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THE PROOF OF THE NON-EXISTENCE OF LIMIT CYCLES FOR A QUADRATIC DIFFERETIAL SYSTEM***

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Abstract

the authors give some results by using Dulac function method to prove the non-existence of limit cycles for a quadratic differetial system.

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In many cases the Dulac function method is very useful in the proof of non-existence or non-coexistence of limit cycles (LC, for abbreviation) for quadratic differential systems (QDS, for abbreviation). See the following two examples.

Example 1.[1, §16] For the system

$$\begin{cases} \dot{x} = -y + lx^2 + mxy + ny^2 = P(x, y), \\ \dot{y} = x(1 + ax + by) = Q(x, y), \end{cases}$$

assume

$$m(l + n) - a(b + 2l) \neq 0.$$

If (a) $m^2 + 4n(n + b) \geq 0$, i.e., $N(0, \frac{1}{n})$ is a saddle or node, put

$$\sigma = \sqrt{m^2 + 4n(n + b)},$$

$$L_+ = (m + \sigma)(ny - 1) - 2n(n + b)x, \quad L_- = (m - \sigma)(ny - 1) - 2n(n + b)x,$$

then the Dulac function

$$B_1 = L_+^{-1+\frac{(\sigma+m)b}{2n\sigma}} L_-^{-1+\frac{(\sigma-m)b}{2n\sigma}} \tag{2}$$

gives

$$\frac{\partial(B_1 P)}{\partial x} + \frac{\partial(B_1 Q)}{\partial y} = [Cx + A(1 - ny)]x^2 B_1 L_+{}^{-1} L_-{}^{-1}, \tag{3}$$

where

$$A = m(l + n) - a(b + 2l) \neq 0, \ C = am(2l + b + n) - (2l + b)(n + b)(n + l). \tag{4}$$

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Since $Cx + A(1 - ny) = 0$ passes through $N(0, \frac{1}{n})$ but not $O(0, 0)$, around O and the anti-saddle lying on $1 + ax + by = 0$ LC cannot co-exsit, otherwise there would be three contact points on the line $Cx + A(1 - ny) = 0$, this is impossible.

If (b) $m^2 + 4n(n + b) < 0$, i.e., $N(1, \frac{1}{n})$ is a focus, then LC may exist both around O and N, as was shown by the famous example of (1,2) distribution in [2]. Now assume that aside from O and N there exists a third finite critical point $M(x_1, y_1)$. Putting $n = 1$, without loss of generality, and

$$L'_i = y - y_1 - \theta_i(x - x_1), \quad i = 1, 2,$$

where θ_i are roots of

$$(-1 + mx_1 + 2y_1)\theta^2 + [(2l - b)x_1 + my_1]\theta - ax_1 = 0,$$

and using the Dulac function

$$B_2 = [y - y_1 - \theta_1(x - x_1)]^{k_1}[y - y_1 - \theta_2(x - x_1)]^{k_2},$$

where

$$k_1 = \frac{(\theta_1 x_1 - y_1)[m - (b + 2l)\theta_2]}{\theta_2 - \theta_1}, \quad k_2 = \frac{(\theta_2 x_1 - y_1)[m - (b + 2l)\theta_1]}{\theta_1 - \theta_2},$$

we have

$$\frac{\partial(B_2 P)}{\partial x} + \frac{\partial(B_2 Q)}{\partial y} = K{L'_1}^{k_1 - 1}{L'_2}^{k_2 - 1}(y_1 x - x_1 y)^2(A'x + B'y + C'), \tag{5}$$

where

$$A' = al(b + 2l)\frac{x_1}{y_1} + am(1 + b + 2l),$$
$$B' = \frac{my_1}{x_1} + mb + a(b + 2l) - \frac{m}{x_1^2} + \frac{m^2}{x_1},$$
$$C' = m(l + 1) - a(b + 2l) \neq 0.$$

It can be proved that $A'x + B'y + C' = 0$ passes through the point (x_1, y_1), so around O and N LC cannot co-exist.

