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THE GOLDBACH-VINOGRDOV THEOREM WITH THREE PRIMES IN A THIN SUBSET

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Abstract

It is proved constructively that there exists a thin subset S of primes, satisfying

$$|S \cap [1, x]| \ll x^{\frac{9}{10}} \log^c x$$

for some absolute constant $c > 0$, such that every sufficiently large odd integer N can be represented as

$$\begin{cases} N = p_1 + p_2 + p_3, \\ p_j \in S, \quad j = 1, 2, 3. \end{cases}$$

Let r be prime, and b_j positive integers with $(b_j, r) = 1, j = 1, 2, 3$. It is also proved that, for almost all prime moduli $r \leq N^{\frac{3}{20}} \log^{-c} N$, every sufficiently large odd integer $N \equiv b_1 + b_2 + b_3 (\bmod r)$ can be represented as

$$\begin{cases} N = p_1 + p_2 + p_3, \\ p_j \equiv b_j (\bmod r), \quad j = 1, 2, 3, \end{cases}$$

where $c > 0$ is an absolute constant.

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§1. Introduction and Statement of Results

In 1937, Vinogradov[7] proved that $J(N)$, the number of representations of an integer N as sums of three primes, satisfies the following asymptotic formula

$$J(N) = \sigma(N) \frac{N^2}{2 \log^3 N} (1 + o(1)), \tag{1.1}$$

where $\sigma(N)$ is the singular series, and $\sigma(N) \gg 1$ for odd N. It therefore follows that every sufficiently large odd integer is the sum of three primes. This settled the ternary Goldbach problem, and the result is referred to as the Goldbach-Vinogradov theorem.

Many authors have considered the corresponding problems with restricted conditions posed on the three primes in the Goldbach- Vinogradov theorem. One of these generalizations was proposed by Wirsing[8] in 1986. He proved that there exists a thin subset S of

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primes with

$$|S \cap [1, x]| \ll (x \log x)^{\frac{1}{3}} \tag{1.2}$$

such that every sufficiently large odd integer N can be written as

$$\begin{cases} N = p_1 + p_2 + p_3, \\ p_j \in S, \quad j = 1, 2, 3. \end{cases} \tag{1.3}$$

This result, which is best possible apart from the logarithmic factor, is based on probabilistic arguments; hence the set S above is nonconstructive. It was Wolke[9] who suggested the problem of constructing S explicitly, and proved constructively that one can take

$$|S \cap [1, x]| \ll x^{\frac{15}{16}}. \tag{1.4}$$

The purpose of the present paper is to make further contribution to this problem. Our main result is

Theorem 1.1. *We can construct a thin subset S of primes, with $|S \cap [1, x]| \ll x^{\frac{9}{10}} \log^B x$ for some absolute constant $B > 0$, such that every sufficiently large odd integer N can be writen in the form of* (1.3).

To prove this theorem, we investigate another generalization of the Goldbach-Vinogradov theorem, i.e., the representation of an odd integer as sums of three primes in arithmetic progressions. Let k be a fixed positive integer, $b_j, j = 1, 2, 3$, integers with $(b_j, k) = 1$, and $J(N; k, b_1, b_2, b_3)$ the number of solutions of the equation

$$\begin{cases} N = p_1 + p_2 + p_3, \\ p_j \equiv b_j (\bmod k), \quad j = 1, 2, 3 \end{cases} \tag{1.5}$$

with N odd and p_j prime. In 1926, Rademacher[6] showed that, subject to the generalized Riemann hypothesis,

$$J(N; k, b_1, b_2, b_3) = \sigma(N; k) \frac{N^2}{2 \log^3 N} (1 + o(1)). \tag{1.6}$$

Here for odd N satisfying $N = b_1 + b_2 + b_3 (\bmod k)$,

$$\sigma(N; k) = \frac{C(k)}{k^2} \prod_{p|k} \frac{p^3}{(p-1)^3 + 1} \prod_{p|N, p\nmid k} \frac{(p-1)((p-1)^2 - 1)}{(p-1)^3 + 1} \prod_{p>2} \left(1 + \frac{1}{(p-1)^3}\right), \tag{1.7}$$

where $p > 2$ throughout, $C(k) = 2$ for odd k, and $C(k) = 8$ for even k. If N fails to satisfy the above conditions, then $\sigma(N; k) = 0$. This result was established unconditionally in [1, 11].

To prove Theorem 1.1, we need k to be as large as possible with respect to N. The methods of [11] and [1] manage to give (1.6) for all $k \le \log^A N$, where $A > 0$ is arbitrary, but fail to work when k is larger than N^ε. Recently, Wolke[9] developed a new approach to the problem under consideration, by which he proved that (1.6) is true for almost all prime moduli $k \le N^{\frac{1}{11}}$.

