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FINITENESS CONDITIONS FOR GENERALIZED EXPONENTS OF DIGRAPHS**

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Abstract

Necessary and sufficient conditions are given for the finiteness of the generalized exponents $\exp_D(k)$, $f(D,k)$ and $F(D,k)$ for digraphs which are not necessarily primitive. Also the largest finite value of the generalized exponent $\exp_D(k)$ for digraphs of order n is determined and the complete characterizations of the extreme digraphs are given.

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§1. Introduction

R. A. Brualdi and Bolian Liu introduced in 1990[1] the concept of generalized exponents for primitive digraphs. This concept is a generalization of the traditional concept of the exponents for primitive digraphs and has backgrounds in memoryless communication systems associated with digraphs. In this paper we show that generalized exponents can also be defined for digraphs which are not necessarily primitive. We will give necessary and sufficient conditions for the finiteness of the generalized exponents $\exp_D(k)$, $f(D,k)$ and $F(D,k)$. These ideas and results suggest that the study of generalized exponents will not be necessarily restricted in the scope of the primitive digraphs. As an example, we also obtain in §4 the largest finite value of the generalized exponents $\exp_D(k)$ for D ranging over all digraphs of order n (which are not necessarily primitive) with the finite $\exp_D(k)$ value, and give the complete characterizations of the extreme digraphs with the largest finite $\exp_D(k)$ value.

Let D be a digraph in which loops (cycles of length one) are permitted. D is called primitive if there exists a positive integer k such that for each ordered pair of vertices x and y (not necessarily distinct) there is a walk of length k from x to y. The smallest such k is called the exponent of D, denoted by $\gamma(D)$.

It is well known that a digraph D is primitive if and only if D is strongly connected and the greatest common divisor of the lengths of its cycles is 1.

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Definition 1.1. *Let D be a digraph and x be a vertex of D. The "vertex exponent" $\gamma_D(x)$ is defined to be the smallest positive integer p such that there are walks of length p from x to all vertices of D. If no such a p exists, then we define $\gamma_D(x) = \infty$.*

Definition 1.2.[1] *Let D be an arbitrary digraph of order n. If we choose to order the n vertices $v_1, v_2, \cdots, v_n$ of D in such a way that*

$$\gamma_D(v_1) \leqslant \gamma_D(v_2) \leqslant ... \leqslant \gamma_D(v_n),$$

then we call $\gamma_D(v_k)$ the k^{th} generalized exponent of D, denoted by $exp_D(k)$.

It is obvious that

$$\exp_D(1) \leqslant \exp_D(2) \leqslant \cdots \leqslant \exp_D(n). \tag{1.1}$$

If D is primitive, then $\exp_D(n) = \gamma(D)$. So the concept of the generalized exponents is a generalization of the traditional concept of the exponents for primitive digraphs.

Definition 1.3.[1] *Let D be a digraph and $X \subseteq V(D)$ be a subset of $V(D)$. The "set exponent" $exp_D(X)$ is defined to be the smallest positive integer p such that for each vertex y of D there exists a walk of length p from at least one vertex in X to y. If no such a p exists, then we define* $\exp_D(X) = \infty$.

It is not difficult to verify that if for each vertex y in D there exists a walk of length p from at least one vertex in X to y, then for each vertex z in D there also exists a walk of length $p+1$ from at least one vertex in X to z.

We have the following relation between the vertex exponent and set exponent:

$$\gamma_D(x) = \exp_D(\{x\}). \tag{1.2}$$

Definition 1.4.[1] *Let D be an arbitrary digraph of order n and $1 \leqslant k \leqslant n$. Then we define*

$$f(D,k) = \min\{\exp_D(X) \mid X \subseteq V(D) \text{ and } \mid X \mid = k\} \tag{1.3}$$

$$F(D,k) = \max\{\exp_D(X) \mid X \subseteq V(D) \text{ and } \mid X \mid = k\}. \tag{1.4}$$

$f(D,k)$ and $F(D,k)$ are called the "k^{th} lower generalized exponent"and "k^{th} upper generalized exponent"of D, respectively.

It is easy to see from (1.2) that

$$f(D,1) = \exp_D(1) \quad , \quad F(D,1) = \exp_D(n). \tag{1.5}$$

If D is primitive, then all the three types of the generalized exponents $\exp_D(k), f(D,k)$ and $F(D,k)$ are finite. But for an arbitrary digraph D they are not necessarily finite. In §3 of this paper, we will give the necessary and sufficient conditions for the finiteness of the generalized exponents $\exp_D(k), f(D,k)$ and $F(D,k)$ which will allow us to study the generalized exponents for digraphs which are not necessarily primitive later.

We would also like to point out that, by using the basic connections between digraphs and matrices, all the concepts and results in this paper can be expressed in the corresponding matrix version.

§2. Preliminaries

In this section we give some graph theoretical preliminaries about the "imprimitive partition"of a nontrivial strongly connected digraph and the "condensation digraph"of a digraph, and give some number theoretical preliminaries about the Frobenius set and Frobenius number of a given set of positive integers.

