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REMARKS ON h-TRANSFORM AND DRIFT**

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Abstract

The author studies the h-transforms of symmetric Markov processes and corresponding Dirichlet spaces, and also discusses the drift transformation of Fukushima and Takeda's type[2] and improves their result by a different approach.

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§1. h-Transforms

In this paper we are going to give a formula characterizing Revuz measures under h-transform. Assume that X is a right Markov process with state space $(E, \mathcal{B})$ which is metrizable, constructed on the canonical space Ω of right continuous paths, and (P_t) and (U_q) are the semigroup and resolvent of Xrespectively. Let h be an excessive function and let $E_h := \{0 < h < \infty\}$. Define kernels P_t^h by

$$\begin{aligned} P_t^h(x, dy) &= \frac{1}{h(x)} P_t(x, dy) h(y), \ x \in E_h; \\ &= \epsilon_x(dy), \ x \in E - E_h. \end{aligned} \tag{1.1}$$

Then it is well known and easy to check that (P_t^h) is a sub-Markovian semigroup on E. It is also known (see, e.g. [6, 7]) that there exist probabilities $P^{x/h}$ on Ω for $x \in E$ such that $X^h := (X_t, P^{x/h})$ is a right process with state space $(E, \mathcal{B})$ and semigroup (P_t^h). Clearly $X = X^1$. We call X^h the h-transform of X (by h) and denote its resolvent by (U_q^h). We make the assumption that $E_h = E$ in this paper just for convenience. The notations $\mathbf{E}^q(h)$ and $\mathrm{Exc}^q(h)$, $q \geq 0$, are used for the classes of functions and measures respectively excessive relative to the semigroup $(e^{-qt} P_t^h)$ (or called q, h-excessive). Particularly $\mathbf{E}^q := \mathbf{E}^q(1)$ and $\mathrm{Exc}^q := \mathrm{Exc}^q(1)$. (By convention q will be erased if $q = 0$.) Recall that $v \in \mathbf{E}(h)$ if and only if $vh \in \mathbf{E}$ while $\xi \in \mathrm{Exc}$ if and only if $h\xi \in \mathrm{Exc}(h)$. Also known as in [7] if L_h denotes the energy functional of X^h, $L_h(h\xi, v) = L(\xi, hv)$ where $\xi \in \mathrm{Exc}$, $v \in \mathbf{E}(h)$ and $L := L_1$, the energy functional of X.

As a convention for notations, the 'p' and 'b' before a class of functions stand for 'non-negative' and 'bounded' respectively. For any measure μ and function f, $\mu(f)$ is a shorthand

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for the integral $\int f d\mu$. For two functions f, g on E, $(x, y) \mapsto f(x)g(y)$ defines a function on $E \times E$, which is denoted by $f \otimes g$.

§2. Revuz Measures

In this section we bring in weak duality. However some of the results are true even without duality and we shall not bother to indicate them explicitly. Assume that, with respect to a σ-finite measure m on E, X has a weak duality $\hat{X} = (\hat{X}_t, \hat{P}^x)$, which is also a right process on $(E, \mathcal{B})$, with semigroup $(\hat{P}_t)$. As a convention the hat sign "ˆ" is always used on notations to indicate that they are with respect to $\hat{X}$. Clearly for $h \in \mathbf{E}$, $\hat{h} \in \hat{\mathbf{E}}$, X^h and $\hat{X}^{\hat{h}}$ are in duality with respect to the measure $h\hat{h}m$.

Let A be an additive functional of X, namely A is an increasing adapted process and for almost every $\omega \in \Omega$, (i) $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \geq 0$; (ii) $A_t(\omega) < \infty$ for $t < \zeta(\omega)$. Let U_A (resp. U_A^h) denote the potential operator of A under X (resp. X^h). Let also, for $\xi \in \mathrm{Exc}$, ν_A^ξ denote the bivariate Revuz measure of A relative to X and ξ; precisely for any nonnegative measurable function G on $E \times E$,

$$\nu_A^\xi(G) := \uparrow \lim_{t \downarrow 0} \frac{1}{t} P^\xi \int_0^t G(X_{s-}, X_s) dA_s.$$

Similarly the bivariate Revuz measure of A relative to X^h and an excessive measure ξ of X^h is denoted by $\nu_A^{\xi/h}$. Obviously the Revuz measure $\rho^x i_A$ of A relative to X and ξ is the right marginal measure of corresponding bivariate Revuz measure, i.e., $\rho_A^\xi = \nu_A^\xi(1 \otimes \cdot)$. We say A is integrable to ξ if ρ_A^ξ is finite, and s-integrable to xi if $A = \sum_n A^{(n)}$ where each $A^{(n)}$ is an additive functional of X integrable to ξ.