In this paper we improve the above $\frac{1}{11}$ to $\frac{3}{20}$. Precisely speaking, we have the following

Theorem 1.2. *Let $A > 0$ be arbitrary and r prime. Then there exists $B = B(A) > 0$ such that*

$$\sum_{r \le R} r \max_{(b_j, r) = 1} \Big| \sum_{\substack{N = n_1 + n_2 + n_3 \\ n_j \equiv b_j (\bmod r)}} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3) - \sigma(N; r) \frac{N^2}{2} \Big| \ll N^2 \log^{-A} N \tag{1.8}$$

holds for $R = N^{\frac{3}{20}} \log^{-B} N$.

The basic tool of the proof is the circle method. On the minor arcs, one needs a nontrivial estimate for exponential sums over primes in arithmetic progressions to large moduli r. All published results of this kind are, however, trivial unless the set of minor arcs is chosen very "thin". Consequently, the set of major arcs is much "larger" than usual, hence more difficult to treat. In fact, the set of major arcs in the present situation is so "large" that the integral on it can only be controlled by an average process over some kind of special moduli r. This is the reason why we get an "almost-all" result on prime moduli r. The proof follows Wolke's idea in [9], while the improvement comes from a mean-value theorem in [5, 10] on exponential sums over primes.

Theorem 1.2 will be established in the following three sections. To be brief, the proof of Theorem 1.1 is omitted, since it follows directly from [9, Theorem 2 and Lemma 2].

Remark 1.1. The method of this paper has been modified in [5] to establish Theorem 1.2 for almost all positive integers $r \le N^{\frac{1}{8}-\varepsilon}$, where $\varepsilon > 0$ is arbitrary.

Remark 1.2. It should be mentioned that Balog and Friedlander[2] gave another approach to the problem dealt with in Theorem 1.1. They showed that the S in (1.2) can be taken as the set of Piateski-Shapiro primes, i.e., $S = \{p : p = [n^c], n$ runs over all the positive integers$\}$ with $c = \frac{21}{20} - \varepsilon$. It thus follows that $|S \cap [1, x]| \ll x^{\frac{20}{21}+\varepsilon}$. The exponent $\frac{20}{21}$ has been improved to $\frac{15}{16}$ by Jia [4].

We use a standard notation in number theory. In particular, the letter r in this paper stands always for primes, while L for $\log N$. The expression $r \sim R$ means $R < r \le 2R$.

§2. Reduction of Theorem 1.2

Let

$$R \le N^{\frac{3}{20}} L^{-B} \tag{2.1}$$

as in Theorem 1.2. In view of the result of [1, 11], we can further assume that R is larger than an arbitrary power of L. Also, let

$$P_1 = R^{\frac{4}{3}} L^{3C}, \quad P_2 = R^2 L^{3C}, \quad Q = \frac{N}{R^2} L^{-4C}; \tag{2.2}$$

the constants B and C will be specified later.

Write $\alpha \in \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right]$ in the form

$$\alpha = \frac{a}{q} + \lambda, \quad 1 \le a \le q, \quad (a, q) = 1. \tag{2.3}$$

For each prime $r \sim R$, the set of major arcs of the circle method is defined as $E_1(r) \cup E_2(r)$, where

$$E_1(r) = \bigcup_{\substack{q \le P_1 \\ r \nmid q}} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right], \quad E_2(r) = \bigcup_{\substack{q \le P_2 \\ r | q}} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right].$$

Since $2P_1 < Q, 2P_2 < Q$, no two major arcs intersect. The set of minor arcs is defined as $\left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] - E_1(r) - E_2(r)$. It follows from Dirichlet's lemma on rational approximation that, the set of minor arcs is the union of $E_3(r)$ and $E_4(r)$, with

$$E_3(r) = \left\{\alpha : P_1 < q \le Q,\ r \nmid q,\ |\lambda| \le \frac{1}{qQ}\right\}, \quad E_4(r) = \left\{\alpha : P_2 < q \le Q,\ r | q,\ |\lambda| \le \frac{1}{qQ}\right\}.$$

Let $\Lambda(n)$ be the von Mangoldt function, $e(\alpha) = e^{2\pi i\alpha}$ as usual, and $S(\alpha;r,b) = \sum_{\substack{n\le N\\ n\equiv b(\bmod r)}} \Lambda(n)e(n\alpha)$. Then the statement of Theorem 1.2 is equivalent to

$$\sum_{r\sim R} r \max_{(b_j,r)=1} \Big| \int_0^1 S(\alpha;r,b_1)S(\alpha;r,b_2)S(\alpha;r,b_3)e(-N\alpha)d\alpha - \sigma(N;r)\frac{N^2}{2}\Big| \ll N^2L^{-A}.$$

It thus suffices to prove

$$\sum_{r\sim R} r \max_{(b_j,r)=1} \Big| \int_{E_1(r)\cup E_2(r)} S(\alpha;r,b_1)S(\alpha;r,b_2)S(\alpha;r,b_3)e(-N\alpha)d\alpha - \sigma(N;r)\frac{N^2}{2}\Big| \ll N^2L^{-A}, \tag{2.4}$$

and

$$\sum_{r\sim R} r \max_{(b_j,r)=1} \Big| \int_{E_3(r)\cup E_4(r)} S(\alpha;r,b_1)S(\alpha;r,b_2)S(\alpha;r,b_3)e(-N\alpha)d\alpha\Big| \ll N^2L^{-A}. \tag{2.5}$$

The estimate of $S(\alpha;r,b)$ with $(b,r)=1$ on the minor arcs is given in the following lemma.