A digraph D is called a trivial digraph, if D contains no arc (a loop is also considered as an arc). Otherwise D is called nontrivial. It is obvious that a nontrivial strongly connected digraph always contains a cycle.

Definition 2.1. *Let D be a nontrivial strongly connected digraph. Then the greatest common divisor (abbreviated as "g.c.d.") of the lengths of all cycles of D is called the period of D, denoted by $p(D)$.*

Since any cycle of D is a closed walk and the length of any closed walk is a sum of the lengths of some cycles of D, we conclude that the period $p(D)$ is also the g.c.d. of the lengths of all closed walks of D.

By the primitivity creteria mentioned in §1 we know that a nontrivial strongly connected digraph D is primitive if and only if $p(D) = 1$.

In the following, we always use the notation $| W |$ to denote the length of the walk W.

Lemma 2.1. *Let D be a nontrivial strongly connected digraph with the period $p(D) = p$. Then, for any $u, v \in V(D)$ and any two walks W_1,W_2 from u to v, we have*

$$| W_1 | \equiv | W_2 | \pmod p. \tag{2.1}$$

Proof. By the strong connectivity of D there is a walk W from v to u in D. Then $W + W_1$ and $W + W_2$ are two closed walks of D and so we have $| W | + | W_i | \equiv 0 \pmod p$ for $i = 1, 2$. Subtracting these two relations we obtain (2.1).

The following notion about the "imprimitive partition" will play an important role in our main results.

Lemma 2.2. *Let D be a nontrivial strongly connected digraph with the period $p(D) = p$, and let v be a fixed vertex of D. Let*

$$V_i = \{u \in V(D) \mid | W | \equiv i \pmod p \text{ for any walk } W \text{ from } v \text{ to } u\} \quad (i = 1, \cdots, p). \tag{2.2}$$

Then we have

(1) *$V_1, \cdots, V_p$ form a partition of the vertex set $V(D)$ (called the "imprimitive partition" of D and $V_1, \cdots, V_p$ are called the "imprimitive sets" of D).*

(2) *For any walk W from some vertex of V_i to some vertex of V_j, we have*

$$| W | \equiv j - i \pmod p. \tag{2.3}$$

Proof. (1) By Lemma 2.1 we know the sets $V_1, \cdots, V_p$ are well-defined. Clearly $V_1, \cdots,$ V_p are pairwisely disjoint and $V(D) = V_1 \cup V_2 \cup \cdots \cup V_p$. Now D is strongly connected, so there exists a walk W^* of length p starting from the vertex v. By taking the subwalks of W^* we can obtain a walk of length i starting from the vertex v for any $i = 1, \cdots, p$. This shows that each set V_i is not empty and thus they form a partition of $V(D)$.

(2) Suppose W is a walk from vertex $u_i \in V_i$ to vertex $u_j \in V_j$. Take a path P from the vertex v to vertex u_i. Then by the definition of V_i and V_j we have

$$| P | \equiv i \pmod p \tag{2.4}$$

$$| P | + | W | \equiv j \pmod p. \tag{2.5}$$

Subtracting the two relations we obtain (2.3).

We point out that the imprimitive partition of a strongly connected digraph D is unique up to the cyclic shiftings of the sets $V_1, \cdots, V_p$. The corresponding matrix version of the imprimitive partition is the "imprimitive normal form"for nonnegative irreducible matrices[3].

We also need the notion of the "condensation digraph"of a digraph and some facts about the acyclic digraphs.

Definition 2.2.[4] *Let D be a digraph, then the condensation digraph $\hat{D}$ is the digraph with the vertex set*

$$\hat{V} = \{\hat{F} \mid F \text{ is a strong component of } D\} \tag{2.6}$$

and there is an arc from $\hat{F_1}$ to $\hat{F_2}$ in $\hat{D}$ if and only if $F_1 \neq F_2$ and there is at least one arc from some vertex of F_1 to some vertex of F_2 in D.

It is easy to see that the condensation digraph $\hat{D}$ of any digraph D is an acyclic digraph (see [4, p.173,10.1.9). Namely, $\hat{D}$ contains no cycle.

In the following, we denote the indegree and outdegree of a vertex u in a digraph by $d^-(u)$ and $d^+(u)$, respectively.

Lemma 2.3. *Let Γ be an acyclic digraph. Then we have*

(1) *There exists some vertex x in Γ with the indegree $d^-(x) = 0$.*

(2) *For any vertex v in Γ, there exists some path in D from some vertex u with the indegree $d^-(u) = 0$ to the vertex v.*

Proof. (1) Let Q be the longest path in Γ and suppose that the initial vertex of Q is x. Then x must be a vertex with $d^-(x) = 0$ since Γ is an acyclic digraph.

(2) Take a path P with the terminal vertex v such that P is the longest path among all the paths with the terminal vertex v. Then the initial vertex of P must have the indegree zero.

Now we introduce some number theoretical notions which will be used in the proof of our main results.

