Chin. Ann. of Math.
19B: 4(1998),433-444.

THE ZEROS AND ORDER OF MEROMORPHIC SOLUTIONS OF $f^{(k)}+Bf=H(z)$***

CHEN ZONGXUAN*   YU JIARONG**

Abstract

Suppose that B is a rational function having a pole at ∞ of order $n>0$ and that $H\not\equiv 0$ is a meromorphic function satisfying $\sigma(H)=\beta\neq(n+k)/k$. If the differential equation $f^{(k)}+Bf=H(z)$ has a meromorphic solution f, then all meromorphic solutions f satisfy

$$\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\max\{\beta,(n+k)/k\},$$

except at most one exceptional meromorphic solution f_0.

Keywords Linear differential equation, Meromorphic solution, Zero, Order
1991 MR Subject Classification 34A20, 30D35
Chinese Library Classification O175.11, O175.51

§1. Introduction and Results

Consider non-homogeneous linear differential equations of the form

$$f^{(k)}+Bf=H\quad(k\geq 2).\tag{1.1}$$

I. Laine proved in [7]

Theorem A. *Let* $B(z),P_0(z),P_1(z)\not\equiv 0$ *be polynomials such that* $\deg B=n\geq 1$, $\deg P_0=\beta<(n+k)/k$ *and* $H=P_1(z)e^{P_0(z)}$, *then*

(a) *If* $\deg P_1<n$, *then all solutions of* (1.1) *satisfy*

$$\lambda(f)=\bar{\lambda}(f)=\sigma(f)=(n+k)/k.\tag{1.2}$$

(b) *If* $\deg P_1\geq n$, *then apart from one possible exception, all solutions satisfy* (1.2). *The possible exceptional solution is of the form* $f_0=Q\cdot\exp(P_0)$, *where* Q *is a polynomial of degree* $\deg Q=\deg P_1-n$.

Gao Shian had earlier addressed the case when $k=2$ in Theorem A.

In this paper, we use the same notations as in [1], i.e., we use $\lambda(f)$ and $\bar{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and the sequence of distinct zeros of $f(z)$ and $\lambda(1/f)$ and $\bar{\lambda}(1/f)$ to denote respectively the exponents of convergence of the pole-sequence and the sequence of distinct poles of a meromorphic function $f(z),\sigma(f)$

Manuscript received May 27, 1996.
*Department of Mathematics, Jiangxi Normal University, Nanchang 330027, China.
**Department of Mathematics, Wuhan University, Wuhan 430072, China.
***Project supported by the National Natural Science Foundation of China.

to denote the order of growth of $f(z)$. We use the standard notations of the Nevanlinna's theory (e. g. see [5]).

In [2], Chen Zongxuan proved

Theorem B. *Let B be a rational function having a pole at ∞ of order $n>0$, $H(z)$ be a transcendental meromorphic function satisfying $\sigma(H)=\beta<(n+k)/k$. If all solutions of* (1.1) *are meromorphic functions, then apart from one possible exception f_0, all solutions satisfy* (1.2) *and $\lambda(1/f)=\lambda(1/H)$. The possible exceptional solution f_0 satisfies $\beta\le\sigma(f_0)<(n+k)/k$.*

Theorem C. *Let B be a rational function having a pole at ∞ of order $n>0$, $H(z)$ be a meromorphic function satisfying $(n+k)/k<\sigma(H)=\beta<\infty$. If all solutions of* (1.1) *are meromorphic functions, then*

(a) $\sigma(f)=\beta,\quad \lambda(1/f)=\lambda(1/H)$;

(b) *If $\beta=\lambda(H)>\lambda(1/H)$, then $\lambda(f)=\beta$;*

(c) *If $\beta>\max\{\lambda(H),\lambda(1/H)\}$, then apart from one possible exception f_0 having $\lambda(f_0)<\beta$, all solutions satisfy*

$$\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\beta. \tag{1.3}$$

By the fundamental theory of the differential equation with complex coefficients, we know that all solutions of linear differential equation with entire coefficients are entire functions. But a solution of linear differential equation with meromorphic coefficients is not perhaps a meromorphic function. For example, $f=\exp\{\frac{1}{z}\}$ is a solution of

$$f''-\Big(\frac{1}{z^4}+\frac{2}{z^3}\Big)f=0,$$

but $\exp\{\frac{1}{z}\}$ is not a meromorphic function. Therefore in Theorems B and C, the condition that all solutions of (1.1) are meromorphic functions is very rigorous. In this paper, we shall subtract this condition in Theorems B and C, generalize Theorems B and C, and obtain the precise estimate of the order of exceptional solution f_0. We shall prove the following theorems.

Theorem 1.1. *Suppose that B is a rational function having a pole at ∞ of order $n>0$, $H\not\equiv 0$ is a meromorphic function satisfying $\sigma(H)=\beta$. If* (1.1) *has a meromorphic solution f, then*

(a) *If $\beta<(n+k)/k$, then all meromorphic solutions f of* (1.1) *satisfy* (1.2) *with at most one possible exceptional meromorphic solution f_0. The possible exceptional solution f_0 satisfies $\sigma(f_0)=\beta$.*

(b) *If $\beta=(n+k)/k$, then all meromorphic solutions f of* (1.1) *satisfy $\sigma(f)=(n+k)/k$ and*

$$\max\{\lambda(f),\lambda(1/f)\}\ge\max\{\lambda(H),\lambda(1/H)\}. \tag{1.4}$$

Theorem 1.2. *Suppose that B is a rational function having a pole at ∞ of order $n>0$, H is a meromorphic function satisfying $(n+k)/k<\sigma(H)=\beta<\infty$. If* (1.1) *has a meromorphic solution f, then*

(a) *$\sigma(f)=\beta$;*

(b) *If $\beta=\lambda(H)>\lambda(1/H)$, then $\lambda(f)=\beta$.*

(c) *If* $\beta > \max\{\lambda(H), \lambda(1/H)\}$, *then all meromorphic solutions* f *of* (1.1) *satisfy* (1.3) *with at most one possible exceptional meromorphic solution* f_0. *The possible exceptional solution* f_0 *satisfies* (1.4).