Lemma 2.1. *If $h \in \mathbf{E}$ $\hat{h} \in \hat{\mathbf{E}}$ and A is s-integrable to m, then*

$$\nu_A^{h\hat{h}m/h} = (\hat{h} \otimes h) \cdot \nu_A^m.$$

Proof. Let G be any nonnegative measurable function on $E \times E$. Set

$$\kappa(]s, t]) := \int_{]s,t]} G(X_{u-}, X_u) dA_u,$$

for $0 \leq s < t$. Then κ is an s-integrable homogeneous random measure. By (4.8) in [3], we have

$$\nu_A^{h\hat{h}m/h}(G) = \rho_\kappa^{h\hat{h}m/h}(1_E) = \rho_\kappa^{\hat{h}m}(h) = \nu_A^{\hat{h}m}(1 \otimes h \cdot G).$$

Similarly by (8.12) in [5],

$$\begin{aligned}
\nu_A^{\hat{h}m}(G) &= \lim_{t \downarrow 0} t^{-1} P^m[\hat{h}(X_0)\kappa(]0, t])] \\
&= \lim_{t \downarrow 0} t^{-1} P^m[\int_0^t \hat{h}(X_{s-})\kappa(ds)] \\
&= \nu_A^m(\hat{h} \otimes 1 \cdot G).
\end{aligned}$$

Therefore

$$\begin{aligned}
\nu_A^{h\hat{h}m/h} &= 1 \otimes h \cdot \nu_A^{\hat{h}m} \\
&= (1 \otimes h)(\hat{h} \otimes 1) \cdot \nu_A^m = \hat{h} \otimes h \cdot \nu_A^m.
\end{aligned}$$

That completes the proof.

Finally we take a look at the canonical measures of X and its h-transform. Let d be a metric on E and for positive integer n, $D_n := \{(x,y) \in E\times :\ d(x,y) \le \frac{1}{n}\}$,

$$A_t^{(n)} := \sum_{s\le t} 1_{D_n^c}(X_{s-}, X_s).$$

It is clear that each $A^{(n)}$ is an additive functional of X and s-integrable since the jumps are uniformly bounded. The canonical measure of X relative to m is the increasing limit of the bivariate Revuz measure of $A^{(n)}$ relative to X and m as n goes to infinity. Thus the following result is immediate from Lemma 2.1.

Corollary 2.1. *Let ν^m and $\nu^{h\hat{h}m/h}$ be the canonical measures of X and X^h relative to m and $h\hat{h}m$, respectively. Then*

$$\nu^{h\hat{h}m/h} = (\hat{h} \otimes h) \cdot \nu^m. \tag{2.1}$$

Remark 2.1. If (N, H) is a Lévy system of X, then it is not hard to check that a Lévy system of X^h can be taken as (N^h, H), where

$$N^h(x, dy) := N(x, dy)h(y)/h(x).$$

The corollary above follows easily from this fact.

§3. Dirichlet Form Associated with h-Transform

In this section let X be an m-symmetric right process associated with a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. As usual $\mathcal{E}_q := \mathcal{E} + q(\cdot,\cdot)$ and $(\mathcal{E}_q, D(\mathcal{E}))$ is nothing but the Dirichlet space associated with q-subprocess of X. It is well known that the h-transform X^h is an h^2m-symmetric right process associated with a Dirichlet form $(\mathcal{E}^h, D(\mathcal{E}^h))$ which is defined to be

$$\begin{aligned} D(\mathcal{E}^h) &:= \{u \in L^2(E; h^2m) :\ uh \in D(\mathcal{E})\}; \\ \mathcal{E}^h(u,v) &:= \mathcal{E}(uh, vh),\ u, v \in D(\mathcal{E}^h). \end{aligned} \tag{3.1}$$

