdots z_r\}) \subset U(L).$$

Then

$$\mathcal{O}(L, L') \cong F[z_1, \cdots, z_r] \otimes_F U(L').$$

Let V be a GR L'-module. The action of $\mathfrak{A}(L')$ on V can be extended to $\mathcal{O}(L,L')$ by letting the polymonial algebras $F[z_1,\cdots,z_r]$ operate via its canonical supplementation. Furthermore, there exists a natural $U(L)$-module isomorphism

$$U(L)\otimes_{\mathcal{O}(L,L')}V\cong\mathfrak{A}\otimes_{\mathfrak{A}'}V \tag{3.5.1}$$

defined by $u\otimes v\mapsto\pi(u)\otimes v$. The verification of the above is the same as in [8, p.158]. According to Jacobson's refinement of the PBW's theorem, $U(L)$ is a free $\mathcal{O}(L,L')$-module with basis $\{e^\alpha:0\leqslant\alpha\leqslant\tau\}$. By (3.5.1) and [8, 2.1], we have

Theorem 3.5.1. $H^n(L,\mathfrak{A}\otimes_{\mathfrak{A}'}V)\cong\bigoplus\limits_{i+j=n}\Lambda^i(L/L')\otimes_F H^j(L',{}_\theta V)$.

§4. Application: The Determination of the First Cohomology Groups for Graded Cartan Type Lie Algebra

The cohomology of graded Lie algebras of Cartan type was first studied by Dzhumadil'daw. In [5], he gave the structure of $H^1(W(1,\mathbf{n}),U_t)$. Afterwards, based on the determination of irreducible graded module of graded Cartan type Lie algebras (see [18–20]), Chiu, S. and Shen, G. then determined in [4] the structures of the first cohomology groups for $X=W,S,H$ and small m with coefficients in a (universal) graded L-modules $\widetilde{V_0}$ associated to an irreducible $L_{[0]}$-module.

In this section, we shall apply the reduction theorems obtained in the above section to determine the first cohomology groups $H^1_{p^s}(L,Z_{p^s}(\lambda))$ and $H^1_{p^s}(L,Z_{p^s}(\lambda))$ for

$$L=X(m:\mathbf{n})^{(2)},\quad X\in\{W,S,H,K\}.$$

Throughout this section, F is assumed to be algebraically closed.

4.1. Definition 4.1.1. *A finite-dimensional GR Lie algebra L associated with $(E,\varphi_\mathbf{s})$ is said to be toral if L is abelian and $e_i^{\varphi_\mathbf{s}}=e_i$ for any $e_i\in E$.*

Special examples of toral GR Lie algebras are those abelian restricted Lie algebras with non-singular p-mappings over an algebraically closed field, in the situation of which L posseses a basis consisting of toral elements, i.e. with $x^{[p]}=x$ (see [17, (3.6), p.82]).

Let L be toral and $\rho:L\to\mathfrak{gl}(V)$ a finite-dimensional GR representation of L. Then the GR enveloping algebra $\mathfrak{A}:=\mathfrak{A}_{p^s}(L,E)$ is commutative. Moreover $\rho(\mathfrak{A})$ can be diagnolized simultaneously because $\rho(e_i)^{p^{s_i}}=\rho(e_i)$. Hence, $\mathfrak{A}$ is semisimple. Consequently F, as an $\mathfrak{A}$-module via the supplementation ϵ, is projective. We have $H^n_{p^s}(L,V)=0$ for $n>0$ and any $\mathfrak{A}$-module V.

4.2. Call $L'\subset L$ a GR ideal if L' is a GR subalgebra as well as an ideal. Let L' be a GR ideal of L and V a GR L'-module. Consider the natural action of L/L' on $H^*_{p^s}(L',V)$.

Proposition 4.2.1. *Let L' be a GR ideal of a finite-dimensional GR Lie algebra L and suppose L/L' is toral. Then we have a natural isomorphism*

$$H^1_{p^s}(L,V)\cong H^1_{p^s}(L',V)^{L/L'}$$

for a GR L'-module V.

Before giving the proof, we need a lemma. By the definition of GR ideals, L/L' is also a GR Lie algebra associated with $(E^{''},\pi'\varphi\mid_{E''})$, where $E^{''}=E\backslash E'$, π' is the canonical projection. Thus a GR L-module V admits a GR L/L'-module structure. Let

$$V^{L'} := \{v \in V \mid L'v = 0\},$$

which is an L submodule since L' is an ideal.