Definition 2.3. *Let $r_1, \cdots, r_k$ be positive integers. The Frobenius set $S(r_1, \cdots, r_k)$ of the numbers $r_1, \cdots, r_k$ is defined as*

$$S(r_1, \cdots, r_k) = \Big\{ \sum_{i=1}^{k} a_i r_i \mid a_1, \cdots, a_k \text{ are nonnegative integers} \Big\}. \tag{2.7}$$

It is well known, by a lemma of Schur, that if $\gcd(r_1, \cdots, r_k) = 1$, then $S(r_1, \cdots, r_k)$ contains all the sufficiently large nonnegative integers. In this case we define the Frobenius number $\varphi(r_1, \cdots, r_k)$ to be the least integer φ such that $m \in S(r_1, \cdots, r_k)$ for all integers $m \geqslant \varphi$. In the general case where $\gcd(r_1, \cdots, r_k) = p$, we have

$$S(r_1, \cdots, r_k) = p \cdot S\Big(\frac{r_1}{p}, \cdots, \frac{r_k}{p}\Big) \tag{2.8}$$

and we define the generalized Frobenius number $\overline{\varphi}(r_1, \cdots, r_k)$ as

$$\overline{\varphi}(r_1, \cdots, r_k) = p \cdot \varphi\Big(\frac{r_1}{p}, \cdots, \frac{r_k}{p}\Big). \tag{2.9}$$

Thus, $\overline{\varphi}(r_1, \cdots, r_k)$ is the least multiple of p above which all multiples of p belong to $S(r_1, \cdots, r_k)$.

For the case $k = 2$, it is well known that if a and b are relatively prime positive integers, then the Frobenius number is

$$\varphi(a, b) = (a - 1)(b - 1). \tag{2.10}$$

§3. The Finiteness Conditions for Generalized Exponents $\exp_D(k)$, $f(D, k)$ and $F(D, k)$

In order to give the finiteness conditions for generalized exponents $\exp_D(k)$, $f(D, k)$ and $F(D, k)$, we first give the necessary and sufficient conditions for the finiteness of the "set exponent" $\exp_D(X)$.

Theorem 3.1. *Let D be a digraph, $F_1, \cdots, F_r$ be those strong components of D such that $\hat{F}_1, \cdots, \hat{F}_r$ are all the vertices with indegree zero in the condensation digraph $\hat{D}$. Let p_i be the period of the strongly connected subdigraph F_i (if F_i is nontrivial), and let*

$$V(F_i) = V_{i1} \dot{\cup} \cdots \dot{\cup} V_{ip_i} \qquad (i = 1, \cdots, r) \tag{3.1}$$

be the imprimitive partition of the digraph F_i (if F_i is nontrivial), $i = 1, \cdots, r$. Then for any vertex subset $X \subseteq V(D)$, the set exponent $\exp_D(X)$ is finite if and only if all the strong components $F_1, \cdots, F_r$ are nontrivial and

$$X \cap V_{ij} \neq \emptyset \qquad (i = 1, \ldots, r;\ j = 1, \cdots, p_i) \tag{3.2}$$

for any indices i and j with $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant p_i$.

Proof. Necessity. Suppose some strong component F_i is trivial. Then F_i must be a single vertex (say z_i) without loops. Since $\hat{F}_i$ has indegree zero in $\hat{D}$, z_i also has indegree zero in D, and thus for any $Y \subseteq V(D)$ and any positive integer p there is no walk of length p from any vertex of Y to the vertex z_i. So $\exp_D(Y) = \infty$ for any $Y \subseteq V(D)$, a contradiction. Now suppose there exist indices i_0 and j_0 with $1 \leqslant i_0 \leqslant r$ and $1 \leqslant j_0 \leqslant p_{i_0}$ such that $X \cap V_{i_0 j_0} = \emptyset$. For an arbitrary positive integer m, we take j with $1 \leqslant j \leqslant p_{i_0}$ such that

$$j - j_0 \equiv m \pmod{P_{i_0}}, \tag{3.3}$$

and then take a vertex y in $V_{i_0 j}$. We will show that there is no walk of length m from any vertex of X to y. Suppose not, let $x \in X$ and W be a walk of length m from x to y. Since $y \in V_{i_0 j} \subseteq V(F_{i_0})$ and $\hat{F}_{i_0}$ has indegree zero in $\hat{D}$, we must have $x \in V(F_{i_0})$ and thus all the vertices of the walk W are in the strong component F_{i_0}. Now assume $x \in V_{i_0 t}$, where $1 \leqslant t \leqslant p_{i_0}$. By using Lemma 2.2 for the digraph F_{i_0} we have

$$j - t \equiv m \pmod{p_{i_0}}. \tag{3.4}$$

Comparing (3.3) and (3.4) we get $t \equiv j_0 \pmod{p_{i_0}}$, and so $t = j_0$ since $1 \leqslant t \leqslant p_{i_0}$ and $1 \leqslant j_0 \leqslant p_{i_0}$. Therefore $x \in V_{i_0 j_0} \cap X$, contradicting the assumption $X \cap V_{i_0 j_0} = \emptyset$. Thus we have proved that there is no walk of length m from any vertex of X to y, and so $\exp_D(X) > m$. But m is an arbitrary integer, so $\exp_D(X)$ must be infinite. This proves the necessity part of the theorem.

