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THE INVARIANT CONTINUOUS-TRACE C^*-ALGEBRAS BY THE ACTIONS OF COMPACT ABELIAN GROUPS**

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Abstract

The author studies the relation of continuous-trace property between C^*-algebra A and the fixed point C^*-algebra A^α in certain C^*-dynamic system (A, G, α) by introducing an α-invariant continuous trace property. For separable C^*-dynamic system (A, G, α) with G compact and abelian,A liminal, $\alpha_t \in \mathrm{Aut}_{C_b(\hat{A})}(A)$ and pointwise unitary, the necessary and sufficient condition for A to be continuous-trace, which contains A^α continuous-trace, is obtained.

Keywords Continuous-Trace, Action of Group, C^*-Algebra
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§1. Introduction

Let (A, G, α) be a C^*-dynamic system. The relation between $A \times_\alpha G$ and A has been studied for a long time, and considerable progress has been made. Specially if A is a continuous-trace C^*-algebra, I. Raeburn and his collaborators got rich results several years ago. For example, [17] and [19] say that if $\hat{A}$ is paracompact, G is abelian and α is locally unitary, then A is continuous-trace iff $A \times_\alpha G$ is continuous-trace. In other direction, the relation between $A \times_\alpha G$ and A^α with G compact has also been studied for many years. The movement of this paper is to investigate the continuous-trace relation between $A(A \times_\alpha G)$ and A^α and to characterize the pointwise unitary property in a C^*-dynamic system (A, G, α) with A continuous-trace. If A is separable and G is generated by a compact subset, then α is locally unitary iff α is pointwise unitary by Rosenberg's results.In this paper we prove that the pointwise unitary property is equivalent to α-invariant continuous-trace property, if G is compact and abelian, and $\alpha_t \in \mathrm{Aut}_{C_b(\hat{A})}(A)$. By the way, throughout the paper we always assume $\alpha_t \in \mathrm{Aut}_{C_b(\hat{A})}(A)$, which enables us to consider the action in fiber level and whose other equivalent description can be found in [16]. It is easy to see that if α is pointwise unitary then $\alpha_t \in \mathrm{Aut}_{C_b(\hat{A})}(A)$.

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§2. α-Invariant Contionuous-Trace Property and Pointwise Unitary Property

In this part, we give the definitions and some properties of $\alpha(\cdot)$-invariant Fell's condition and α-invariant continuous-trace C^*-algebra.

Definition 2.1. *Let X be a locally compact Hausdoff space, $\mathbf{A} = (A(x), \Lambda)$ be a continuous field of elementary C^*-algebras in the sense of* [2]. *For every $x \in X$, $(A(x), G, \alpha(x))$ is a C^*-dynamic system such that $\forall f(\cdot) \in \Lambda, t \in G$, $\alpha(\cdot)_t(f(\cdot)) \in \Lambda$, where $\alpha(\cdot)_t(f(\cdot))(x) = \alpha(x)_t(f(x))$. We say that $\mathbf{A}$ satisfies $\alpha(\cdot)$-invariant Fell's condition if for every $x \in X$ there is a neighbourhood N of x and a continuous field $p(\cdot)$ of projections of rank 1 over N such that $\alpha(y)_t(p(y)) = p(y)$, $\forall y \in N$, $t \in G$.*

Naturally we want to know, when $\mathbf{A}$ is locally constructed by a continuous field of Hilbert spaces as in [2], what is the group action.

Let A be an elementary C^*-algebra, G be a locally compact Hausdorff group, (A, G, α) be a C^*-dynamic system, and $p \in A$ be a projection of rank 1 which is invariant under the action α. (H, ξ, U) is defiend by (A, p, α), where $H = Ap$ is a Hilbert space, whose inner product is given by $\langle a, b\rangle = \mathrm{tr}(b^*a)$, $\xi = p \in H$, and $U_t(ap) = \alpha_t(a)p$, $\forall t \in G$. By

$$\|ap\|_H^2 = \mathrm{tr}(pa^*ap) = \|pa^*ap\|_A = \|ap\|_A^2 \ \text{ and } \alpha_t(a)p = \alpha_t(ap),$$

we see that U is a strongly continuous homomorphism from G to $U(H)$(the unitary group of H) and $U_t\xi = \xi$, $\forall t \in G$. We denote the map from (A, p, α) to (H, ξ, U) by β.