§2. Lemmas and Preliminaries

Theorem D (Borel, see Theorem 5.13 in [8]). *Suppose that* $Q(z)$ *is canonical product formed by* $\{z_n; n = 1, 2, \cdots\}$ $(z_n \neq 0)$ *and* $\lambda(Q) = \beta < \infty$. *Set* $O_n = \{z : |z - z_n| < |z_n|^{-\alpha}\}$ $(\alpha(> \beta)$ *is a constant*), *then for any given* $\epsilon > 0$, $|Q(z)| \geq \exp\{-|z|^{\beta+\epsilon}\}$ *holds for* $z \notin \bigcup\limits_{n=1}^{\infty} O_n$.

Theorem E (See [6, p.19]). *Suppose that* $w(z)$ *is a finite order entire function,* $\mu(r)$ *is the maximum term of the power series of* $w(z)$, *then* $\lim\limits_{r\to\infty} \log M(r, w)/\log \mu(r) = 1$.

Lemma 2.1. *Suppose that* $H(z)$ *is a meromorphic function,* $\sigma(H) = \beta < \infty$. *Then for any given* $\epsilon > 0$, *there is a set* $E_1 \subset (1, \infty)$ *that has finite linear measure and finite logarithmic measure such that* $|H(z)| \leq \exp\{r^{\beta+\epsilon}\}$ *holds for* $|z| = r \notin [0, 1] \cup E_1$, *and* $r \to \infty$.

Proof. If H has only finitely many poles, then Lemma 2.1 holds obviously. Now assume that $H(z)$ has infinitely many poles. Set $H(z) = h(z)/[z^{k_1} \cdot Q(z)]$, where k_1 is nonnegative integer, $h(z)$ is an entire function, $Q(z)$ is a canonical product formed by the nonzero poles $\{z_j : j = 1, 2, \cdots; |z_j| = r_j,\ 0 < r_1 \leq r_2 < \cdots\}$ of $H(z)$. Hence $\sigma(h) \leq \sigma(H) = \beta,\ \sigma(Q) = \lambda(Q) \leq \beta$.

For any given $\epsilon > 0$, set $O_j = \{z : |z - z_j| \leq r_j^{-(\beta+\epsilon/2)}\}$ $(j = 1, 2, \cdots)$ and $O = \bigcup\limits_{j=1}^{\infty} O_j$. Set

$$E_1 = \bigcup_{j=1}^{\infty} (r_j - r_j^{-(\beta+\epsilon/2)}, r_j + r_j^{-(\beta+\epsilon/2)}).$$

Since

$$\sum_{j=1}^{\infty} 1/r_j^{\beta+\epsilon/2} = d < \infty, \tag{2.1}$$

we see that the linear measure of E_1, $mE_1 = 2d < \infty$. For $|z| = r \notin E_1 \cup [0, 1]$, we have by Theorem D, $|Q(z)| \geq \exp\{-r^{\beta+\epsilon/2}\}$. Hence

$$|H(z)| \leq \exp\{2r^{\beta+\epsilon/2}\}/r^{k_1} \leq \exp\{r^{\beta+\epsilon}\}$$

holds for $|z| = r \notin E_1 \cup [0, 1],\ r \to \infty$.

Now we prove logarithmic measure of E_1, lm $E_1 < \infty$. From

$$\operatorname{lm} E_1 = \sum_{j=1}^{\infty} [\log(r_j + r_j^{-(\beta+\epsilon/2)}) - \log(r_j - r_j^{-(\beta+\epsilon/2)})] = \sum_{j=1}^{\infty} \log\Big(1 + \frac{2r_j^{-(\beta+\epsilon/2)}}{r_j - r_j^{-(\beta+\epsilon/2)}}\Big),$$

and for sufficiently large r_j

$$\log\Big(1 + \frac{2r_j^{-(\beta+\epsilon/2)}}{r_j - r_j^{-(\beta+\epsilon/2)}}\Big) \leq \frac{2r_j^{-(\beta+\epsilon/2)}}{r_j - r_j^{-(\beta+\epsilon/2)}} \leq 2r_j^{-(\beta+\epsilon/2)},$$

we have $\operatorname{lm} E_1 \leq \infty$ by (2.1).

Lemma 2.2. *Suppose that $g(z)$ is a transcendental entire function, $\sigma(g)=\alpha<\infty$. Then there is a set $E_2\subset(1,\infty)$ that has infinite logarithmic measure such that*

$$\lim_{\substack{r\to\infty\\ r\in E_2}}\frac{\log\log M(r,g)}{\log r}=\lim_{\substack{r\to\infty\\ r\in E_2}}\frac{\log\nu_g(r)}{\log r}=\alpha,$$

where $\nu_g(r)$ denotes the centralindex of $g(z)$.

Proof. By $\sigma(g)=\alpha$, there exists $\{r_n\}$ $(r_n\to\infty)$, such that

$$\lim_{r_n\to\infty}\frac{\log\log M(r_n,g)}{\log r_n}=\alpha. \tag{2.2}$$

Set $E_2\subset(1,+\infty)$. E_2 has the following properties: (a) If the sequence $\{r_n\}$ satisfies (2.2), then $\{r_n\}\subset E_2$. (b) If a sequence $\{r_n\}\subset E_2$ $(r_n\to\infty)$, then (2.2) holds for $\{r_n\}$. Now we affirm that the logarithmic measure of E_2, lm $E_2=\infty$. In fact, if lm $E_2=\delta<\infty$, then by the definition of E_2, we have

$$\overline{\lim_{\substack{r\to\infty\\ r\in(1,\infty)-E_2}}}\frac{\log\log M(r,g)}{\log r}=\alpha_1<\alpha. \tag{2.3}$$

Now for a given $\{r'_n\}\subset(1,\infty)$, $r'_n\to\infty$, there exists a point $r''_n\in[r'_n,(\delta+1)r'_n]-E_2$. From

$$\frac{\log\log M(r''_n,g)}{\log r''_n}\ge\frac{\log\log M(r'_n,g)}{\log[(\delta+1)r'_n]}=\frac{\log\log M(r'_n,g)}{\log r'_n+\log(\delta+1)},$$

we have

$$\begin{aligned}\overline{\lim_{r'_n\to\infty}}\frac{\log\log M(r'_n,g)}{\log r'_n}&=\overline{\lim_{r'_n\to\infty}}\frac{\log\log M(r'_n,g)}{\log r'_n+\log(\delta+1)}\\&\le\overline{\lim_{r''_n\to\infty}}\frac{\log\log M(r''_n,g)}{\log r''_n}\\&\le\overline{\lim_{\substack{r\to\infty\\ r\in(1,+\infty)-E_2}}}\frac{\log\log M(r,g)}{\log r}.\end{aligned}$$

Since $\{r'_n\}$ is arbitrary, we have $\alpha\le\alpha_1$. This is a contradiction, hence lm$E_2=\infty$.