Sufficiency. Suppose the strong components $F_1, \cdots, F_r$ are all nontrivial and (3.2) holds. Take $x_{ij} \in X \cap V_{ij}$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant p_i$. Let

$$\overline{X} = \{x_{11}, \cdots, x_{1p_1}; x_{21}, \cdots, x_{2p_2}; \cdots; x_{r1}, \cdots, x_{rp_r}\} \subseteq X. \tag{3.5}$$

We want to show that $\exp_D(\overline{X}) < \infty$.

For $i = 1, \cdots, r$, let

$$V_i = \{u \in V(D) \mid \text{ there exists a path from some vertex of } F_i \textit{ to } u\}. \tag{3.6}$$

Then $V_i \supseteq V(F_i)$. Since $\hat{F}_1, \cdots, \hat{F}_r$ are all the vertices with indegree zero in the condensation digraph $\hat{D}$, we know that for any vertex u in D there exists a path from some vertex of some F_i to u (by (2) of Lemma 2.3). This tells us that

$$V(D) = V_1 \cup V_2 \cup \cdots \cup V_r, \tag{3.7}$$

but here $V_1, \cdots, V_r$ are not necessarily pairwisely disjoint.

Now we fix the index i. For any vertex u in V_i, let u^* be a fixed vertex in F_i such that there exists a path $P(u^*, u)$ from u^* to u, and let the length of the path $P(u^*, u)$ be $l_i(u)$. Also for any j with $1 \leqslant j \leqslant p_i$, take a fixed walk $P_{ij}(u^*)$ from the vertex x_{ij} to the vertex u^* such that $P_{ij}(u^*)$ passes through all vertices of the subdigraph F_i (we can do this since F_i is a strongly connected subdigraph), and let the length of $P_{ij}(u^*)$ be $l_{ij}(u^*)$. Now let $d_{ij}(u) = l_{ij}(u^*) + l_i(u)$. Then the walk $W_{ij}(u) = P_{ij}(u^*) + P(u^*, u)$ is a walk from the vertex x_{ij} to the vertex u with length $d_{ij}(u)$ which passes through all vertices of F_i. Let $\{r_{i1}, \cdots, r_{i\lambda_i}\}$ be the set of the distinct lengths of the cycles of the subdigraph F_i (by the assumption F_i is nontrivial). Then we have

$$\gcd(r_{i1}, \cdots, r_{i\lambda_i}) = p(F_i) = p_i. \tag{3.8}$$

Now let

$$m_i = \max_{\substack{1 \leqslant j \leqslant p_i \\ u \in V_i}} d_{ij}(u) + \overline{\varphi}(r_{i1}, \cdots, r_{i\lambda_i}), \tag{3.9}$$

where $\overline{\varphi}(r_{i1}, \cdots, r_{i\lambda_i})$ is the generalized Frobenius number defined in (2.9). We will show that for any integer $m \geqslant m_i$ and any vertex $v \in V_i$ there exists a walk of length m from some vertex of $\overline{X}$ to the vertex v.

By (2) of Lemma 2.2 we know that $\{l_{ij}(u^*) \mid j = 1, \cdots, p_i\}$ is a complete system of residues $\bmod\, p_i$, so $\{d_{ij}(u) \mid j = 1, \cdots, p_i\}$ is also a complete system of residues mod p_i. Thus for any integer $m \geqslant m_i$ and any vertex $v \in V_i$, there exists some index j_0 with $1 \leqslant j_0 \leqslant p_i$ such that

$$m \equiv d_{ij_0}(v) \pmod{p_i}. \tag{3.10}$$

But we also have

$$m - d_{ij_0}(v) \geqslant m_i - \max_{\substack{1 \leqslant j \leqslant p_i \\ u \in V_i}} d_{ij}(u) = \overline{\varphi}(r_{i1}, \cdots, r_{i\lambda_i}), \tag{3.11}$$

so by (3.10), (3.11) and the definition of the generalized Frobenius number we have

$$m - d_{ij_0}(v) \in S(r_{i1}, \cdots, r_{i\lambda_i})$$

which implies that there exist nonnegative integers $a_{i1}, \cdots, a_{i\lambda_i}$ such that

$$m - d_{ij_0}(v) = \sum_{h=1}^{\lambda_i} a_{ih} r_{ih}. \tag{3.12}$$

Now the walk $W_{ij_0}(v)$ passes through all the vertices of F_i, so we can add a_{ih} times of the cycles of length $r_{ih} (h = 1, \cdots, \lambda_i)$ to the walk $W_{ij_0}(v)$ to obtain a new walk $W'_{ij_0}(v)$ from the vertex $x_{ij_0} \in \overline{X}$ to the vertex v with the length

$$| W'_{ij_0}(v) | = | W_{ij_0}(v) | + \sum_{h=1}^{\lambda_i} a_{ih} r_{ih} = d_{ij_0}(v) + \sum_{h=1}^{\lambda_i} a_{ih} r_{ih} = m.$$

Thus $W'_{ij_0}(v)$ is the desired walk of length m from some vertex x_{ij_0} in $\overline{X}$ to the given vertex v in V_i.