The form $(\mathcal{E}^h, D(\mathcal{E}^h))$ is called the h-transform of $(\mathcal{E}, D(\mathcal{E}))$. Clearly they are both quasi-regular and if h is bounded, then $bD(\mathcal{E}) \subset D(\mathcal{E}^h)$. For convenience any element in Dirichlet spaces assumes automatically its quasi-continuous version. Let $\mathcal{E}^{(c)}$, ν and k be the diffusion part of $\mathcal{E}$, the canonical measure and the killing measure of X relative to m respectively. Then the Beurling-Deny formula for $\mathcal{E}$ reads as, for $u \in D(\mathcal{E})$,

$$\mathcal{E}(u,u) = \mathcal{E}^{(c)}(u,u) + \frac{1}{2}\int (u(x) - u(y))^2 \nu(dx, dy) + k(u^2). \tag{3.2}$$

Let $\mathcal{E}^{h,(c)}$, ν^h and k^h be the counterparts of $(\mathcal{E}^h, D(\mathcal{E}^h))$.

Theorem 3.1. *If $h \in \mathbf{E} \cap D(\mathcal{E})$ and $0 < h < \infty$ a.e. m, then for $u \in bD(\mathcal{E}^h)$*

$$\begin{aligned} \mathcal{E}^{h,(c)}(u,u) &= \mathcal{E}^{(c)}(uh, uh) - \mathcal{E}^{(c)}(u^2h, h); \\ \nu^h &= (h \otimes h) \cdot \nu; \\ k^h(u^2) &= \mathcal{E}(u^2h, h). \end{aligned}$$

Proof. The second formula is immediate by Corollary 2.1. For the other two, since

$h \in D(\mathcal{E})$, we see $1 \in D(\mathcal{E}^h)$ and

$$\begin{aligned} k^h(u^2) &= \mathcal{E}^h(u^2,1) = \mathcal{E}(u^2h,h) \\ &= \mathcal{E}^{(c)}(u^2h,h) + \frac{1}{2}\int [u^2(x)h(x) - u^2(y)h(y)][h(x)-h(y)]\nu(dx,dy) \\ &\quad + k(u^2h^2) \\ &= \mathcal{E}^{(c)}(u^2h,h) + \int u^2(x)h(x)[h(x)-h(y)]\nu(dx,dy) + k(u^2h^2). \end{aligned}$$

Hence we have

$$\begin{aligned} \mathcal{E}^h(u,u) &= \mathcal{E}(uh,uh) \\ &= \mathcal{E}^{(c)}(uh,uh) + \frac{1}{2}\int [u(x)h(x) - u(y)h(y)]^2\nu(dx,dy) + k(u^2h^2) \\ &= \mathcal{E}^{(c)}(uh,uh) + \frac{1}{2}\int [u(x)-u(y)]^2 h(x)h(y)\nu(dx,dy) \\ &\quad + \frac{1}{2}\int [u^2(x)h(x) - u^2(y)h(y)][h(x)-h(y)]\nu(dx,dy) + k(u^2h\cdot h) \\ &= \mathcal{E}^{(c)}(uh,uh) - \mathcal{E}^{(c)}(u^2h,h) + \frac{1}{2}\int [u(x)-u(y)]^2\nu^h(dx,dy) + \mathcal{E}(u^2h,h). \end{aligned}$$

It follows immmediately that $\mathcal{E}^{h,(c)}(u,u) = \mathcal{E}^{(c)}(uh,uh) - \mathcal{E}^{(c)}(u^2h,h)$, of which the strong locality can be easily verified.

§4. Drift Transformation and Distorted Dirichlet Space

Now we fix $\alpha > 0$ and consider the Ito-Watanabe's factorization of h-transforms. Let $h := U^\alpha g$ with $g \in L^2(E,m) \cap b\mathcal{B}(E)$ strictly positive and

$$M_t := e^{-\alpha t}\frac{h(X_t)}{h(X_0)}. \tag{4.1}$$

Then M is a supermartingale multiplicative functional of X and the transformation carried by M is actually an h-transform for α-subprocess X^α of X. Let $M^{[h]}$ be the martingale part in Fukushima's decomposition of $A^{[h]} := h(X.) - h(X_0)$ and define formally (see §6.3 in [2] for details)

$$Z_t^{[h]} := \int_0^t \frac{dM_s^{[h]}}{h(X_{s-})}, \tag{4.2}$$

which is also a martingale additive functional of X. Let $Z^{[h]} = Z^{[h],c} + Z^{[h],d}$ be the decomposition as continuous and purely discontinuous parts. Denote by $L^{[h]}$ the Doleans-Dade's exponential martingale of $Z^{[h]}$. Then it admits a representation as follows:

$$L_t^{[h]} = \exp\Big(Z_t^{[h],c} - \frac{1}{2}\langle Z^{[h],c}\rangle_t\Big)e^{Z_t^{[h],d}}\prod_{s\le t}\frac{h(X_s)}{h(X_{s-})}e^{-(\frac{h(X_s)}{h(X_{s-})}-1)}1_{\{t<\zeta\}}. \tag{4.3}$$