Lemma 2.1. *Let $A>0$ be arbitrary and $\alpha\in E_3(r)\cup E_4(r)$. If C is sufficiently large, then $S(\alpha;r,b)\ll \frac{N}{r\log^A N}$, uniformly for $r\sim R$.*

Proof. We need the following result of Balog and Perelli[3]: For $M\le N$ and $h=(r,q)$,

$$\sum_{\substack{n\le M\\ n\equiv b(\bmod r)}} \Lambda(n)e\Big(\frac{a}{q}n\Big) \ll L^3\Big(\frac{hN}{rq^{\frac12}} + \frac{q^{\frac12}N^{\frac12}}{h^{\frac12}} + \frac{N^{\frac45}}{r^{\frac25}}\Big). \tag{2.6}$$

The desired result now follows from partial summation.

We can now give

Proof of (2.5). It follows from Lemma 2.1 that the integral over $E_3(r)\cup E_4(r)$ is

$$\int_{E_3(r)\cup E_4(r)} S(\alpha;r,b_1)S(\alpha;r,b_2)S(\alpha;r,b_3)e(-N\alpha)d\alpha$$

$$\ll \max_{\alpha\in E_3(r)\cup E_4(r)} |S(\alpha;r,b_1)| \left(\int_0^1 |S(\alpha;r,b_2)|^2d\alpha\right)^{\frac12} \left(\int_0^1 |S(\alpha;r,b_3)|^2d\alpha\right)^{\frac12} \ll \frac{N^2}{r^2L^A},$$

uniformly for $r\sim R$. Hence the quantity on the left-hand side of (2.5) is $\ll N^2L^{-A}$, which proves (2.5).

Theorem 1.2 now reduces to (2.4), which will be established in the following two sections.

§3. Mean-Value Estimates

The next lemma is the key element to establish (2.4), hence Theorem 1.2. For the proof (see [10] or [5]).

Lemma 3.1. *Let $Z\ge 1$ be arbitrary. For any $A>0$, there exists $D=D(A)>0$ such that if $1\le K\le Z^{\frac23}N^{\frac13}L^{-D}$, $\theta = Z^2K^{-3}L^{-D}$, then*

$$\sum_{q\le K} \max_{|\lambda|\le\theta} \frac{1}{\varphi(q)} \sum_{\chi \bmod q} |\tau(\bar\chi)| \Big|\sum_{n\le N}\chi(n)\Lambda(n,\chi)e(n\lambda)\Big| \ll ZNL^{-A}.$$

Here and in the sequel $\tau(\chi)$ denotes the Gauss sum, i.e., $\tau(\chi) = \sum_{a=1}^{q}\chi(a)e\left(\frac{a}{q}\right)$, while $\Lambda(n,\chi^0)=\Lambda(n)-1$, and $\Lambda(n,\chi)=\Lambda(n)$ for $\chi\ne\chi^0$.

For $(a,q)=1, r$ prime and $b \not\equiv 0 (\bmod r)$, define

$$f(r,q,a,b)=\begin{cases}\frac{\mu(q)}{\varphi(rq)}, & \text{if } r \nmid q,\\ \frac{\mu(q_1)}{\varphi(q)}e(\frac{ab\bar{q}_1}{r}), & \text{if } q=rq_1, r \nmid q_1, q_1\bar{q}_1 \equiv 1(\bmod r),\\ 0, & \text{if } r^2|q,\end{cases} \tag{3.1}$$

and let

$$E(r,q,a,b,\lambda)=\sum_{\substack{n\le N\\ n\equiv b(\bmod r)}}\Lambda(n)e\Big(n\Big(\frac{a}{q}+\lambda\Big)\Big)-f(r,q,a,b)\sum_{n\le N}e(n\lambda), \tag{3.2}$$

$$E^*(r,q)=\max_{(a,q)=1}\max_{(b,r)=1}\max_{|\lambda|\le\frac{1}{qQ}}|E(r,q,a,b,\lambda)|. \tag{3.3}$$

Lemma 3.2. *Let R, P_1, P_2 and Q be defined as in* (2.1) *and* (2.2), *while f, E and E^* as in* (3.1), (3.2) *and* (3.3). *Then for any $A>0$, there are constants B and C such that*

$$\sum_{r\sim R}\Big\{\sum_{\substack{q\le P_1\\ r\nmid q}}+\sum_{\substack{q\le P_2\\ r|q}}\Big\}E^*(r,q)\ll NL^{-A}.$$

Proof. Consider three cases separately.