Finally we take

$$M = \max_{1 \leqslant i \leqslant r} m_i.$$

For any vertex y in D, there exists an index i_0 with $1 \leqslant i_0 \leqslant r$ such that $y \in V_{i_0}$ since $V(D) = V_1 \cup V_2 \cup \cdots \cup V_r$. Now $M \geqslant m_{i_0}$ and $y \in V_{i_0}$, so by the above arguments we know that there exists a walk of length M from some vertex of $\overline{X}$ to the vertex y. Therefore by the definition of the set exponent we have $\exp_D(\overline{X}) \leqslant M$ and so by the fact $\overline{X} \subseteq X$ we obtain

$$\exp_D(X) \leqslant \exp_D(\overline{X}) \leqslant M < \infty.$$

This proves the sufficiency part and completes the proof of the theorem.

From Theorem 3.1 we can obtain the finiteness conditions for the generalized exponents $\exp_D(k), f(D,k)$ and $F(D,k)$.

Theorem 3.2. *Let D be a digraph of order n where $F_1, \cdots, F_r; p_1, \cdots, p_r$ and $V_{ij} (1 \leqslant i \leqslant r, 1 \leqslant j \leqslant p_i)$ are as defined in Theorem 3.1. Let k be an integer with $1 \leqslant k \leqslant n$. Then $f(D,k)$ is finite if and only if all the strong components $F_1, \cdots, F_r$ are nontrivial and $k \geqslant \sum_{i=1}^{r} p_i$.*

Proof. $f(D,k) < \infty \Longleftrightarrow$ there exists some subset $X \subseteq V(D)$ with $| X | = k$ and $\exp_D(X) < \infty$

$\Longleftrightarrow F_1, \cdots, F_r$ are all nontrivial and there exists some subset $X \subseteq V(D)$ with $| X | = k$ and $X \cap V_{ij} \neq \emptyset$ for $i = 1, \cdots, r; j = 1, \cdots, p_i$ (by Theorem 3.1)

$\Longleftrightarrow F_1, \cdots, F_r$ are all nontrivial and $k \geqslant \sum_{i=1}^{r} \sum_{j=1}^{p_i} 1 = \sum_{i=1}^{r} p_i$.

Theorem 3.3. *Let D be a digraph as in Theorem* 3.2. *Then $F(D,k)$ is finite if and only if $F_1, \cdots, F_r$ are all nontrivial and*

$$k > n - \min_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant p_i}} | V_{ij} | . \tag{3.13}$$

Proof. $F(D,k) < \infty \Longleftrightarrow$ for any subset $X \subseteq V(D)$ with $| X | = k$, we have $\exp_D(X) < \infty$

$\Longleftrightarrow F_1, \cdots, F_r$ are all nontrivial and for any subset $X \subseteq V(D)$ with $| X | = k$ we have $X \cap V_{ij} \neq \emptyset$ for $i = 1, \cdots, r; j = 1, \cdots, p_i$ (by Theorem 3.1)

$\Longleftrightarrow F_1, \cdots, F_r$ are all nontrivial and $| V_{ij} | > n - k$ for $i = 1, \cdots, r$ and $j = 1, \cdots, p_i$

$\Longleftrightarrow F_1, \cdots, F_r$ are all nontrivial and $\min_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant p_i}} | V_{ij} | > n - k$.

Since $\exp_D(1) = f(D,1)$, by taking $k = 1$ in Theorem 3.2 we have the following corollary.

Corollary 3.1. *Let D be a digraph as in Theorem* 3.2. *Then $\exp_D(1)$ is finite if and only if D satisfies the following two conditions:*

(1) *D has a unique strong component (say,* F_1*) such that* $\hat{F_1}$ *has the indegree zero in* $\hat{D}$.

(2) *The subdigraph* F_1 *is a primitive digraph (i.e.,* F_1 *is nontrivial and* $p(F_1) = 1$*).*

Proof. By taking $k = 1$ in Theorem 3.2 we have

$\exp_D(1) < \infty \Longleftrightarrow f(D, 1) < \infty$

$\Longleftrightarrow F_1, \cdots, F_r$ are nontrivial and $1 \geqslant \sum_{i=1}^{r} p_i$

$\Longleftrightarrow r = 1, p_1 = 1$ and F_1 is nontrivial

$\Longleftrightarrow$ The conditions (1) and (2) hold.

Now we can give the finiteness condition for $\exp_D(k)$.

Theorem 3.4. *Let D be a digraph of order n as in Theorem 3.2, and* $1 \leqslant k \leqslant n$*. Then* $\exp_D(k)$ *is finite if and only if D satisfies the following three conditions:*

(1) *D has a unique strong component (say,* F_1*) such that* $\hat{F_1}$ *has the indegree zero in* $\hat{D}$.

(2) *The subdigraph* F_1 *is a primitive digraph.*

(3) $| V(F_1) | \geqslant k$.