By $\sigma(g)=\alpha<\infty$ and Theorem E, we have

$$\lim_{r\to\infty}\frac{\log M(r,g)}{\log\mu(r)}=1, \tag{2.4}$$

where $\mu(r)$ is the maximum term of the power series of $g(z)$, $\mu(r)=|a_{\nu_g(r)}|r^{\nu_g(r)}$. By (2.4), for sufficiently large r,

$$\log M(r,g)\le2\log\mu(r)\le2\log^+|a_{\nu_g}|+2\nu_g(r)\cdot\log r.$$

From

$$\frac{\log\log M(r,g)}{\log r}\le\frac{\log\nu_g(r)}{\log r}+\frac{\log^+\log^+|a_{\nu_g}|+2\log2+\log\log r}{\log r},$$

we have

$$\alpha=\lim_{\substack{r\to\infty\\ r\in E_2}}\frac{\log\log M(r,g)}{\log r}=\underline{\lim}_{\substack{r\to\infty\\ r\in E_2}}\frac{\log\log M(r,g)}{\log r}\le\underline{\lim}_{\substack{r\to\infty\\ r\in E_2}}\frac{\log\nu_g(r)}{\log r}\le\overline{\lim_{\substack{r\to\infty\\ r\in(0,\infty)}}}\frac{\log\nu_g(r)}{\log r}=\alpha,$$

i.e.,

$$\lim_{\substack{r\to\infty\\ r\in E_2}}\frac{\log\nu_g(r)}{\log r}=\alpha.$$

Lemma 2.3. *Suppose that $g(z)$ is an entire function with $\sigma(g)=\infty$. Then there is a set $E_2\subset(1,\infty)$ that has infinite logarithmic measure such that*

$$\lim_{\substack{r\to\infty\\ r\in E_2}}\frac{\log\nu_g(r)}{\log r}=\infty.$$

Proof. Using the same proof as in the proof of the upper half part of Lemma 2.2, we can prove Lemma 2.3.

Lemma 2.4.[2] *Suppose that B is a rational function having a pole at ∞ of order $n>0$. If $f\not\equiv 0$ is a meromorphic solution of the homogeneous equation*

$$f^{(k)}+Bf=0, \tag{2.5}$$

then $\sigma(f)=(n+k)/k$.

Lemma 2.5.[4] *Suppose that $u(z)$ is a meromorphic function with $\sigma(u)=\beta<\infty,\ \epsilon>0$ is a given constant. Then there exists a set $E_3\subset(1,\infty)$ that has finite logarithmic measure such that*

$$\Big|\frac{u^{(j)}(z)}{u(z)}\Big|\le r^{j(\beta-1+\epsilon)}\quad(j=1,\cdots,k) \tag{2.6}$$

hold for all z satisfying $|z|=r\notin[0,1]\cup E_3$.

Lemma 2.6. *Suppose that $u(z)$ is a meromorphic function with $\sigma(u)=\beta<\infty$, m is integer, $\epsilon>0$ is a given constant. Then there exists a set $E_3\subset(1,\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z|=r\notin[0,1]\cup E_3$, we have*

$$|u(z)\cdot(u^{-1}(z))^{(m)}|\le r^{m(\beta-1+\epsilon)}. \tag{2.7}$$

Proof. Firstly we use induction to prove

$$u\Big(\frac{1}{u}\Big)^{(m)}=\sum_{(j_1,\cdots,j_m)}\alpha_{(j_1\cdots j_m)}\Big(\frac{u'}{u}\Big)^{j_1}\cdots\Big(\frac{u^{(m)}}{u}\Big)^{j_m}, \tag{2.8}$$

where $\alpha_{(j_1\cdots j_m)}$ is a constant, $j_1,\cdots,j_m$ satisfying $1\cdot j_1+2\cdot j_2+\cdots+m\cdot j_m=m$. For $n=1$, (2.8) holds obviously. For $n=m$, assume that (2.8) holds. So, we have for $n=m+1$,

$$\begin{aligned}
\Big(\frac{1}{u}\Big)^{(m+1)}&=\Big[\Big(\frac{1}{u}\Big)^{(m)}\Big]'=\Big[\frac{1}{u}\sum_{(j_1\cdots j_m)}a_{(j_1\cdots j_m)}\Big(\frac{u'}{u}\Big)^{j_1}\cdots\Big(\frac{u^{(m)}}{u}\Big)^{j_m}\Big]'\\
&=-\frac{u'}{u^2}\sum_{(j_1\cdots j_m)}a_{(j_1\cdots j_m)}\Big(\frac{u'}{u}\Big)^{j_1}\cdots\Big(\frac{u^{(m)}}{u}\Big)^{j_m}\\
&\quad+\frac{1}{u}\sum_{(j_1\cdots j_m)}a_{(j_1\cdots j_m)}\Big\{\sum_{i=1}^{m}\Big(\frac{u'}{u}\Big)^{j_1}\cdots\Big(\frac{u^{(d-1)}}{u}\Big)^{j_d-1}\\
&\quad\cdot\Big[j_d\Big(\frac{u^{(d)}}{u}\Big)^{j_d-1}\Big(\frac{u^{(d+1)}}{u}\Big)-j_d\Big(\frac{u^{(d)}}{u}\Big)^{j_d}\Big(\frac{u'}{u}\Big)\Big]\cdot\Big(\frac{u^{(d+1)}}{u}\Big)^{j_{d+1}}\cdots\Big(\frac{u^{(m)}}{u}\Big)^{j_m}\Big\}\\
&=\frac{1}{u}\sum_{(j_1\cdots j_m)}a_{(j_1\cdots j_m)}\Big(\frac{u'}{u}\Big)^{j_1+1}\cdots\Big(\frac{u^{(m)}}{u}\Big)^{j_m}+\frac{1}{u}\sum_{(j_1\cdot j_m)}a_{(j_1\cdots j_m)}\\
&\quad\cdot\Big\{\sum_{i=1}^{m}\Big[-j_d\Big(\frac{u'}{u}\Big)^{j_1+1}\Big(\frac{u''}{u}\Big)^{j_2}\cdots\Big(\frac{u^{(m)}}{u}\Big)^{j_m}\\
&\quad+j_d\Big(\frac{u'}{u}\Big)^{j_1}\cdots\Big(\frac{u^{(d)}}{u}\Big)^{j_d-1}\Big(\frac{u^{(d+1)}}{u}\Big)^{j_{d+1}+1}\cdots\Big(\frac{u^{(m)}}{u}\Big)^{j_m}\Big]\Big\},
\end{aligned}$$