Case 1. $r\nmid q$. Denoting by S the first sum on the right-hand side of (3.2), and then introducing the Dirichlet characters ξ mod r and η mod q, one has

$$\begin{aligned}S=&\sum_{\substack{c=1\\(c,q)=1}}^{q}e\left(\frac{ac}{q}\right)\sum_{\substack{n\le N\\ n\equiv b(\bmod r)\\ n\equiv c(\bmod q)}}\Lambda(n)e(n\lambda)+O(L^2)\\ =&\frac{1}{\varphi(r)\varphi(q)}\sum_{\xi \bmod r}\bar{\xi}(b)\sum_{\eta \bmod q}\sum_{\substack{c=1\\(c,q)=1}}^{q}\bar{\eta}(c)e\left(\frac{ac}{q}\right)\sum_{n\le N}\xi\eta(n)\Lambda(n)e(n\lambda)+O(L^2)\\ =&I+J+K+O(L^2),\end{aligned}$$

say, where I, J and K are the sums corresponding to

(i) $\xi=\xi^0 \mod r, \quad \eta=\eta^0 \mod q$,

(j) $\xi=\xi^0 \mod r, \quad \eta\ne\eta^0 \mod q, \quad$ (k) $\xi\ne\xi^0 \mod r$

respectively. It is easily seen that

$$\begin{aligned}I=&\frac{\mu(q)}{\varphi(r)\varphi(q)}\sum_{\substack{n\le N\\ \chi=\chi^0 \bmod rq}}\chi(n)\Lambda(n)e(n\lambda)\\ =&\frac{\mu(q)}{\varphi(rq)}\sum_{n\le N}e(n\lambda)+O\Big\{\frac{1}{\varphi(rq)}\sum_{\substack{n\le N\\ \chi=\chi^0 \bmod qr}}\chi(n)(\Lambda(n)-1)e(n\lambda)\Big\}+O\Big\{\frac{L^2}{\varphi(rq)}\Big\}.\end{aligned}$$

We also have

$$\begin{aligned}J\ll&\frac{1}{\varphi(rq)}\sum_{\eta\ne\eta^0 \bmod q}|\tau(\bar{\eta})|\Big|\sum_{\substack{n\le N\\(n,r)=1}}\eta(n)\Lambda(n)e(n\lambda)\Big|\\ \ll&\frac{1}{\varphi(rq)}\sum_{\eta\ne\eta^0 \bmod q}|\tau(\bar{\eta})|\Big|\sum_{n\le N}\eta(n)\Lambda(n)e(n\lambda)\Big|+\frac{q^{\frac12}L^2}{\varphi(r)}.\end{aligned}$$

To estimate K, one notes that every $\xi \bmod r \neq \xi^0 \bmod r$ is primitive since r is prime. Therefore $|\tau(\bar{\xi})| = r^{\frac{1}{2}}$, and consequently $|\tau(\bar{\chi})| = |\tau(\bar{\xi\eta})| = |\bar{\xi}(q)\bar{\eta}(r)\tau(\bar{\xi})\tau(\bar{\eta})| = |\tau(\bar{\xi})||\tau(\bar{\eta})| = r^{\frac{1}{2}}|\tau(\bar{\eta})|$. Hence

$$\begin{aligned} K &\ll \frac{1}{\varphi(rq)} \sum_{\substack{\chi \bmod rq \\ \chi=\xi\eta \\ \xi\neq\xi^0}} |\tau(\bar{\eta})| \Big| \sum_{n\le N} \chi(n)\Lambda(n)e(n\lambda)\Big| \\ &\ll \frac{1}{r^{\frac{3}{2}}\varphi(q)} \sum_{\chi \bmod rq \neq \chi^0 \bmod rq} |\tau(\bar{\chi})| \Big| \sum_{n\le N} \chi(n)\Lambda(n)e(n\lambda)\Big|. \end{aligned}$$

One thus concludes that

$$\begin{aligned} E(r,q,a,b,\lambda) &= S - f(r,q,a,b) \sum_{n\le N} e(n\lambda) \\ &\ll \frac{1}{r\varphi(q)} \sum_{\eta\neq\eta^0 \bmod q} |\tau(\bar{\eta})| \Big| \sum_{n\le N} \eta(n)\Lambda(n)e(n\lambda)\Big| \\ &\quad + \frac{1}{r^{\frac{3}{2}}\varphi(q)} \sum_{\chi \bmod rq} |\tau(\bar{\chi})| \Big| \sum_{n\le N} \chi(n)\Lambda(n,\chi)e(n\lambda)\Big| + \frac{q^{\frac{1}{2}}L^3}{r} + L^2. \end{aligned} \tag{3.4}$$