Proof. If $\exp_D(k) < \infty$, then $\exp_D(1) \leqslant \exp_D(k) < \infty$ and so conditions (1) and (2) hold by Corollary 3.1. Now we will prove that, under the assumption that conditions (1) and (2) hold, $\exp_D(k) < \infty$ if and only if condition (3) holds.

Suppose conditions (1) and (2) hold. Then we have $r = 1$ (r is as defined in Theorem 3.2) and $p_1 = 1$, and $V(F_1) = V_{11}$ is the imprimitive partition of F_1. Thus for any vertex $x \in V(D)$, we have

$$\gamma_D(x) < \infty \Longleftrightarrow \exp_D(\{x\}) < \infty \Longleftrightarrow \{x\} \cap V_{11} \neq \emptyset \Longleftrightarrow x \in V(F_1). \tag{3.14}$$

So

$\exp_D(k) < \infty \Longleftrightarrow | \{x \in V(D) \mid \gamma_D(x) < \infty\} | \geqslant k \Longleftrightarrow | V(F_1) | \geqslant k.$

If D is a strongly connected digraph, then the conditions (1) and (3) in Theorem 3.4 hold trivially. Thus we have the following:

Corollary 3.2. *Let D be a nontrivial strongly connected digraph of order n and* $1 \leqslant k \leqslant n$*. Then we have*

(1) $\exp_D(k) < \infty$ *if and only if D is a primitive digraph.*

(2) *Let p be the period of D. Then* $f(D, k) < \infty$ *if and only if* $k \geqslant p$.

(3) *Let* $V(D) = V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_p$ *be the imprimitive partition of the digraph D. Then* $F(D, k) < \infty$ *if and only if* $k > n - \min_{1 \leqslant i \leqslant p} | V_i |$.

§4. The Largest Finite Generalized Exponent $\exp_D(k)$ with Characterizations of the Extreme Digraphs

In §3 we have obtained the necessary and sufficient conditions for the finiteness of the generalized exponents $\exp_D(k)$, $f(D, k)$ and $F(D, k)$. This suggests that from now on it will no longer be necessary to restrict the study of the generalized exponents in the scope of the primitive digraphs, and we can consider various problems related to generalized exponents for general digraphs which are not necessarily primitive, and generalize the corresponding results in primitive cases to general (non-primitive) cases. As an example of this idea of generalizations, we determine in this section the largest finite value of the generalized exponent $\exp_D(k)$ for D ranging over all digraphs of order n (with the finite $\exp_D(k)$ value) and give the complete characterizations for the extreme digraphs with the largest finite $\exp_D(k)$ value.

Definition 4.1. *A digraph D is called "k-generalized primitive"(or simply "k-primitive") if $\exp_D(k) < \infty$. The set of all k-primitive digraphs of order n is denoted by $GP(n,k)$.*

Similarly, D is called k-lower primitive if $f(D,k) < \infty$ and D is called k-upper primitive if $F(D,k) < \infty$.

Obviously, a $(k+1)$-primitive digraph is always k-primitive, and a digraph D of order n is primitive if and only if it is n-primitive. A primitive digraph of order n is k-primitive for all $1 \leqslant k \leqslant n$.

Lemma 4.1. *Let D be a primitive digraph of order n and $s = s(D)$ be the length of the shortest cycle of D. Then*

$$\exp_D(1) \leqslant s(n-2)+1. \tag{4.1}$$

Proof. See [2], Theorem 3.4.4.

Lemma 4.2. *Let D be a k-primitive digraph of order n. Then we have*

$$\exp_D(i+1) \leqslant \exp_D(i)+1 \qquad (1 \leqslant i \leqslant k-1). \tag{4.2}$$

Proof. Since $\exp_D(k) < \infty$, D satisfies the three conditions in Theorem 3.4. Let F_1 be the unique strong component of D such that $\hat{F_1}$ has the indegree zero in $\hat{D}$. Let $|V(F_1)| = m$ (where $m \geqslant k$ by Theorem 3.4). We may order the m vertices $v_1, \cdots, v_m$ of F_1 in such a way that

$$\gamma_D(v_1) \leqslant \gamma_D(v_2) \leqslant \cdots \leqslant \gamma_D(v_m).$$

Notice that $\gamma_D(x) < \infty$ if and only if $x \in V(F_1)$ (see (3.14)), so we have

$$\exp_D(j) = \gamma_D(v_j) \quad (j = 1, \cdots, m).$$

Now $1 \leqslant i \leqslant k-1 \leqslant m-1$ and F_1 is strongly connected, so there exist indices j, t such that $(v_t, v_j) \in E(F_1)$ and $1 \leqslant j \leqslant i < t \leqslant m$. Thus

$$\exp_D(i+1) \leqslant \exp_D(t) = \gamma_D(v_t) \leqslant \gamma_D(v_j)+1 = \exp_D(j)+1 \leqslant \exp_D(i)+1.$$

The Lemma is proved.

We also need to prove the following number theoretical lemma, which will be used in the proof of Theorem 4.1.