Example 2. In [3] we use the Dulac function

$$B_3 = (y - kx - \frac{1}{n})^{\frac{m}{nk}}(1 - y)^{-1 + 2l - \frac{m}{k}}$$

to study the non-existence of LC around O or $S_1(x_1, y_1)$ for the quadratic system (1), in which we assume[1]

$$b = -1, n > 1, l < 0, a < 0$$

and k is a root of the equation

$$nk^2 + mk + 1 - n = 0$$

or

$$k = k_i = \frac{-m \mp \sqrt{m^2 + 4n(n - 1)}}{2n}, \quad i = 1, 2,$$

[1]Under these conditions $N(0, \frac{1}{n})$ is a saddle. Let $S_i(x_i, y_i)$ be the two critical points on 1+ax-y=0. Then they are both anti-saddles (with $x_1 > 0, x_2 < 0$) when $na^2 + ma + l < 0$. If $na^2 + ma + l > 0$, then $S_1(x_1, y_1)$ is still an anti-saddle, but $S_2(x_2, y_2)$ becomes a saddle with $x_2 > x_1 > 0$. When $na^2 + ma + l = 0, S_2$ goes to infinity.

which determines the slopes of tangents L and L':

$$y - k_i x - \frac{1}{n} = 0$$

of separatrices at $N(0, \frac{1}{n})$. Now

$$\frac{\partial(B_3P)}{\partial x} + \frac{\partial(B_3Q)}{\partial y} = B_3\Big(y - kx - \frac{1}{n}\Big)^{-1}(1-y)^{-1}x^2E(x,y),$$

where

$$E(x,y) = \frac{m}{nk}[a-(l+n)k](1-y) + a\Big(1-2l+\frac{m}{k}\Big)\Big(y-kx-\frac{1}{n}\Big).$$

We can rewrite E(x,y)=0 as

$$\overline{L}: \frac{a(2l-1)-m(l+n)}{n} + \frac{am(n-1)+k_1[m(l+n)-an(2l-1)]}{nk_1}y - a[m+k_1(1-2l)]x = 0 \tag{6}$$

or

$$\overline{L'}: \frac{a(2l-1)-m(l+n)}{n} + \frac{am(n-1)+k_2[m(l+n)-an(2l-1)]}{nk_2}y - a[m+k_2(1-2l)]x = 0. \tag{7}$$

Here $\overline{L}$ ($\overline{L'}$) passes through the intersection point $R(R')$ of $y = 1$ and $L(L')$. The two straight lines $y = 1$ and L (or L') divide the (x, y) plane into four regions $G_1, \cdots, G_4$ (or $G'_1, \cdots, G'_4$). Since trajectories all go from one side of $y = 1$ (or L, L') to the other side (except contact at $(0, 1)$ and $(0, \frac{1}{n})$ resp.), LC cannot intersect $y = 1$ and $L(L')$ and exist in any region $G_i(G'_i)$, if $\overline{L}(\overline{L'})$ does not meet it.

Notice that in these two examples the assumption

$$m(l+n) - a(b+2l) \neq 0,$$

i.e., $W_1 \neq 0$, or $O(0, 0)$ is a weak focus of order 1, is important. If $A = 0$ in (3), then the expression in the [1] is Cx, so the conclusion in Example 1 (a) is not valid. If $C' = 0$ in (5), then the straight line $A'x + B'y = 0$ passes through O and (x_1, y_1), so around $N(0, 1)$ no LC exists. We get the following

Theorem 1. *If in* (1), $O(0, 0)$ *is a weak focus of order* ≥ 2, $N(0, \frac{1}{n})$ *is also a focus, and there exists a third finite critical point* (x_1, y_1), *then around* N *no LC can exist.*

Maybe, we can prove more: (1,2) distribution of LC can occur for (1) only when there are two finite critical points (foci).

Finally, if $W_1 = 0$ in (6) or (7), then $\overline{L}(\overline{L'})$ passes through $O(0, 0)$, and we can get no conclusion on the non-existence of LC around O.

Now, let us study the quadratic system

$$\begin{cases} \dot{x} = -y + lx^2 + xy, \\ \dot{y} = x(1 + ax + by) \end{cases} \tag{8}$$

under the condition[2]

$$b \neq 0, \quad a \neq 0, \quad \frac{1}{5}; \quad l - a(b+2l) = 0, \quad bl - a \neq 0. \tag{9}$$

[2]The first equality means $W_1 = 0$, under which $W_2 = al(5a-1)(bl-a)$, so the last inequality means $W_2 \neq 0$.

In [4] it was proved that (8) has no LC around $O(0,0)$ when

$$a < 0 \quad \text{or} \quad a > \frac{1}{5}. \tag{10}$$

The method is to transform (8) into $\dot{x} = y,\ \dot{y} = -g(x) - f(x)y$, then to prove that

$$F(z) = F(w) \textit{ and } G(z) = G(w) \ \Big(\text{where } F(z) = \int_0^z f(\xi)d\xi, G(z) = \int_0^z g(\xi)d\xi\Big)$$

have no intersection point in the region $-\infty < z < 0, 0 < w < 1$.