Case 2. If $q = rq_1, r \nmid q_1$, then

$$\begin{aligned} S &= \sum_{\substack{c=1 \\ (c,q_1)=1}}^{q_1} \sum_{\substack{d=1 \\ (d,r)=1}}^{r} e\left(\frac{a(cr+dq_1)}{q}\right) \sum_{\substack{n\le N \\ n\equiv b (\bmod r) \\ n\equiv cr (\bmod q_1) \\ n\equiv dq_1 (\bmod r)}} \Lambda(n)e(n\lambda) + O(L^2) \\ &= \sum_{\substack{c=1 \\ (c,q_1)=1}}^{q_1} \sum_{\substack{d=1 \\ (d,r)=1 \\ d\equiv b\bar{q}_1 (\bmod r)}}^{r} e\left(\frac{ac}{q_1}\right) e\left(\frac{ad}{r}\right) \sum_{\substack{n\le N \\ n\equiv b (\bmod r) \\ n\equiv cr (\bmod q_1)}} \Lambda(n)e(n\lambda) + O(L^2) \\ &= e\left(\frac{ab\bar{q}_1}{r}\right) \sum_{\substack{c=1 \\ (c,q_1)=1}}^{q_1} e\left(\frac{ac}{q_1}\right) \sum_{\substack{n\le N \\ n\equiv b (\bmod r) \\ n\equiv cr (\bmod q_1)}} \Lambda(n)e(n\lambda) + O(L^2). \end{aligned}$$

Introducing characters, one sees, as in the first case,

$$\begin{aligned} S - f(r,q,a,b) \sum_{n\le N} e(n\lambda) &\ll \frac{1}{r\varphi(q_1)} \sum_{\eta \bmod q_1 \neq \eta^0 \bmod q_1} |\tau(\bar{\eta})| \Big| \sum_{n\le N} \eta(n)\Lambda(n)e(n\lambda)\Big| \\ &\quad + \frac{1}{r^{\frac{3}{2}}\varphi(q_1)} \sum_{\chi \bmod rq_1 \neq \chi^0 \bmod rq_1} |\tau(\chi)| \Big| \sum_{n\le N} \xi(n)\Lambda(n)e(n\lambda)\Big| + L^2. \end{aligned} \tag{3.5}$$

Case 3. If $q = r^2 q_2, r \nmid q_2$, then on putting $q_2\bar{q}_2 \equiv 1 (\bmod r)$, $q_2\tilde{q}_2 \equiv 1 (\bmod r^2)$ one sees that

$$S = \sum_{\substack{c=1 \\ (c,q_2)=1}}^{q_2} \sum_{\substack{d=1 \\ (d,r^2)=1}}^{r^2} e\left(\frac{a(cr^2+dq_2)}{q}\right) \sum_{t=1}^{r} \sum_{\substack{n\le N \\ n\equiv b+tr (\bmod r^2) \\ n\equiv dq_2 (\bmod r^2) \\ n\equiv cr^2 (\bmod q_2)}} \Lambda(n)e(n\lambda) + O(L^2)$$

$$= \sum_{\substack{c=1\\(c,q_2)=1}}^{q_2} \sum_{t=1}^{r} e\left(\frac{ab\tilde{q}_2}{r^2}+\frac{at\bar{q}_2}{r}+\frac{ac}{q_2}\right) \sum_{\substack{n\le N\\ n\equiv b+tr(\bmod r^2)\\ n\equiv cr^2(\bmod q_2)}} \Lambda(n)e(n\lambda)+O(L^2)$$

$$= \frac{1}{\varphi(r^2)}\frac{1}{\varphi(q_2)} \sum_{\substack{c=1\\(c,q_2)=1}}^{q_2} \sum_{t=1}^{r} e\left(\frac{ab\tilde{q}_2}{r^2}+\frac{at\bar{q}_2}{r}+\frac{ac}{q_2}\right)$$

$$\cdot \sum_{\xi \bmod r^2} \bar{\xi}(b+tr) \sum_{\eta \bmod q_2} \bar{\eta}(cr^2) \sum_{n\le N} \xi\eta(n)\Lambda(n)e(n\lambda)+O(L^2)$$

$$= \frac{1}{\varphi(r^2q_2)} e\left(\frac{ab\tilde{q}_2}{r^2}\right) \sum_{t=1}^{r} e\left(\frac{at\bar{q}_2}{r}\right) \sum_{\xi \bmod r^2} \bar{\xi}(b+tr) \sum_{\eta \bmod q_2} \bar{\eta}(r^2)\tau(\bar{\eta})$$

$$\cdot \sum_{n\le N} \xi\eta(n)\Lambda(n)e(n\lambda)+O(L^2).$$

If a character $\xi \bmod r^2$ can be defined $\bmod r$, and then $\bar{\xi}(b+tr)=\bar{\xi}(b)$, hence the sum over t vanishes. We can therefore restrict our discussion to primitive characters $\xi \bmod r^2$. Hence

$$E(r,q,a,b,\lambda) \ll \frac{r}{\varphi(q)} \sideset{}{^*}\sum_{\xi \bmod r^2} \sum_{\eta \bmod q_2} |\tau(\bar{\eta})| \Big| \sum_{n\le N} \xi\eta(n)\Lambda(n)e(n\lambda)\Big| + L^2$$

$$\ll \frac{1}{\varphi(q)} \sum_{\chi \bmod q \ne \chi^0 \bmod q} |\tau(\bar{\chi})| \Big| \sum_{n\le N} \chi(n)\Lambda(n)e(n\lambda)\Big| + L^2. \tag{3.6}$$

For any $m \ge 3$, it is obvious that $r^m \nmid q$; otherwise one should have $q \ge R^3$, which contradicts the fact $q \le P_2$.