Lemma 4.2.1. *Let (L', E') be a GR ideal of a GR Lie algebra (L, E). If V is an injective $\mathfrak{A}$-module, then $V^{L'}$ is an injective $\mathfrak{A}'' := \mathfrak{A}_{p^s}(L/L', E\backslash E')$-module.*

Proof. Let $i : W' \to W$ be an injective $\mathfrak{A}''$-module homomorphism. Suppose $f : W' \to V^{L'}$ is an $\mathfrak{A}''$-module homomorphism. For our purpose, it is sufficient to find an $\mathfrak{A}''$-module homomorphism $\tilde{f}$ of W to $V^{L'}$ such that the following diagram is commutative:

$$\begin{array}{ccc} W' & \overset{i}{\rightarrowtail} & W \\ f\big\downarrow & \swarrow\tilde{f} & \big\downarrow \\ V^{L'} & \rightarrowtail & V \end{array}.$$

Notice that $\mathfrak{A}'' \cong \mathfrak{A}/\mathfrak{A}L'$ and that an $\mathfrak{A}''$-module (resp. an $\mathfrak{A}''$-module homomorphism) may be regarded as an $\mathfrak{A}$-module (resp. an $\mathfrak{A}$-module homomorphism). The condition that V is an injective $\mathfrak{A}$-module implies that there is an $\mathfrak{A}$-module homomorphism $\tilde{f} : W \to V$ extending f. But

$$x'\tilde{f}(W) = \tilde{f}(x'W) = 0 \quad \text{for } x' \in L',$$

so $\mathrm{im}\tilde{f} \subset V^{L'}$. Hence $\tilde{f}$ is an $\mathfrak{A}'$-module homomorphism of W to $V^{L'}$. The proof is completed.

Proof of Proposition 4.2.1. Let $\mathfrak{L}$ be the category of GR L-modules and $\mathfrak{L}''$ the category of GR L/L'-modules. As $\mathrm{Hom}_{\mathfrak{A}'}(F, V) = V^{L'}$ for any $\mathfrak{A}$-module V,

$$\mathfrak{G} := \mathrm{Hom}_{\mathfrak{A}}(F, -)$$

is a functor of $\mathfrak{L}$ to $\mathfrak{L}''$. Let

$$\mathfrak{F} := \mathrm{Hom}_{\mathfrak{A}''}(F, -),$$

a functor of $\mathfrak{L}''$ to the abelian group category **Ab**. $\mathfrak{G}$ and $\mathfrak{F}$ are left exact, being Hom functors. By the above lemma, $\mathfrak{G}(V) = V^{L'}$ is right $\mathfrak{F}$-acyclic. According to cohomology five-term sequence (see [16, (11.2), p.304]), we have an exact sequence

$$\begin{aligned} 0 \to H^1_{p^s}(L/L', V^{L'}) &\to H^1_{p^s}(L, V) \to H^1_{p^s}(L', V)^{L/L'} \\ &\to H^2_{p^s}(L/L', V^{L'}) \to H^2(L, V). \end{aligned}$$

As L/L' is toral, the earlier argument in Section 4.1 shows that

$$H^1_{p^s}(L/L', V^{L'}) = H^2_{p^s}(L/L', V^{L'}) = 0.$$

We obtain the desired conclusion.

Remark 4.2.1. If $(L, E, \varphi_{\mathbf{s}})$ is a finite-dimensional GR Lie algebra with $\mathbf{s} = (s, \cdots, s)$, i.e.

$$s_1 = s_2 = \cdots = s_l = s,$$

then there is a stronger result

$$H^n_{p^s}(L, V) \cong H^n_{p^s}(L', V)^{L/L'}, \quad \forall n \in \mathbb{N}^+,$$

where L' is as in Proposition 4.2.1.

Proof. As L' is a GR ideal of L, it is readily verified that $P(L')$ is a p-ideal of $P(L)$, and that $P(L)/P(L') \cong P(L/L')$. Furthermore the condition $s_1 = s_2 = \cdots = s_l$ implies that

$$P(L)/P(L') \cong P(L/L')$$

is a torus. By [9, (3.8), p.2884], we have

$$\begin{aligned} H^n_{p^s}(L,V) = H^n_p(P(L),V) &\cong H^n_p(P(L'),V)^{P(L)/P(L')} \\ &= H^n_{p^s}(L',V)^{L/L'}, \quad \forall n \in \mathbb{N}^+. \end{aligned}$$

The final equality follows from the facts that

$$P(L)/P(L') \cong P(L/L')$$

and $M^{\mathfrak{L}} = M^{P(\mathfrak{L})}$ for a GR Lie algebra $\mathfrak{L}$ and a GR $\mathfrak{L}$-module M.