where the indexs satisfy $1\cdot(j_1+1)+2\cdot j_2+\cdots+m\cdot j_m=m+1$, or $1\cdot j_1+\cdots+d\cdot(j_d-1)+(d+1)\cdot(j_{d+1}+1)+\cdots+m\cdot j_m=m+1$. Therefore (2.8) holds.

Now by (2.8) and Lemma 2.5, it is easy to see that Lemma 2.6 holds.

Lemma 2.7. *Suppose that $b_0,\cdots b_{k-1}$, $H\not\equiv 0$ are meromorphic functions, $\sigma(H)=\beta<\infty$, and that there are a set $E_3\subset(1,+\infty)$ that has finite logarithmic measure and a constant number $C_1>0$ such that for $|z|=r\notin[0,1]\cup E_3$,*

$$|b_j(z)|\le r^{C_1}\quad (j=0,\cdots,k-1) \tag{2.9}$$

hold. If an entire function $g(z)$ is a solution of the equation

$$g^{(k)}+b_{k-1}g^{(k-1)}+\cdots+b_0g=H, \tag{2.10}$$

then $\sigma(g)<\infty$.

Proof. Assume that $\sigma(g)=\infty$, $\mu(r)$ denotes the maximum term of the power series of $g(z)$, and $\nu_g(r)$ denotes the centralindex of $g(z)$. By Lemma 2.3, we know that there is a set $E_2\subset(1,\infty)$ that has infinite logarithmic measure such that

$$\lim_{\substack{r\to\infty\\ r\in E_2}}\frac{\log\nu_g(r/2)}{\log(r/2)}=\infty. \tag{2.11}$$

Since $\nu_g(r)$ is a step function to r, we can assume that t_j $(j=0,1\cdots,0=t_0<t_1<t_2<\cdots)$ are discontinuous points of $\nu_g(r)$. As $t\in(t_j,t_{j+1})$, we have $\mu(t)=|a_{\nu_g(t)}|\cdot t^{\nu_g(t)}$, where the centralindex $\nu_g(t)=m$ is a fixed constant. Hence

$$\mu'(t)=m|a_m|t^{m-1}=\mu\cdot\nu_g(t)/t$$

holds for $t\in(t_j,t_{j+1})$. Since $\mu(t)$ is a continuous function, we have for $r>2$

$$\log\mu(r)-\log\mu(1)=\int_1^r[\mu'(t)/\mu(t)]dt=\int_1^r[\nu(t)/t]dt>\int_{\frac{r}{2}}^r(\nu(t)/t)dt\ge\nu(r/2)\cdot\log 2.$$

By Cauchy's inequality, it is easy to see that $\mu(r)\le M(r,g)$. So,

$$\nu(r/2)\cdot\log 2\le\log M(r.g)-\log\mu(1). \tag{2.12}$$

For a given large α such that

$$\alpha>\max\{C_1,\beta\}+k, \tag{2.13}$$

by (2.11) and (2.12), we obtain

$$\nu(r)=(r/2)\ge(r/2)^\alpha=C_2r^\alpha, \tag{2.14}$$

$$M(r,g)\ge C_3\exp\{C_4r^\alpha\} \tag{2.15}$$

for $r\in E_2,\ r\to\infty$, where C_2,C_3,C_4 are positive constants.

By the Wiman-Valiron theory (see [6, 9, 10]) we have basic formulas

$$\frac{g^{(j)}(z)}{g(z)}=\Big(\frac{\nu_g(r)}{z}\Big)^j(1+o(1))\quad (j=1,\cdots,k), \tag{2.16}$$

where $|z|=r,|g(z)|=M(r,g),r\notin E_4,\ \int_{E_4}\frac{dr}{r}<\infty$.

By Lemma 2.1, we have

$$|H(z)|\le\exp\{r^{\beta+\frac{1}{2}}\} \tag{2.17}$$

for $|z|=r\in[1,+\infty]-E_1,\ \int_{E_1}\frac{dr}{r}<\infty$.

Now, we take sufficiently large $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4)$, $|g(z)| = M(r, g)$, and logarithmic measure lm $[E_2 - (E_1 \cup E_3 \cup E_4) = \infty$. By (2.10) and (2.16), we have

$$\Big(\frac{\nu_g(r)}{z}\Big)^k (1 + o(1)) + b_{k-1}\Big(\frac{\nu_g(r)}{z}\Big)^{k-1}(1 + o(1)) + \cdots + b_0 = \frac{H(z)}{g(z)},$$

$$\begin{aligned}&\frac{\nu(r)}{z^k}(1 + o(1))\\ &= \frac{H(z)}{g(z)\nu^{k-1}(r)} - \frac{b_{k-1}}{z^{k-1}}(1 + o(1)) - \frac{b_{k-2}}{z^{k-2}\nu(r)}(1 + o(1)) - \cdots - \frac{b_0}{\nu^{k-1}(r)}.\end{aligned} \tag{2.18}$$