Lemma 4.3. *Let a, b be relatively prime positive integers, let $S(a,b)$ be the Frobenius set of a and b and $\varphi(a,b) = (a-1)(b-1)$ be the Frobenius number of a and b (namely, $\varphi(a,b)-1$ is the largest integer not in $S(a,b)$). If m is an integer not in $S(a,b)$, then $\varphi(a,b)-1-m \in S(a,b)$.*

Proof. Let $\varphi = \varphi(a,b)$. Since a, b are relatively prime, there exists an integer x such that $0 \leqslant x \leqslant b-1$ and $xa \equiv m \pmod{b}$. Write $xa - m = yb$. If $y \leqslant 0$, then we would have $m = xa - yb \in S(a,b)$, a contradiction. So $y > 0$ and we obtain

$$\varphi - 1 - m = (b-1-x)a + (y-1)b \in S(a,b).$$

Now we prove our main results in this section.

Theorem 4.1. *Let $D \in GP(n,k)$ be a k-primitive digraph of order n where $n \geqslant 4$ and $1 \leqslant k \leqslant n$. Then we have*

$$\exp_D(k) \leqslant (n-1)(n-2)+k, \tag{4.3}$$

and the equality holds in (4.3) if and only if D is isomorphic to the following (primitive) digraph F_n.

Fig. 4.1 The Digraph F_n

Proof. Firstly we will show that if $D \in GP(n,k)$ and $D \ncong F_n$, then $\exp_D(k) < (n-1)(n-2)+k$.

Case 1. D is a primitive digraph with the shortest cycle length $s \leqslant n-2$.

Then by Lemma 4.1 and Lemma 4.2 we have

$$\begin{aligned}\exp_D(k) &\leqslant \exp_D(1)+(k-1) \leqslant s(n-2)+1+(k-1)\\ &\leqslant (n-2)^2+k < (n-1)(n-2)+k.\end{aligned}$$

Case 2. D is a primitive digraph with the shortest cycle length $s \geqslant n-1$.

Then D must contain some cycle of length $n-1$ and also contain some cycle of length n by the primitivity of D. By direct verifications we can see that there are only two such digraphs of order n (up to isomorphism): one is just the digraph F_n in Fig.4.1, and the other is the digraph $H_n = F_n + \{(2,n)\}$ (obtained by adding a new arc $(2,n)$ to the digraph F_n). By our assumption of this step, $D \ncong F_n$, so in this case we have $D \cong H_n$.

Now we want to show that there is a walk of length $\varphi(n,n-1) = (n-1)(n-2)$ from vertex 1 to any vertex i in the digraph H_n.

Subcase 2.1. $i = n-1$.

By taking $m=1$ and $a=n, b=n-1$ in Lemma 4.3 (noticing that $n \geqslant 4$ implies $1 \notin S(n,n-1)$), we know $\varphi(n,n-1)-2 \in S(n,n-1)$, so we can write $\varphi(n,n-1)-2 = xn+y(n-1)$, where x and y are nonnegative integers. Taking the path $Q=(1,n)+(n,n-1)$ of length 2 from vertex 1 to vertex $n-1$, and adding x cycles of length n and y cycles of length $n-1$, we obtain a walk of length $2+xn+y(n-1)=\varphi(n,n-1)$ from vertex 1 to vertex $i=n-1$ in H_n.

Subcase 2.2. $i = n$.

Taking the path

$$P = (1,n-1)+(n-1,n-2)+\cdots+(3,2)+(2,n)$$

of length $n-1$ from vertex 1 to vertex n, and adding $n-3$ cycles of length $n-1$ to P, we obtain a walk of length $(n-1)(n-2)$ from vertex 1 to vertex $i=n$.

Subcase 2.3. $1 \leqslant i \leqslant n-2$.

Then $1 \leqslant n-i-1 \leqslant n-2$, so $n-i-1 \notin S(n,n-1)$ and by Lemma 4.3 we have

$$\varphi(n,n-1)-(n-i)=\varphi(n,n-1)-1-(n-i-1)\in S(n,n-1).$$

Thus we can write $\varphi(n,n-1)-(n-i)=x_i n+y_i(n-1)$, where x_i and y_i are nonnegative integers. Taking the path

$$P_i=(1,n-1)+(n-1,n-2)+\cdots+(i+1,i)$$

of length $n-i$ from vertex 1 to vertex i, and adding x_i cycles of length n and y_i cycles of length $n-1$, we obtain a walk of length $(n-i)+x_i n+y_i(n-1)=\varphi(n,n-1)$ from vertex 1 to vertex i in H_n.

Combining the three subcases we obtain

$$\gamma_{H_n}(1)\leqslant(n-1)(n-2),$$

so in this case (where $D\cong H_n$) we have

$$\exp_D(1)\leqslant\gamma_D(1)\leqslant(n-1)(n-2)$$

and thus

$$\exp_D(k)\leqslant\exp_D(1)+(k-1)<(n-1)(n-2)+k.$$

Case 3. D is not primitive.

Then D is k-primitive but not primitive, so $1\leqslant k\leqslant n-1$. By Corollary 3.2 we also know that D is not strongly connected. Now D satisfies the three conditions in Theorem 3.4, so there is a unique strong component F_1 of D such that $\hat{F_1}$ has the indegree zero in $\hat{D}$. Let $|V(F_1)|=m$. Then we have $k\leqslant m\leqslant n-1$ since D is not strongly connected.