Our problem is : Can we prove this result by the Dulac function method? Let us take a Dulac function

$$B(x,y) = (x-1)^{\alpha}(1+by)^{\beta}, \tag{11}$$

then we get

$$\frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} = (x-1)^{\alpha-1}(1+by)^{\beta-1}R(x,y),$$

where $R(x,y)$ is a cubic polynomial without constant term. In order that $R(x,y)$ also contains no linear terms, we must take $\alpha = 1, \beta = -\frac{b+2l}{b}$, and then by (9)

$$R(x,y) = x^2[ab + 2al - l - a(b+2l)x - bly] = -x^2[a(b+2l)x + bly].$$

This shows that $R(x,y)$ will change sign in any neighbourhood of $O(0,0)$, so we cannot ensure the non-existence of LC even in a small neighbourhood of O.

Notice that there are two important special cases of (8):

Case 1. When $\frac{a}{l} = b$, i.e., $a = bl$, (8) can be transformed into a quadratic system of type (I) by a linear transformation. There are only two finite critical points $O(0,0)$ and $(\frac{1}{1-a}, \frac{1}{b(a-1)})$. So, if $W_1 = 0$, then $W_2 = W_3 = 0, O$ is a center.

Case 2. When

$$b < 0, \qquad (b-l)^2 + 4a < 0, \tag{12}$$

(8) is a bounded system. There is only one critical point at infinity $(0,1,0)$, and in general there are three finite critical points, one saddle and two anti-saddles. Since

$$(b-l)^2 + 4a = (b+l)^2 + 4(a-bl),$$

(12) gives

$$a - bl < -\frac{(b+l)^2}{4} \le 0.$$

Consequently, if $a - bl \ge -\frac{(b+l)^2}{4}$, then (8) is not a bounded quadratic system. In case $a - bl > 0$, aside from $O(0,0)$, there are two sadddles; moreover, there are three critical points at infinity.

The bad situation of using Dulac function (11) cannot be improved if we use more complicated form of Dulac function with three linear factors.

For example, for the system

$$\begin{cases} \dot{x} = -y + \dfrac{3x^2}{4} + xy, \\ \dot{y} = x\Big(1 + x - \dfrac{3y}{4}\Big), \end{cases} \tag{13}$$

$O(0,0)$ is a stable weak focus of order 2, $S_1(\frac{4}{5},\frac{12}{5})$ and $S_2(-\frac{4}{5},\frac{4}{15})$ are saddles. There are nine lines having the property discribed before for L and L' in Example 2:

$$L_1: y-k_1x-l_1=0, L_2: y-k_2x-l_2=0, k_{1,2}=\frac{-23\pm 5\sqrt{73}}{54}, l_i=\frac{4(3k_i+1)}{15},$$

$$L_3: y-k_3x-l_3=0, L_4: y-k_4x-l_4=0, k_{3,4}=\frac{21\pm 5\sqrt{17}}{2}, l_i=\frac{4(3-k_i)}{5},$$

these are tangent lines of separatrices at S_2 and S_1;

$$L_5: 4y+3x+3=0, \quad L_6: 2y-x-2=0,$$
$$L_7: 3y-4=0, \quad L_8: y+2x+4=0, \quad L_9: x-1=0.$$

For any

$$B(x,y)=L_i^\alpha L_j^\beta L_k^\gamma, \quad i\neq j\neq k, \tag{14}$$

when we adjust α,β,γ so that $\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}$ contains no linear terms in x, y, we will always see that the coefficients of x^2 and y^2 are equal in absolute value but opposite in sign. So (14) cannot be used to prove the non-existence of LC in any small neighbourhood of O.

If we take

$$B(x,y)=e^{\alpha(3x+4y)}L_i^\beta,$$

the result will be similar. But for

$$\dot{x}=-y+x^2+xy, \quad \dot{y}=x(1+2x+2y),$$

which belongs to Case 1, the Dulac function

$$B(x,y)=e^{\alpha(y-2x)}\Big(y-kx-k-\frac{1}{2}\Big)^\gamma,$$

where

$$k=\frac{-1-\sqrt{33}}{8}\geq -0.843, \quad \alpha=\frac{1-4k}{k-2}, \quad \gamma=\frac{7(2k+1)}{2(2-k)},$$

gives

$$\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}$$
$$=\frac{7e^{\alpha(y-2x)}}{(k-2)}\Big(y-kx-\frac{1}{2}-k\Big)^{\gamma-1}\Big[\Big(k^2-\frac{1}{2}k-1\Big)x^2-\Big(\frac{3k}{2}+1\Big)xy-ky^2\Big]\leq 0$$

for all x, y, so there is no LC around O.