By Ito's formula, we have

$$\frac{h(X_t)}{h(X_0)} = L_t^{[h]}\cdot e^{\int_0^t \frac{Ah(X_s)}{h(X_s)}ds}, \tag{4.4}$$

where A is the generator of $(\mathcal{E}, D(\mathcal{E}))$. Clearly $Ah = -g + \alpha h$. Let $B_t := \int_0^t \frac{g(X_s)}{h(X_s)}ds$. Then B is a PCAF and the Ito-Watanabe's factorization of M is

$$M_t = L_t^{[h]}\cdot e^{-B_t}. \tag{4.5}$$

This factorization may be extended to $h \in \mathbf{E}^\alpha \cap D(\mathcal{E})$ (set of α-potentials in terms of [2]). In this case there exists a measure ξ of finite energy with a corresponding PCAF N such that $h = U^\alpha \xi = U_N^\alpha 1$ a.e. Then the factorization above holds with $B_t = \int_0^t \frac{1}{h(X_s)} dN_s$.

In some sense the transformation involving $L^{[h]}$ is more important than h-transform since $L^{[h]}$ may still be well defined as a martingale multiplicative functional and represented as (4.3) as long as h admits the Fukushima's decomposition, for instance when h is only a nonnegative element in $D(\mathcal{E})$ (though M, defined by (4.1), is no longer a supermartingale). By Lemma 6.3.1 of [2] the transformed process $\tilde{X}$ of X by $L^{[h]}$ is h^2m-symmetric. We denote by $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$ the Dirichlet space on $L^2(E, h^2m)$ associated with $\tilde{X}$ and let $X^{\alpha,h}$ be the transformed process of X by M. Then $X^{\alpha,h}$ may be recovered from $\tilde{X}$ by a killing transform associated with B. The following result generalizes slightly Theorem 6.3.1 in [2].

Theorem 4.1. *Let $h \in \mathbf{E}^\alpha \cap D(\mathcal{E})$ and be strictly positive.*

(i) *$D(\mathcal{E}^h)$ is densely contained in $D(\tilde{\mathcal{E}})$.*

(ii) *For any $u \in bD(\mathcal{E}^h)$,*

$$\tilde{\mathcal{E}}(u,u) = \mathcal{E}^{h,(c)}(u,u) + \frac{1}{2}\int (u(x)-u(y))^2 h(x)h(y)\nu(dx,dy).$$

(iii) *$1 \in D(\tilde{\mathcal{E}})$ and $\tilde{\mathcal{E}}(1,1) = 0$.*

(iv) *If, in addition, h is bounded, then $D(\mathcal{E}) \subset D(\tilde{\mathcal{E}})$ and $u \in D(\mathcal{E})$,*

$$\tilde{\mathcal{E}}(u,u) = \int h^2 d\mu^c_{\langle u\rangle} + \frac{1}{2}\int (u(x)-u(y))^2 h(x)h(y)\nu(dx,dy), \tag{4.6}$$

where $\mu^c_{\langle u\rangle}$ is the Revuz measure of $\langle M^{[u],c}\rangle$.