By (2.13)–(2.15), (2.17), we have

$$\frac{|H(z)|}{|g(z)|} = \frac{|H(z)|}{M(r, g)} \le \frac{1}{C_3}\exp\{r^{\beta+\frac{1}{2}} - C_4 r^\alpha\} \to 0, \tag{2.19}$$

$$\Big|\frac{b_j(z)}{z^j \nu^{k-1-j}(r)}\Big| \to 0 \quad (j = 0, \cdots, k-2) \tag{2.20}$$

for $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4)$, $r \to \infty$. And (2.14) (2.19) and (2.20) give

$$\begin{aligned}&\Big|\frac{H(z)}{g(z)\cdot \nu^{k-1}(z)} - \frac{b_{k-1}}{z^{k-1}}(1 + o(1)) - \frac{b_{k-2}}{z^{k-2}\nu(r)}(1 + o(1)) - \cdots - \frac{b_0}{\nu^{k-1}(r)}\Big|\\ &= O\Big(\frac{b_{k-1}}{z^{k-1}}\Big) = O(r^{C_1-k+1}).\end{aligned} \tag{2.21}$$

On the other hand, by (2.14), we have

$$\Big|\frac{\nu(r)}{z^k}(1 + o(1))\Big| \ge C_2 r^{\alpha-k} > r^{C_1} \tag{2.22}$$

for $r \in E_2$, $r \to \infty$. (2.21) contradicts (2.22) by (2.18). Therefore $\sigma(g) < \infty$.

Lemma 2.8. *Suppose that B is a rational function having a pole at ∞ of order $n > 0$, and $H \not\equiv 0$ is a meromorphic function with $\sigma(H) = \beta$. If* (1.1) *has a meromorphic solution f, then*

(a) *If $\beta < (n + k)/k$, then all meromorthic solutions f of* (1.1) *satisfy $\sigma(f) = (n + k)/k$, with at most one exceptional meromorthic solution f_0 with $\sigma(f_0) = \beta$.*

(b) *If $(n + k)/k \le \beta < \infty$, then $\sigma(f) = \beta$.*

Proof. By (1.1), we have $\sigma(f) \ge \beta$. By (1.1) and the fact that B has only finitely many poles, we know that if $|z|$ $(< \infty)$ is sufficiently large, then either f and H are both analytic at z, or f has a pole at z of order m_1 if and only if H has a pole at z of order $m_1 + k$. So, $\bar\lambda(1/f) = \bar\lambda(1/H)$. By $n(r.f) \le n(r, H) + O(1)$ and $n(r, H) \le (k + 1)n(r, f) + O(1)$, it follows that

$$\lambda(1/f) = \lambda(1/H). \tag{2.23}$$

Set $f(z) = g(z)/(z^{m_2}u(z)) = g(z)/u_1(z)$, where m_2 is nonnegative integer, $g(z)$ is an entire function, $u(z)$ is a canonica1 product (or polynomial) formed by the nonzero poles $\{z_j : j = 1, 2, \cdots\}$ $(|z_j| = r_j,\ 0 < r_1 \le r_2 \le \cdots)$ of f, $u_1(z) = z^{m_2}u(z)$, then $\lambda(u_1) = \sigma(u_1) = \lambda(1/f) = \lambda(1/H) \le \beta$.

(a) First we prove that if $\sigma(f) = \alpha > \beta$, then $\sigma(f) = (n + k)/k$. By $\sigma(f) > \beta$, we have $\sigma(g) = \sigma(f) = \alpha$. For any given ϵ $(0 < 2\epsilon < \min\{\alpha - \beta, ((n + k)/k) - \beta\})$, by Lemma 2.1, it follows that there is a set $E_1 \subset (1, +\infty)$ that has finite logarithmic measure, such that

$$|1/u_1(z)| \le \exp\{r^{\beta+\epsilon}\} \tag{2.24}$$

holds for $|z| = r \notin [0,1] \cup E_1,\ r \to \infty$. By (2.23) and the fact that the poles of f can only occur at poles of H except at most finitely many poles, it follows that

$$|H(z)| \le \exp\{r^{\beta+\epsilon}\} \tag{2.25}$$

holds for $|z| = r \notin [0,1] \cup E_1,\ r \to \infty$. Substituting $f(z) = g(z)/u_1(z)$ into (1.1), we have

$$\frac{g^{(k)}}{g} + C_k^1 u_1 (u_1^{-1})' \frac{g^{(k-1)}}{g} + \cdots + C_k^{k-1} u_1 (u_1^{-1})^{(k-1)} \frac{g'}{g} + u_1 (u_1^{-1})^{(k)} + B = \frac{H \cdot u_1}{g}, \tag{2.26}$$

where C_k^j $(j = 1, \cdots, k-1)$ are the usua1 notation of the binomial coefficients. By $\sigma(u_1) \le \beta$ and Lemma 2.6, there is a set $E_3 \subset (1, +\infty)$ that has finite logarithmic measure, such that for $|z| = r \notin E_3 \cup [0,1]$, for $j = 1, \cdots, k$, we have

$$|u_1(z)(u_1^{-1}(z))^{(j)}| \le r^{j(\beta-1+\epsilon)}. \tag{2.27}$$

By $0 < 2\epsilon < [(n+k)/k] - \beta$, we have $k(\beta - 1 + \epsilon) < n$. So

$$|u_1(z)(u_1^{-1}(z))^{(k)} + B| = O(r^n). \tag{2.28}$$

By Lemma 2.7 and (2.26)–(2.28), we have $\sigma(g) = \alpha < \infty$.