Now, for the system (8) with O as a weak focus of order 2, let us take a quartic polynomial

$$B(x,y)=1+a_1x+b_1y+a_2x^2+b_2xy+c_2y^2+a_3x^3+b_3x^2y$$
$$+c_3xy^2+d_3y^3+a_4x^4+b_4x^3y+c_4x^2y^2+d_4xy^3+e_4y^4 \tag{15}$$

as a Dulac function. We get

$$\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}=F(x,y), \tag{16}$$

where $F(x,y)$ is a polynomial of fifth degree.

If we set the coefficients of x and y in $F(x,y)$ equal to zero, we get

$$a_1=1, \ b_1=-b-2l.$$

By this, we can find that the coefficient of x^2 in $F(x,y)$ is

$$b+3l+b_2-a(b+2l)=b+2l+b_2+W_1$$

and the coefficient of y^2 in $F(x,y)$ is $-b-2l-b_2$.

Since in system (13), $W_1=0$, we see the reason why for (14) the coefficients of x^2 and y^2 in $\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}$ are always equal in absolute value but opposite in sign.

If we set now also the coefficients of xy, y^2, x^3, x^2y, xy^2 and y^3 in $F(x,y)$ equal to zero, we can determine the values of a_2, b_2, a_3, b_3, c_3 and d_3 in (15). Since $W_1=0$, the coefficient of x^2 in $F(x,y)$ is also zero. Next let us set the coefficients of x^4 and y^4 in (16) equal to A and B respectively, meanwhile set the coefficients of x^3y and xy^3 equal to zero. Then we can determine the values of a_4, b_4, d_4 and e_4 in (15). By using these values, we can get the coefficient of x^2y^2 in $F(x,y)$ as

$$-3(A+B)+6c_2[l-a(b+2l)]+(bl-a)(3l-b).$$

Since $W_1=l-a(b+2l)=0$ and $W_2=al(5a-1)(bl-a)$, this expression is

$$-3(A+B)+\frac{W_2}{a^2}, \tag{17}$$

and

$$\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}=Ax^4+\Big[\frac{W_2}{a^2}-3(A+B)\Big]x^2y^2+By^4+G(x,y), \tag{18}$$

where

$$G(x,y)=k_1x^5+k_2x^4y+k_3x^3y^2+k_4x^2y^3+k_5xy^4+k_6y^5.$$

Moreover we can take the same Dulac function (15) for the system

$$\begin{cases}\dot{x}=-y+lx^2+mxy+ny^2,\\ \dot{y}=x(1+ax+by),\end{cases} \tag{19}$$

and let $\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}$ contain no linear terms in x and y. Then we get similar results about the coefficients of x^2 and y^2, which are

$$b_2-mn+m(b+2l)+m(l+n)-a(b+2l)=b_2-mn+m(b+2l)+W_1$$

and $-b_2+mn-m(b+2l)$, respectively. This is the reason why many Dulac functions in [1] and [5] are not available in the case $W_1=0$.

When we apply the same method as above to the coefficients in $\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}$, we get the coefficient of x^2y^2 as

$$-3(A+B)+6c_2[m(l+n)-a(b+2l)]+K, \tag{20}$$

where

$$\begin{aligned}K=&\,5ab^3+35ab^2l+70abl^2+40al^3+5a^2bm+10a^2lm-6b^2lm\\&-22bl^2m-20l^3m+2abm^2-6alm^2-lm^3+2ab^2n-5a^2mn-6b^2mn\\&-20blmn-19l^2mn-9am^2n-m^3n+2bmn^2+6lmn^2+5mn^3\end{aligned}$$

and c_2 is an undetermined coefficient. If $W_1=m(l+n)-a(b+2l)=0$ and $a\neq 0$, we can rewrite (20) as

$$-3(A+B)+\frac{am(5a-m)[(l+n)^2(b+n)-a^2(b+2l+n)]}{a^2}=-3(A+B)+\frac{W_2}{a^2}. \tag{21}$$

So, for the $\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}$ of system (19), it is similar to (18) under the condition $W_1=0$. By the above results we have the following two theorems.