We thus conclude from (3.4), (3.5) and (3.6) that

$$\sum_{r\sim R} \Big\{ \sum_{\substack{q\le P_1\\ r\nmid q}} + \sum_{\substack{q\le P_2\\ r\mid q}} \Big\} E^*(r,q)$$

$$\ll \sum_{r\sim R} \Big\{ \sum_{\substack{q\le P_1\\ r\nmid q}} \max_{|\lambda|\le \frac{1}{qQ}} \Big(\frac{1}{r\varphi(q)} \sum_{\chi \bmod q \ne \chi^0 \bmod q} |\tau(\bar{\chi})| \Big| \sum_{n\le N} \chi(n)\Lambda(n)e(n\lambda)\Big|$$

$$+ \frac{1}{r^{\frac{3}{2}}\varphi(q)} \sum_{\chi \bmod rq} |\tau(\bar{\chi})| \Big| \sum_{n\le N} \chi(n)\Lambda(n,\chi)e(n\lambda)\Big| \Big) \Big\}$$

$$+ \sum_{r\sim R} \Big\{ \sum_{\substack{q_1\le \frac{P_2}{r}\\ r\nmid q_1}} \max_{|\lambda|\le \frac{1}{qQ}} \Big(\frac{1}{r\varphi(q_1)} \sum_{\chi \bmod q_1 \ne \chi^0 \bmod q_1} |\tau(\bar{\chi})| \Big| \sum_{n\le N} \chi(n)\Lambda(n)e(n\lambda)\Big|$$

$$+ \frac{1}{r^{\frac{3}{2}}\varphi(q_1)} \sum_{\chi \bmod rq_1 \ne \chi^0 \bmod rq_1} |\tau(\bar{\chi})| \Big| \sum_{n\le N} \chi(n)\Lambda(n)e(n\lambda)\Big| \Big) \Big\}$$

$$+ \sum_{r\sim R} \sum_{\substack{q_2\le \frac{P_2}{r}\\ r\nmid q_2}} \max_{|\lambda|\le \frac{1}{qQ}} \frac{1}{r^2\varphi(q_2)} \sum_{\chi \bmod r^2q_2 \ne \chi^0 \bmod r^2q_2} |\tau(\bar{\chi})| \Big| \sum_{n\le N} \chi(n)\Lambda(n)e(n\lambda)\Big|$$

$$+ O(P_1^{\frac{3}{2}}L^3 + P_1RL^2 + P_2L^2)$$

$$\ll \sum_{q\le P_2} \max_{|\lambda|\le \frac{1}{qQ}} \frac{1}{\varphi(q)} \sum_{\chi \bmod q} |\tau(\bar{\chi})| \Big| \sum_{n\le N} \chi(n)\Lambda(n;\chi)e(n\lambda)\Big|$$

$$
\begin{aligned}
&+\frac{1}{R^{\frac12}}\sum_{q\le P_1R}\max_{|\lambda|\le\frac{R}{qQ}}\frac{1}{\varphi(q)}\sum_{\chi\bmod q}|\tau(\bar\chi)|\Big|\sum_{n\le N}\chi(n)\Lambda(n;\chi)e(n\lambda)\Big|\\
&+O(P_1^{\frac32}L^3+P_1RL^2+P_2L^2)\\
\ll\ & L^2\sum_{q\sim U}\max_{|\lambda|\le\frac{1}{UQ}}\frac{1}{\varphi(q)}\sum_{\chi\bmod q}|\tau(\bar\chi)|\Big|\sum_{n\le N}\chi(n)\Lambda(n;\chi)e(n\lambda)\Big|\\
&+\frac{L^2}{R^{\frac12}}\sum_{q\sim V}\max_{|\lambda|\le\frac{R}{VQ}}\frac{1}{\varphi(q)}\sum_{\chi\bmod q}|\tau(\bar\chi)|\Big|\sum_{n\le N}\chi(n)\Lambda(n;\chi)e(n\lambda)\Big|+O(P_1RL^2),
\end{aligned}\tag{3.7}
$$

where
$$U\le P_2=R^2L^C,\ \ V\le P_1R=R^{\frac73}L^C.\tag{3.8}$$

By Lemma 3.1 with $Z=1$, the first sum on the right-hand side of (3.7) is admissible if

$$U\le N^{\frac13}L^{-D},\ \frac{1}{UQ}\le U^{-3}L^{-D}.\tag{3.9}$$

Taking $Z=R^{\frac12}$ in Lemma 3.1, we see that the second term on the right-hand side of (3.7) is admissible if

$$V\le N^{\frac13}R^{\frac13}L^{-D},\ \frac{R}{VQ}\le RV^{-3}L^{-D}.\tag{3.10}$$

In view of the definitions of Q,U and V (see (2.2) and (3.8)), the optimal choice of R satisfying (3.9) and (3.10) is $R\le N^{\frac{3}{20}}L^{-B}$, as stated in (2.1). This proves the lemma.