Proof. It is easily seen that the Revuz measure of B relative to h^2m is $h\xi$ and the Dirichlet space associated with $X^{\alpha,h}$ is $(\mathcal{E}^h_\alpha, D(\mathcal{E}^h)$. By results in §6.1 of [2] we find that (i) is true and for any $u \in D(\mathcal{E}^h)$,

$$\mathcal{E}^h_\alpha(u,u) = \tilde{\mathcal{E}}(u,u) + \xi(u^2h). \tag{4.7}$$

However by Theorem 3.1

$$\mathcal{E}^h_\alpha(u,u) = \mathcal{E}^{h,(c)}(u,u) + \frac{1}{2}\int (u(x)-u(y))^2 \nu^h(dx,dy) + \mathcal{E}(u^2h,h) + \alpha(hu,hu) \tag{4.8}$$

and

$$\mathcal{E}(u^2h,h) + \alpha(hu,hu) = \mathcal{E}_\alpha(u^2h,h) = \mathcal{E}_\alpha(u^2h,U^\alpha\xi) = \xi(u^2h).$$

Then (ii) easily follows from (4.7) and (4.8). Finally $1 \in D(\mathcal{E}^h) \subset D(\tilde{\mathcal{E}})$ and $\tilde{\mathcal{E}}(1,1) = 0$ obviously.

If h is bounded, it is easily seen that $bD(\mathcal{E}) \subset D(\tilde{\mathcal{E}})$ and by Lemma 3.2.5 in [2] we find that for $u \in bD(\mathcal{E})$,

$$\mathcal{E}^{h,(c)} = \mathcal{E}^{(c)}(uh,uh) - \mathcal{E}^{(c)}(u^2h,h) = \int h^2 d\mu^c_{\langle u\rangle}.$$

Therefore

$$\tilde{\mathcal{E}}(u,u) = \int h^2 d\mu^c_{\langle u\rangle} + \frac{1}{2}\int (u(x)-u(y))^2 h(x)h(y)\nu(dx,dy).$$

Then we know that $\tilde{\mathcal{E}}(u,u) \le ||h||^2_\infty \mathcal{E}(u,u)$ for $u \in bD(\mathcal{E})$ and it follows that $D(\mathcal{E}) \subset D(\tilde{\mathcal{E}})$ and (4.6) holds for $u \in D(\mathcal{E})$. That completes the proof.

Remark 4.1. The problem was addressed detailedly in §6.3 of [2], where $h = U^\alpha g$ with $g \in L^2(E, m)$ strictly positive and bounded. However our approach is rather different.

Example 4.1. Suppose that X is a diffusion, h is locally in $D(\mathcal{E})$ and strictly positive. Let $D_l(\mathcal{E})$ be the totality of functions locally in $D(\mathcal{E})$. Since the Fukushima's decomposition of $A^{[h]}$ still exists uniquely, we may still construct $L^{[h]}$ as above and

$$L_t^{[h]} = \exp\Big(\int_0^t \frac{dM_s^{[h]}}{h(X_s)} - \frac{1}{2}\int_0^t \frac{d\langle M^{[h]}\rangle_s}{h^2(X_s)}\Big)1_{t<\zeta}. \tag{4.9}$$

It is known by the recent work of Fitzsimmons[1] that $D_l(\mathcal{E}) = D_l(\tilde{\mathcal{E}})$ and for $u \in D_l(\tilde{\mathcal{E}})$

$$\tilde{\mathcal{E}}(u,u) = \int h^2 \mu_{\langle u\rangle}. \tag{4.10}$$

Example 4.2. More precisely let X be a Brownian motion on R^d and h a nonnegative function locally in $H^1(R^d)$. Set $l(x) = \ln h(x)$. Then l is also locally in $H^1(R^d)$. It follows from Ito's formula that

$$L_t^{[h]} = \exp\Big(\int_0^t \nabla l(X_s)\cdot dX_s - \frac{1}{2}\int_0^t |\nabla l(X_s)|^2 ds\Big),$$

which gives us a drift (or distorted) Brownian motion Y on R^d which is h^2m-symmetric and has the Dirichlet form $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$ as for $u, v \in D(\tilde{\mathcal{E}})$,

$$\tilde{\mathcal{E}}(u,v) = \frac{1}{2}\int \nabla u\cdot\nabla v\cdot h^2 dx. \tag{4.11}$$

Moreover the generator of Y, when restricted to smooth functions of compact support, has the form

$$\tilde{A}f = \frac{1}{2}\Delta f + \nabla l\cdot\nabla f,$$

which is a drift to A. In particular, when $l(x) = -\frac{1}{4}|x|^2$, Y is called the Ornstein-Ulenbeck's process on R^d.

The examples hint that it is appropriate to call the transformation induced by $L^{[h]}$ the drift transformation and the corresponding Dirichlet form $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$ the distorted form of $(\mathcal{E}, D(\mathcal{E}))$ whenever it makes sense.

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