By Lemma 2.2 and $\sigma(g) < \infty$, there is a set $E_2 \subset (1, +\infty)$ that has infinite logarithmic measure such that

$$\lim_{\substack{r\to\infty \\ r\in E_2}} \frac{\log\log M(r,g)}{\log r} = \lim_{\substack{r\to\infty \\ r\in E_2}} \frac{\log \nu_g(r)}{\log r} = \alpha. \tag{2.29}$$

By the Wiman-Valiron theory, there is a set $E_4 \subset (1, \infty)$ that has finite logarithmic measure, such that for $|z| = r \notin E_4,\ |g(z)| = M(r,g)$, (2.16) holds. By (2.29), we have

$$M(r,g) \ge \exp\{r^{\alpha-\epsilon}\} \tag{2.30}$$

for $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4)$ and sufficiently large r. By (2.25),(2.30), $|u_1(z)| \le \exp\{r^{\beta+\epsilon}\}$ $(r \to \infty)$ and $\beta + \epsilon < \alpha - \epsilon$, we have for $r \to \infty$

$$\Big|\frac{u_1(z) \cdot H(z)}{g(z)}\Big| = \Big|\frac{u_1(z) \cdot H(z)}{M(r,g)}\Big| \le \exp\{2r^{\beta+\epsilon} - r^{\alpha-\epsilon}\} \to 0. \tag{2.31}$$

Now we take $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4),\ |g(z)| = M(r,g)$. Since the logarithmic measure of $E_2 - (E_1 \cup E_3 \cup E_4)$, $\mathrm{lm}\,[E_2 - (E_1 \cup E_3 \cup E_4)] = \infty$, by (2.16), (2.26)–(2.28) and (2.31), we obtain for $|z| = r \to \infty$

$$\begin{aligned} &\Big(\frac{\nu_g(r)}{z}\Big)^k (1 + o(1)) + O(r^{\beta-1+\epsilon}) \Big(\frac{\nu_g(r)}{z}\Big)^{k-1} (1 + o(1)) + \cdots \\ &+ O(r^{(k-1)(\beta-1+\epsilon)}) \Big(\frac{\nu_g(r)}{z}\Big)(1 + o(1)) + O(r^n) = o(1). \end{aligned} \tag{2.32}$$

For $r \in E_2 - (E_1 \cup E_3 \cup E_4),\ r \to \infty$, we have by (2.29)

$$\nu_g(r) = r^{\alpha+o(1)}. \tag{2.33}$$

By (2.33), and since ϵ is arbitrarily small, it is easy to see that the degrees of all terms of the left of (2.32) are respectively

$$k(\alpha - 1),\quad (k - j)(\alpha - 1) + (\beta - 1)j \quad (j = 1, \cdots, k-1),\ n.$$

By $\beta < \alpha$ and the Wiman-Valiron theory, we get $\alpha = (n+k)/k$, i.e., $\sigma(f) = \sigma(g) = (n+k)/k$. Now if f_0 and $f_1 (f_1 \not\equiv f_0)$ are both meromorphic solutions of (1.1), with $\sigma(f_0) = \sigma(f_1) = \beta$, then $\sigma(f_0 - f_1) < (n+k)/k$. But $f_0 - f_1 \not\equiv 0$ is a meromorphic solution of the corresponding homogeneous equation (2.5) of (1.1). This contradicts Lemma 2.4. Therefore, the equation (1.1) has at most one exceptional meromorphic solution f_0 with $\sigma(f_0) = \beta$.

(b) Since $\sigma(f) \geq \beta \geq (n+k)/k$, we need only to prove that $\sigma(f) = \alpha > \beta$ fails.

Assume that $\alpha > \beta$. Using the same reasoning as in (a), for any given ϵ $(0 < \epsilon < \alpha - \beta)$, we easily see that (2.24)–(2.27) hold. By $\beta \geq (n+k)/k$ and (2.27), it follows that

$$|u_1(z)(u_1^{-1}(z))^{(k)} + B| = \{r^{k(\beta-1+\epsilon)}\} \tag{2.28'}$$

holds for $|z| = r \in (1, +\infty) - E_3$, $r \to \infty$. And (2.26), (2.27), (2.28)' and Lemma 2.7 give $\sigma(g) = \alpha < \infty$. Continually using the same proof as in (a), we easily know that (2.29)–(2.33) hold. For $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4)$, it is easy to see that there is only one term $\left(\frac{\nu(r)}{z}\right)^k (1 + o(1))$ with the degree $k(\alpha - 1)$ being the highest one among all terms of (2.32). This is impossible. Therefore, $\sigma(f) = \beta$.

Lemma 2.9. *Suppose that β is a positive integer and $\beta > 1$, A_{k-j} $(j = 1, \cdots, k)$ are rational functions having a pole at ∞ of order $n_{k-j} = j(\beta - 1)$, and $U \not\equiv 0$ is a meromorphic function with $\sigma(U) < \beta$. If the equation*

$$h^{(k)} + A_{k-1}h^{(k-1)} + \cdots + A_0 h = U \tag{2.34}$$

has a meromorphic solution h, then all meromorphic solutions of (2.34) satisfy $\sigma(h) = \beta$ except at most one possible exceptional solution. The possible meromorphic one h_0 satisfies $\sigma(h_0) = \sigma(U)$.

If $h \not\equiv 0$ is a meromorphic solution of the equation

$$h^{(k)} + A_{k-1}h^{(k-1)} + \cdots + A_0 h = 0 \tag{2.35}$$

that is the corresponding homogeneous differential equation of (2.34), then $\sigma(h) = \beta$.

Proof. Set $\sigma(U) = d$. Then $d < \beta$. By (2.34), $\sigma(h) = \alpha \geq d$ holds. Now assume that $\sigma(h) > d$. Set $h(z) = g(z)/u_1(z)$, where $g(z)$ and $u_1(z)$ are defined in the same way as in the proof of Lemma 2.8. Using the same reasoning as in Lemma 2.8, we have $\sigma(u_1) \leq d$ and

$$|u_1(z)| \leq \exp\{r^{d+\epsilon}\} \ (r \to \infty). \tag{2.36}$$

And there is a set $E_4 \subset (1, +\infty)$ that has finite logarithmic measure, such that (2.16) holds for $|z| = r \notin E_4[0, 1]$. For any given $\epsilon(0 < 2\epsilon < \min\{\alpha - d, \beta - d\})$, there is a set $E_1 \subset (1, +\infty)$ that has finite logarithmic measure such that for $|z| = r \notin E_1 \cup [0, 1]$, $r \to \infty$,