Now take a vertex $x\in V(F_1)$. For any vertex $y\in V(D)$, by the property of the strong component F_1 we know that there exists some vertex z in F_1 such that the distance $d(z,y)\leqslant n-m$. Now both of the two vertices x and z are in F_1, so there exists a walk of length $\gamma_{F_1}(x)+(n-m)-d(z,y)$ from x to z in F_1 by the definition of the vertex exponent $\gamma_{F_1}(x)$. Therefore there exists a walk of length $\gamma_{F_1}(x)+(n-m)$ from x to y in D. Since y is an arbitrary vertex in D, it follows that

$$\gamma_D(x)\leqslant\gamma_{F_1}(x)+(n-m)\quad(\text{for any } x\in V(F_1)).\tag{4.4}$$

But we also know that $k\leqslant m$ (by Theorem 3.4), so from (4.4) we obtain

$$\exp_D(k)\leqslant\exp_{F_1}(k)+(n-m).\tag{4.5}$$

Now F_1 is primitive by Theorem 3.4. If $m\geqslant 2$, then its shortest cycle length $s(F_1)\leqslant m-1$ and so we have (by Lemma 4.1 and Lemma 4.2)

$$\begin{aligned}\exp_{F_1}(k)&\leqslant\exp_{F_1}(1)+(k-1)\leqslant s(F_1)\cdot(m-2)+1+(k-1)\\&\leqslant(m-1)(m-2)+k.\end{aligned}\tag{4.6}$$

We can show by direct verifications that (4.6) also holds for the case $m=1$ (and hence $k=1$).

Combining (4.5),(4.6) and using the fact $m \leqslant n-1$, we have

$$\begin{aligned}\exp_D(k) &\leqslant \exp_{F_1}(k) + (n-m) \leqslant (m-1)(m-2) + k + (n-m) \\ &= (m-2)^2 + n - 2 + k \leqslant (n-3)^2 + n - 2 + k \\ &< (n-1)(n-2) + k. \end{aligned} \tag{4.7}$$

This proves the Case 3.

Combining Cases 1, 2 and 3 we obtain

$$\exp_D(k) < (n-1)(n-2) + k \quad (D \in GP(n,k), D \ncong F_n). \tag{4.8}$$

Secondly we will show that $\exp_{F_n}(k) = (n-1)(n-2)+k$. Since F_n is primitive and its shortest cycle length $s(F_n) = n-1$, we have by Lemma 4.1 and Lemma 4.2 that

$$\begin{aligned}\exp_{F_n}(k) &\leqslant \exp_{F_n}(1) + (k-1) \leqslant s(F_n)(n-2) + 1 + (k-1) \\ &= (n-1)(n-2) + k. \end{aligned} \tag{4.9}$$

On the other hand, there is no walk of length $(n-1)^2$ from vertex n to itself in F_n. For if there exists such a (closed) walk W, then W will be a union of the cycles of F_n which contains the cycle of length n at least once (since the cycle of length n is the only cycle in F_n containing the vertex n), so its length $|W| = (n-1)^2$ would be able to be expressed as

$$(n-1)^2 = n + an + b(n-1),$$

where a, b are nonnegative integers, which implies that

$$\varphi(n, n-1) - 1 = (n-1)^2 - n \in S(n, n-1).$$

This contradicts the definition of the Frobenius number $\varphi(n, n-1)$. Thus we have

$$\exp_{F_n}(n) \geqslant \gamma_{F_n}(n) \geqslant (n-1)^2 + 1 \tag{4.10}$$

and so

$$\exp_{F_n}(k) \geqslant \exp_{F_n}(n) - (n-k) \geqslant (n-1)(n-2) + k. \tag{4.11}$$

Combining (4.9) and (4.11) we have

$$\exp_{F_n}(k) = (n-1)(n-2) + k, \tag{4.12}$$

and the results of the theorem follow from (4.8) and (4.12).

Finally we point out that since the unique extreme digraph F_n in the above theorem is primitive, Theorem 4.1 also gives the largest value of the generalized exponent $\exp_D(k)$ for D ranging over all primitive digraphs of order n and gives the complete characterizations of the extreme primitive digraphs. Thus the results in [1, Theorem 3.4] and in [5, Theorem 2] have been generalized from the primitive cases to the (more general) k-primitive cases.

References

[1] Brualdi, R. A. & Liu Bolian, Generalized exponents of primitive directed graphs, *J. of Graph Theory,* **14** (1990), 483–499.

[2] Liu Bolian, Generalized exponents of primitive matrices, Doctorial thesis, July 1992.

[3] Minc, H., Nonnegative matrices, A Wiley-Interscience Publication, 1988.

[4] Bondy, J. A. & Murty, U. S. R., Graph theory with applications, The Macmillon Press, 1976.

[5] Shao Jiayu, Wang Jianzhong & Li Guirong, Generalized primitive exponents with characterizations of the extreme digraphs, *Chin. Ann. of Math.,* **15A**:5(1994), 518–523.