§4. The Major Arcs

In this section we give

Proof of (2.4). In the course of the proof, the following elementary estimate will be used: If $A_j=B+C, j=1,2,3$, then

$$A_1A_2A_3=B^3+C(A_1^2+B^2+A_1B)=B^3+O(|C||A_1|^2+|C||B|^2).\tag{4.1}$$

If $\alpha\in E_1(r)$, then for $j=1,2,3$,

$$S(\alpha;r,b_j)=\frac{\mu(q)}{\varphi(rq)}\sum_{n\le N}e(n\lambda)+O(E^*(r,q)).\tag{4.2}$$

Applying (4.1), one has

$$
\begin{aligned}
I_1:=&\int_{E_1(r)}S(\alpha;r,b_1)S(\alpha;r,b_2)S(\alpha;r,b_3)e(-N\alpha)d\alpha\\
=&\sum_{\substack{q\le P_1\\ r\nmid q}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\Big\{\frac{\mu^3(q)}{\varphi^3(rq)}e\Big(-\frac{aN}{q}\Big)\int_{|\lambda|\le\frac{1}{qQ}}\Big(\sum_{n\le N}e(n\lambda)\Big)^3e(-N\lambda)d\lambda\\
&+O\Big(E^*(r,q)\int_{|\lambda|\le\frac{1}{qQ}}\Big|S\Big(\frac aq+\lambda;r,b_1\Big)\Big|^2d\lambda\Big)\\
&+O\Big(\frac{E^*(r,q)}{\varphi^2(rq)}\int_{|\lambda|\le\frac{1}{qQ}}\Big|\sum_{n\le N}e(n\lambda)\Big|^2d\lambda\Big)\Big\}.
\end{aligned}
$$

The third integral on the right-hand side above is trivially $\ll N$, while the second integral, when summed over a, can be estimated as

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{|\lambda|\le\frac{1}{qQ}} \Big|S\Big(\frac{a}{q}+\lambda;r,b_1\Big)\Big|^2 d\lambda \le \int_0^1 \Big|S\Big(\frac{a}{q}+\lambda;r,b_1\Big)\Big|^2 d\lambda \le \frac{N}{r}.$$

The first integral is

$$\int_{-\frac12}^{\frac12} \Big(\sum_{n\le N} e(n\lambda)\Big)^3 e(-N\lambda)d\lambda + O\Big(\int_{\frac{1}{qQ}}^{\frac12} \lambda^{-3}d\lambda\Big) = \frac12 N^2 + O((qQ)^2).$$

We thus have

$$\begin{aligned} I_1 &= \sum_{\substack{q\le P_1\\ r\nmid q}} \frac{\mu(q)}{\varphi^3(rq)} c_q(-N)\Big(\frac12 N^2 + O((qQ)^2)\Big) + O\Big(\frac{N}{r}\sum_{\substack{q\le P_1\\ r\nmid q}} E^*(r,q)\Big) \\ &= \frac{N^2}{2\varphi^3(r)} \sum_{\substack{q=1\\ r\nmid q}}^{\infty} \frac{\mu(q)}{\varphi^3(q)} c_q(-N) + O\Big(\frac{N^2L}{r^3P_1}\Big) + O\Big(\frac{P_1Q^2}{r^3}\Big) + O\Big(\frac{N}{r}\sum_{\substack{q\le P_1\\ r\nmid q}} E^*(r,q)\Big), \end{aligned}$$

where $c_q(N)$ is the Ramanujan sum. Denote by $\sigma_1(N,r)\frac{N^2}{2}$ the main term above. Summing over $r\sim R$, and appealing to Lemma 3.2, we conclude that

$$\sum_{r\sim R} r \max_{(b_j,r)=1} \Big|I_1 - \sigma_1(N,r)\frac{N^2}{2}\Big| \ll N^2 L^{-A}, \tag{4.3}$$

if B, C are sufficiently large.