$$|U(z)| \leq \exp\{r^{d+\epsilon}\} \tag{2.37}$$

holds. By (2.34) and the fact that for $m = 1, \cdots, k$,

$$\frac{h^{(m)}}{h} = \sum_{j=0}^{m} C_m^j u_1(z)(u_1^{-1}(z))^{(j)} \frac{g^{(m-j)}}{g} = \frac{g^{(m)}}{g} + \sum_{j=1}^{m} C_m^j u_1(z)(u_1^{-1}(z))^{(j)} \frac{g^{(m-j)}}{g}, \tag{2.38}$$

where C_m^j $(m = 1, \cdots, k;\ j = 0, \cdots, m)$ are the binomial coefficients, we obtain

$$\frac{g^{(k)}}{g} + \sum_{j=1}^{k} [C_k^j u_1(u_1^{-1})^{(j)} + A_{k-1}C_{k-1}^{j-1} u_1(u_1^{-1})^{(j-1)} + \cdots$$
$$+ A_{k-j+1}C_{k-j+1}^1 u_1(u_1^{-1})' + A_{k-j}] \frac{g^{(k-j)}}{g} = \frac{Uu_1}{g}. \tag{2.39}$$

By Lemma 2.6 and the hypotheses, there is a set $E_3 \subset (1, +\infty)$ that has finite logarithmic measure such that for $|z| = r \notin [0,1] \cup E_3$ we have

$$|C_k^j u_1(z)(u_1^{-1}(z))^{(j)}| \le r^{j(d-1+\epsilon)},$$
$$|A_{k-1}(z)C_{k-1}^{j-1} u_1(z)(u_1^{-1}(z))^{(j-1)}| \le r^{(j-1)(d-1+\epsilon)+1(\beta-1)},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$|A_{k-j+1}(z)C_{k-j+1}^{1} u_1(z)(u_1^{-1}(z))'| \le r^{1\cdot(d-1+\epsilon)+(j-1)(\beta-1)}.$$

Since $A_{k-j}(z) = a_{k-j}z^{j(\beta-1)}(1+o(1))$ ($a_{k-j} \neq 0$, is a constant) and $\beta > d+\epsilon$, we have

$$\begin{aligned} &C_k^j u_1(z)(u_1^{-1}(z))^{(j)} + \cdots + A_{k-j+1}(z)C_{k-j+1}^1 u_1(z)(u_1^{-1}(z))' + A_{k-j}(z) \\ &= a_{k-j}z^{j(\beta-1)}(1+o(1)) \end{aligned} \tag{2.40}$$

for $|z| = r \in (1,+\infty) - E_3,\ r \to \infty$. By (2.39), (2.40) and Lemma 2.7, we know that $\sigma(g) = \alpha < \infty$. By Lemma 2.2, there is a set $E_2 \subset (1,+\infty)$ that has infinite logarithmic measure, such that

$$\lim_{\substack{r\to\infty\\ r\in E_2}} \frac{\log\log M(r,g)}{\log r} = \lim_{\substack{r\to\infty\\ r\in E_2}} \frac{\log \nu_g(r)}{\log r} = \alpha. \tag{2.41}$$

By (2.36), (2.37), (2.41), we have

$$M(r,g) \ge \exp\{r^{\alpha-\epsilon}\},$$
$$\Big|\frac{u_1(z)\cdot U(z)}{g(z)}\Big| = \Big|\frac{u_1(z)U(z)}{M(r,g)}\Big| \le \exp\{2r^{d+\epsilon} - r^{\alpha-\epsilon}\} \to 0 \tag{2.42}$$

for $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4),\ |g(z)| = M(r,g),\ r \to \infty$. By (2.39), (2.40), (2.42) and (2.16), we have

$$\begin{aligned} &\Big(\frac{\nu_g(r)}{z}\Big)^k (1+o(1)) + a_{k-1}z^{\beta-1}\Big(\frac{\nu_g(r)}{z}\Big)^{k-1}(1+o(1)) + \cdots \\ &+ a_1 z^{(k-1)(\beta-1)}\Big(\frac{\nu_g(r)}{z}\Big)(1+o(1)) + a_0 z^{k(\beta-1)}(1+o(1)) = o(1) \end{aligned} \tag{2.43}$$

for $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4),\ |g(z)| = M(r,g),\ r \to \infty$. For $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4),\ r \to \infty$, we have by (2.41)

$$\nu_g(r) = r^{\alpha+o(1)}. \tag{2.44}$$

By (2.44), and since ϵ is arbitrarily small, we know that the degrees of all terms of the left of (2.43) are respectively

$$k(\alpha-1), (k-j)(\alpha-1) + (\beta-1)j \quad (j = 1, \cdots, k).$$

By the Wiman-Valiron theory, we get $\alpha = \beta$, i.e., $\sigma(h) = \sigma(g) = \beta$.

Using the same manner as above, we can prove that if $h(z) \not\equiv 0$ is a meromorphic solution of the corresponding homogeneous equation (2.35) of (2.34), then $\sigma(h) = \beta$.

If h_0 and h_1 ($h_1 \not\equiv h_0$) are both meromorphic solutions of (2.34) with $\sigma(h_0) = \sigma(h_1) = \sigma(U) = d < \beta$, then $\sigma(h_1 - h_0) < \beta$. But $h_1 - h_0 \not\equiv 0$ is a meromorphic solution of the corresponding homogeneous equation (2.35) of (2.34). By the above proof, we have $\sigma(h_1 - h_0) = \beta$. This is a contradiction. Therefore (2.34) has at most one exceptional meromorphic solution h_0 with $\sigma(h_0) = \sigma(U)$.

Lemma 2.10. *Suppose that B is a rational function and $H \not\equiv 0$ is a meromorphic* *function with* $\sigma(H)=\beta<\infty$*. If* $f(z)$ *is a meromorphic solution of* (1.1)*, then*

$$\max\{\lambda(f),\lambda(1/f)\}\geq\max\{\lambda(H),\lambda(1/H)\}. \tag{2.45}$$

Proof. Set $f=\frac{g}{u_1}$, where g and u_1 are defined in the same way as in the proof of Lemma 2.8. Using the same proof as in Lemma 2.8, we can prove that (2.26)–(2.28) hold. By Lemma 2.7, we know that $\sigma(g)<\infty$. hence $\sigma(f)<\infty$.