Theorem 2. *For system* (19) *or* (8), *if O is a weak focus of order* 2, *then there is no* LC *in the small neighborhood around O.*

Proof. In (18), since $W_2\neq 0$, on taking $A=B=\frac{W_2}{10a^2}$, we have

$$\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}=\frac{W_2(x^4+4x^2y^2+y^4)}{10a^2}+G(x,y). \tag{22}$$

It is easy to prove that this polynomial has definite sign in a small neighborhood. around O, so there is no LC.

In the above proof, we only use $W_1=0$. If we notice the fact that c_2 is an undetermined coefficient, we have

Theorem 3. *For system* (19), *if O is a weak focus of order* 1, *then there is no* LC *in the small neighborhood around O.*

Proof. Since $W_1\neq 0$, we have

$$\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}=W_1x^2+Ax^4+[-3(A+B)+6c_2W_1+K]x^2y^2+By^4+G(x,y).$$

Here K is a certain number for the system (19). If we take $A=B=W_1$ and $c_2=\frac{7W_1-K}{6W_1}$, we have

$$\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}=W_1(x^2+x^4+x^2y^2+y^4)+G(x,y). \tag{23}$$

It is clear that this polynomial has definite sign in a small neighborhood around O, so there is no LC in this neighborhood.

Remark. Although we have got expressions of the divergence like (22) and (23) in the neighborhood of O when it is a weak focus of order 2 and 1 respectively, but the meaning is not significant. Because the range of the neighborhood depends on the absolute value of W_2 or W_1. We may compare these formulae with formula (9.78) in [1].

In order to prove that for system (8) there is no LC in the large around O, now we are in the situation that for the $F(x,y)$ in (16) there are 6 coefficients in $G(x,y)$ which are undetermined, and in (15) there are two coefficients c_2 and c_4 still undetermined. So it is necessary that we must adjust c_2, c_4, A and B so that (18) becomes a definite sign polynomial in the large around O.

For the system (13), by the help of computer, let us take

$$\begin{aligned}B(x,y)=&1+x-0.75y+0.8449x^2-0.75xy-0.1551y^2\\&+0.4506x^3-1.151025x^2y-0.1551xy^2-0.306125y^3+0.1099125x^4\\&-0.480775x^3y+0.33699375x^2y^2+0.393875xy^3+0.13125y^4.\end{aligned}$$

Then

$$\begin{aligned}&\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}\\&=-0.28x^4-1.7475x^2y^2-0.7y^4-0.068603125x^5+0.14180625x^4y\\&\quad-0.488729687x^3y^2+1.240575x^2y^3+0.4924375xy^4+0.13125y^5.\end{aligned}$$

Now, the main point is using L_6, L_7 and L_9 mentioned before. We can rewrite it into

$$\begin{aligned}
&\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}\\
&=(2+x-2y)(-0.068603125x^4-0.488729687x^2y^2)\\
&\quad+\Big(1-\frac{3y}{4}\Big)(-0.0061\dot{3}x^4-0.350820834\dot{6}x^2y^2-0.175y^4)-(1-x)(0.4924375y^4)\\
&\quad+(-0.13666041\dot{6}x^4-0.419219791\dot{3}x^2y^2-0.0325625y^4).
\end{aligned}$$

It is easily seen that the right side of $\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}$ is negative in the shadowed region in Fig.1, so there is no LC in the large around O.

We have proved the non-existence of LC for the system (8) in the special case (13).

Conjecture. The result got in [4] can be proved by Dulac function method for any l, a and b satisfying (9) and (10).

Fig.1

References

[1] Ye Yanqian and others, Theory of limit cycles, Trans. Math. Monograghs, **66**, 1986, Amer. Math. Soc..

[2] Chen Lansun & Wang Mingshu, Relative position and number of limit cycles of quadratic differential system, *Acta Math. Scinica,* **22**(1979), 751–758.

[3] Ye Yanqian, Qualitative theory of the quadratic differential system (I), *Ann. of Diff. Eqs.,* **13:**4(1997), 395–407.

[4] Cai Suilin & Wang Zhongwei, A quadratic system with second order fine focus, *Chin. Ann. of Math.,* **5A:6**(1984), 765–770.

[5] Ye Yanqian, Qualitative theory of polynomial differential system, Shanghai Scientific and Technical Publishers, 1995.