4.3. In the rest of this paper, let $L = X(m : \mathbf{n})^{(2)}$, $X \in \{W, S, H, K\}$. The notations appearing in the following are the same as in Sections 1.6 and 1.7.

Theorem 4.3.1. $H^1_{p^s}(L, Z_{p^s}(\lambda)) \cong (N^+/((N^+)^{(1)} + \langle (E^+)^{\varphi_s}\rangle))^{\mathfrak{h}}$.

Proof. Applying Theorem 3.4.1 for $L' = B$, we have

$$H^n_{p^s}(L, Z_{p^s}(\lambda)) \cong H^n_{p^s}(B, {}_\theta(F_\lambda)).$$

Since N^+ is a GR ideal of B and $B/N^+ \cong \mathfrak{h}$ is toral, the condition in Proposition 4.2.1 is satisfied. Thereby

$$H^1_{p^s}(L, Z_{p^s}(\lambda)) \cong H^1_{p^s}(N^+, {}_\theta(F_\lambda))^{\mathfrak{h}}.$$

Note that $\mathfrak{A}^+ = F \oplus \mathrm{Nilrad}(\mathfrak{A}^+)$ (see [21, Lemma 4.1]). Hence up to isomorphism, $\mathfrak{A}^+$ admits a unique irreducible (1-dimensional) module F via the canonical supplementation. As a GR N^+-module, ${}_\theta(F_\lambda)$ is precisely the trivial N^+-module F. (This fact can be also seen from another viewpoint, pointed out by Shen, G. Y. According to the relation $u\big|_{{}_\theta(F_\lambda)} = \lambda(\theta(u))$ and the definition of the automomorphism θ, $\theta\big|_{N^+} = \mathrm{id}$ in view of the nilpotency of N^+. So ${}_\theta(F_\lambda)$ coincides with F_λ, as well as F, because $\lambda(N^+) = 0$). By Corollary 3.3.1, we have

$$H^1_{p^s}(N^+, F) \cong N^+/((N^+)^{(1)} + \langle (E^+)^{\varphi_s}\rangle).$$

The proof is completed.

Remark 4.3.1. If $s_1 = s_2 = \cdots = s_l$, according to Remark 4.2.1 and the above argument,

$$H^n_{p^s}(L, Z_{p^s}(\lambda)) \cong H^n_{p^s}(N^+, F)^{\mathfrak{h}} (= H^n_p(P(N^+), F)^{\mathfrak{h}}) \ \text{ for } n \in \mathbb{N}^+,$$

which is independent of λ.

4.4. We finally compute the first ordinary Lie cohomology groups with coefficients in $Z_{p^s}(\lambda)$. By Theorem 3.5.1, we have

$$H^n(L, Z_{p^s}(\lambda)) \cong \bigoplus_{s+t=n} \Lambda^s(L/B) \otimes_F H^t(B, {}_\theta(F_\lambda)). \tag{4.4.1}$$

${}_\theta(F_\lambda)$ determines a Lie algebra homomorphism

$$\theta \circ \lambda : B \to F$$

and a B-module $F_{\lambda\circ\theta}$. Set

$$\begin{aligned} I &:= \mathrm{Ker}(\lambda \circ \theta), \\ B^{(1)}_{\lambda\circ\theta} &:= \{[x,y] - \lambda(\theta(x))y + \lambda(\theta(y))x \bigm| x, y \in B\}. \end{aligned}$$

Then $B^{(1)}_{\lambda\circ\theta} \subset I$. By [8, (3.4), p.165] and (1), we have

Proposition 4.4.1.

$$H^1(L, Z_{p^s}(\lambda) \cong \begin{cases} I/B^{(1)}_{\lambda\circ\theta}, & \textit{if } \lambda(e_i) \neq \mathrm{tr}_L(ade_i) - \mathrm{tr}_{P(\mathfrak{B})}(\mathrm{ad}e_i) \textit{ for some } e_i \in E^0, \\ L/B^{(1)}, & \textit{otherwise.} \end{cases}$$

Proof. It follows from the observation that when $\lambda \circ \theta$ is zero, (4.4.1) implies that

$$H^1(L, Z_{p^s}(\lambda)) \cong L/B \oplus L/B^{(1)} \cong L/B^{(1)}.$$

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