Now f and H can be written as

$$f(z)=z^{m_1}\frac{h_1(z)}{u_1(z)}e^{P_1(z)},\quad H(z)=z^{m_2}\frac{h_2(z)}{u_2(z)}e^{P_2(z)}, \tag{2.46}$$

where m_1,m_2 are integers, $h_1(z)$ and $h_2(z)$ are canonical products (or polynomials) formed respectively by the nonzero zeros of f and H, u_1 and u_2 are canonical products (or polynomials) formed respectively by the nonzero poles of f and H, P_1 and P_2 are polynomials with deg $P_1\leq\sigma(f)$, deg $P_2\leq\sigma(H)$. Substituting (2.46) into (1.1), we obtain

$$F(h_1,u_1)=z^{m_2}\frac{h_2}{u_2}e^{P_2-P_1}, \tag{2.47}$$

where F is a rational function in h_1,u_1. and $h_1^{(j)},u_1^{(j)}$ $(j=1,\cdots,k)$, with polynomial coefficients. (2.47) gives

$$\max\{\sigma(h_1),\sigma(u_1)\}\geq\sigma(F)=\sigma\Big(z^{m_2}\frac{h_2}{u_2}e^{P_2-P_1}\Big)\geq\max\{\sigma(h_2),\sigma(u_2)\}.$$

So (2.45) holds.

Lemma 2.11.[2] *Suppose that* $B_0,\cdots,B_{k-1}$ *are rational functions and* $H\not\equiv0$ *is a meromorphic function. If* $f(z)$ *is a meromorphic solution of the equation*

$$f^{(k)}+B_{k-1}f^{(k-1)}+\cdots+B_0f=H,$$

and $\sigma(H)<\sigma(f)<\infty$*, then* $\bar\lambda(f)=\lambda(f)=\sigma(f)$*.*

§3. Proof of Theorems

Proof of Theorem 1.1. (a) By Lemma 2.8, we know that all meromorphic solutions f of (1.1) satisfy $\sigma(f)=(n+k)/k$, with at most one possible exceptional meromorphic solution f_0 with $\sigma(f_0)=\beta$. By Lemma 2.11, the meromorphic solutions f satisfying $\sigma(f)=(n+k)/k>\beta$ satisfy

$$\bar\lambda(f)=\lambda(f)=\sigma(f)=(n+k)/k.$$

(b) By Lemma 2.8 and Lemma 2.10, it follows that (b) holds.

Proof of Theorem 1.2. (a) From Lemma 2.8, $\sigma(f)=\beta$ holds.

(b) If $\beta=\lambda(H)>\lambda(1/H)$, by Lemma 2.10 and (2.23) in proof of Lemma 2.8, it is easy to see that $\lambda(f)=\beta$.

(c) If $\beta>\max\{\lambda(H),\lambda(1/H)\}$, then set $H=Ue^p$, where $U=z^s\frac{V_1}{V_2}$(s is an integer, V_1 and V_2 are canonical products (or polynomials) formed respectively by the nonzero zeros and nonzero poles of H,

$$\sigma(U)=\max\{\lambda(H),\lambda(1/H)\}<\beta,$$

and P is a polynomial with deg$P=\beta$. Now set $f=h\cdot e^p$. Then $f(z)$ and $h(z)$ have the same zeros and poles. Substituting $f=he^p$, $H=Ue^p$ into (1.1), we have

$$h^{(k)}+A_{k-1}h^{(k-1)}+\cdots+A_0h=U, \tag{3.1}$$

where A_{k-j} $(j=1,\cdots,k)$ are rational functions. To work out the order of pole at ∞ of A_{k-j}, by induction, we have for $m\geq 2$ (see [7]),

$$f^{(m)}=\{h^{(m)}+mP'h^{(m-1)}+\sum_{j=2}^{m}[C_m^j(P')^j+H_{j-1}(P')h^{(m-j)}\}e^p, \tag{3.2}$$

where $H_{j-1}(P')$ are differential polynomials in P' and its derivatives of total degree $j-1$ with constant coefficients. It is easy to see that the derivatives of $H_{j-1}(P')$ with respect to z are of the same form as $H_{j-1}(P')$. C_m^j are binomial coefficients. (3.2) and (1.1) give

$$A_{k-1}=kP',\quad A_{k-j}=C_k^j(P')^j+H_{j-1}(P')\quad (j=2,\cdots,k-1),$$
$$A_0=C_k^k(P')^k+H_{k-1}(P')+B.$$

Obviously, A_{k-j} $(j=1,\cdots,k-1)$ are polynomials and deg $A_{k-j}=j(\beta-1)$. By $\beta>(n+k)/k$, the rational function A_0 has a pole at ∞ of order $k(\beta-1)$. By Lemma 2.9, all meromorphic solutions of (3.1) satisfy $\sigma(h)=\beta>\sigma(U)$, except at most one possible exceptional one. The possible meromorphic one h_0 satisfies $\sigma(h_0)=\sigma(U)$. By Lemma 2.11, it follows that $\bar{\lambda}(h)=\lambda(h)=\sigma(h)=\beta$. Therefore, (1,1) has at most one possible exceptional solution $f_0=h_0e^p$, f_0 satisfies (1.4) by Lemma 2.10. All other meromorphic solutions $f=he^p$ of (1.1) satisfy (1.3).

§4. Examples for the Exceptional Solution

Example 4.1. The equation

$$f''+(z^2+1/z^2)f=(2z^2+\cos^2 z)/(z^2\cos^3 z)$$

satisfies the hypotheses of Theorem 1.1(a), and has solution $f_0(z)=\sec z$, such that $\sigma(f_0)=\sigma(H)=1<(n+2)/2$.

Example 4.2. The equation

$$f''-(4z+2/z^2)f=(4z^2-2-2/z-2/z^2)\exp(z^2)$$

satisfies the hypotheses of Theorem 1.2(c), and has solution $f_0=((1/z)+1)\exp(z^2)$, such that $\lambda(f_0)=\lambda(1/f_0)=0,\ \sigma(f_0)=2$.

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