Let H be a Hilbert space, ξ be a unit vector in H, and U be a unitary representation of G on H such that $U_t\xi = \xi(\forall t \in G)$. We get (A, p, α), where $A = K(H)$, p is the projection of rank 1 on $\mathbf{C}\xi$, $\alpha_t = AdU_t$. $\forall \eta \in H$,

$$\alpha_t(p)\eta = U_t p U_t^* \eta = \langle U_t^*\eta, \xi\rangle U_t\xi = \langle \eta, \xi\rangle\xi = p\eta.$$

So $\alpha_t(p) = p$,$\forall t \in G$. Put $\gamma(H, \xi, U) = (A, p, \alpha)$.

Lemma 2.1. (i) *Let H be a Hilbert space, U be a unitary representation of G on H, ξ be a unit vector in H such that $U_t\xi = \xi$, $\forall t \in G$. Put*

$$\gamma(H, \xi, U) = (A, p, \alpha), \quad \beta(A, p, \alpha) = (H', \xi', U').$$

Let $\varphi(a) = a\xi$, for each $a \in Ap = H'$. Then φ is an isomorphism of the Hilbert spaces such that $\varphi(\xi') = \xi$, $\varphi U_t' = U_t\varphi$.

(ii) *Let A be an elementary C^*-algebra, (A, G, α) be a C^*-dynamic system, p be a projection of A of rank 1 such that $\alpha_t(p) = p$, $\forall t \in G$. Put*

$$\beta(A, p, \alpha) = (H, \xi, U), \quad \gamma(H, \xi, U) = (A', p', \alpha').$$

For every $a \in A$, let $\psi(a)$ be a linear operator on $H = Ap$ defined by $\psi(a)y = ay(\forall y \in Ap)$. Then ψ is an isomorphism from the C^*-algebra A onto C^*-algebra A' such that $\psi(p) = p'$, $\psi\alpha_t = {\alpha'}_t\psi$, $\forall t \in G$.

Proof. By [2, 10.6.6] and the discussion above, it is only necessary to check $\varphi U_t' = U_t\varphi$ and $\psi\alpha_t = {\alpha'}_t\psi$.

Let $\mathbf{H} = (H(x), \Gamma)$ be a continuous field of Hilbert space over locally compact space X with a continuous vector field $\xi(\cdot)$ such that $\|\xi(x)\| = 1$, $U(x)$ be a unitary representation of G on $H(x)$ such that $\forall \eta(\cdot) \in \Gamma$, $t \in G$, $U(\cdot)_t(\eta(\cdot)) \in \Gamma$ and $U(x)_t(\xi(x)) = \xi(x)$,$\forall x \in X$.

We put $\gamma(\mathbf{H}, \xi(\cdot), U(\cdot)) = (\mathbf{A}, p(\cdot), \alpha(\cdot))$ pointwise defined by discussion above in Lemma 2.1 and $\mathbf{A}$ is the continuous field of elementary C^*-algebras $(A(x), \Lambda)$ over X,where Λ is (closed) generated by all sections of the form $\theta_{\xi_1(\cdot),\xi_2(\cdot)} + \theta_{\xi_3(\cdot),\xi_4(\cdot)} + \cdots + \theta_{\xi_{2n-1}(\cdot),\xi_{2n}(\cdot)}$ as in [2, 10.7], where $\xi_i(\cdot) \in \Gamma$. Since for every $\xi_1(\cdot), \xi_2(\cdot) \in \Gamma$,$x \in X$, $t \in G$,

$$\alpha(x)_t(\theta_{\xi_1(x),\xi_2(x)}) = U(x)_t \theta_{\xi_1(x),\xi_2(x)} U(x)_t^* = \theta_{U(x)_t\xi_1(x),U(x)_t\xi_2(x)},$$

for every $f(\cdot) \in \Lambda$, $t \in G$, $\alpha(\cdot)_t(f(\cdot)) \in \Lambda$.