Now we turn to $E_2(r)$. Divide $E_2(r)$ into disjoint union $E_{21}(r) \cup E_{22}(r)$, with

$$\begin{aligned} E_{21}(r) &= \{\alpha \in E_2(r) : q \le P_2,\ \ q = rq_1,\ \ r\nmid q_1\}, \\ E_{22}(r) &= \big\{\alpha \in E_2(r) : q \le P_2,\ \ q = r^2q_2,\ \ r\nmid q_2\big\}. \end{aligned}$$

If $\alpha \in E_{21}(r)$, then

$$S(\alpha;r,b_j) = \frac{\mu(q_1)}{\varphi(q)} e\left(\frac{ab\bar{q}_1}{r}\right) \sum_{n\le N} e(n\lambda) + E^*(r,q).$$

Working analogously to the argument above, one sees that

$$\begin{aligned} I_{21} &:= \int_{E_{21}(r)} S(\alpha;r,b_1)S(\alpha;r,b_2)S(\alpha;r,b_3)e(-N\alpha)d\alpha \\ &= \sum_{\substack{q=rq_1\le P_2\\ r\nmid q_1}} \sum_{\substack{a=1\\(a,q)=1}}^{q} e\Big(\frac{-aN}{q}\Big)\frac{\mu(q_1)}{\varphi^3(q)} e\Big(\frac{a\bar{q}_1}{r}(b_1+b_2+b_3)\Big) \\ &\quad \cdot \int_{|\lambda|\le\frac{1}{qQ}} \Big(\sum_{n\le N} e(n\lambda)\Big)^3 e(-N\lambda)d\lambda + O\Big(\frac{N}{r}\sum_{\substack{q=rq_1\le P_2\\ r\nmid q_1}} E^*(r,q)\Big). \end{aligned}$$

And the main term now becomes

$$\begin{aligned} &\frac{1}{\varphi^3(r)} \sum_{\substack{q_1\le\frac{P_2}{r}\\ r\nmid q_1}} \frac{\mu(q_1)}{\varphi^3(q_1)} \sum_{\substack{c=1\\(c,r)=1}}^{r} \sum_{\substack{d=1\\(d,q_1)=1}}^{q_1} e\left(\frac{c}{r}(b_1+b_2+b_3-N)\right) e\left(-\frac{dN}{q_1}\right) \sum_{\substack{n_1+n_2+n_3=N\\ 1\le n_j\le N}} 1 \\ &\quad = \frac{N^2}{2\varphi^2(r)} \sum_{\substack{q_1=1\\ r\nmid q_1}}^{\infty} \frac{\mu(q_1)}{\varphi^3(q_1)} c_{q_1}(-N) + O\left(\frac{N^2L}{rP_2}\right) + O\left(\frac{P_2Q^2}{r}\right). \end{aligned}$$

Denote by $\sigma_2(N,r)\frac{N^2}{2}$ the main term on the right-hand side above. One therefore has

$$I_{21}-\sigma_2(N;r)\frac{N^2}{2} \ll \frac{N^2L}{rP_2}+\frac{P_2Q^2}{r}+\frac{N}{r}\sum_{\substack{q_1\le \frac{P_2}{r}\\ r\nmid q_1}} E^*(r,rq_1),$$

and consequently,

$$\sum_{r\sim R} r \max_{(b_j,r)=1}\left|I_{21}-\sigma_2(N;r)\frac{N^2}{2}\right| \ll N^2L^{-A}, \tag{4.4}$$

if B,C are sufficiently large.

It remains to treat the case $\alpha\in E_{22}(r)$. In this case we have $|S(\alpha;r,b_j)|\le E^*(r,q)$, hence the integral over $E_{22}(r)$ is

$$\begin{aligned} I_{22} &:= \int_{E_{22}(r)} S(\alpha;r,b_1)S(\alpha;r,b_2)S(\alpha;r,b_3)e(-N\alpha)d\alpha \\ &\ll \sum_{\substack{q=r^2q_2\\ r\nmid q_2}} \sum_{\substack{a=1\\ (a,q)=1}}^{q} E^*(r,q)\int_{|\lambda|\le\frac{1}{qQ}}\left|S\left(\frac{a}{q}+\lambda;r,b_2\right)S\left(\frac{a}{q}+\lambda;r,b_3\right)\right|d\lambda \\ &\ll \sum_{\substack{q=r^2q_2\le P_2\\ r\nmid q_2}} E^*(r,q)\int_0^1 |S(\alpha;r,b_2)S(\alpha;r,b_3)|d\alpha \ll \frac{N}{r}\sum_{\substack{q=r^2q_2\le P_2\\ r\nmid q_2}} E^*(r,q). \end{aligned}$$

Therefore

$$\sum_{r\sim R} r\max_{(b_j,r)=1}|I_{22}| \ll NL\sum_{r\sim R}\sum_{\substack{q=r^2q_2\le P_2\\ r\nmid q_2}} E^*(r,q)\ll N^2L^{-A}. \tag{4.5}$$

It is easily verified that $\sigma(N,r)=\sigma_1(N,r)+\sigma_2(N,r)$. Therefore (2.4) follows from (4.3), (4.4), and (4.5). Theorem 1.2 is proved.

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