Let $\mathbf{A} = (A(x), \Lambda)$ be a continuous field of elementary C^*-algebras over X, $(A(x), G, \alpha(x))$ be a C^*-dynamic system such that $\alpha(\cdot)_t(f(\cdot)) \in \Lambda$, $\forall t \in G, f(\cdot) \in \Lambda$, $p(\cdot)$ be a continuous section of projections of rank 1 such that $\alpha(x)_t(p(x)) = p(x)$, $\forall x \in X, t \in G$. We put $\beta(\mathbf{A}, p(\cdot), \alpha(\cdot)) = (\mathbf{H}, \xi(\cdot), U(\cdot))$, pointwise defined by discussion above in Lemma 2.1, and $\mathbf{H} = (H(x), \Gamma)$ is a continuous field of Hilbert spaces over X, where Γ is the set of all $\eta(\cdot) \in \Lambda$ such that for every $x \in X$, $\eta(x)p(x) = \eta(x)$. For every $\eta(\cdot) \in \Gamma$, $t \in G$, $x \in X$,

$$(U(x)_t(\eta(x)))p(x) = (\alpha(x)_t(\eta(x)))p(x) = \alpha(x)_t(\eta(x)) = U(x)_t(\eta(x)),$$

and so clearly $U(\cdot)_t(\eta(\cdot)) \in \Gamma$. We also have $U(x)_t(\xi(x)) = \xi(x)$, $\forall x \in X$.

Lemma 2.2. (i) *Let $\mathbf{H} = (H(x), \Gamma)$ be a continuous field of Hilbert spaces over X, with a continuous section $\xi(\cdot)$ such that $\|\xi(x)\| = 1$, $U(x)$ be a unitary representation of G on $H(x)$ such that $U(x)_t(\xi(x)) = \xi(x)$ for every $x \in X$, and $U(\cdot)_t(\eta(\cdot)) \in \Gamma$,for every $\eta(\cdot) \in \Gamma$,$t \in G$. Put $(\mathbf{H}', \xi'(\cdot), U'(\cdot)) = \beta\gamma(\mathbf{H}, \xi(\cdot), U(\cdot))$, $\mathbf{H}' = (H'(x), \Gamma')$. For every $x \in X$, let φ_x be an isomorphism from $H'(x)$ onto $H(x)$* (Lemma 2.1). *Then $\varphi = (\varphi_x)_{x\in X}$ is an isomorphism from $\mathbf{H}'$ to $\mathbf{H}$ such that $U(x)\varphi_x = \varphi_x U'(x)$.*

(ii) *Let $\mathbf{A} = (A(x), \Lambda)$ be a continuous field of elementary C^*-algebras over X with a continuous section $p(\cdot)$ of projections of rank 1, $(A(x), G, \alpha(x))$ be a C^*-dynamic system such that for every $t \in G$, $f(\cdot) \in \Lambda$, $\alpha(\cdot)_t(f(\cdot)) \in \Lambda$, and $\alpha(x)_t(p(x)) = p(x)$. Let $(\mathbf{A}', p'(\cdot), \alpha'(\cdot)) = \gamma\beta(\mathbf{A}, p(\cdot), \alpha(\cdot))$, $\mathbf{A}' = (A'(x), \Lambda')$. For every $x \in X$, let ψ_x be the canonical isomorphism from $A(x)$ onto $A'(x)$* (Lemma 2.1). *Then $\psi = (\psi_x)_{x\in X}$ is an isomorphism from $\mathbf{A}$ to $\mathbf{A}'$ such that $\psi_x(p(x)) = p'(x)$, $\alpha'(x)_t\psi_x = \psi_x\alpha(x)_t$, $\forall x \in X, t \in G$.*

Proof. Use Lemma 2.1, the discussion above, and [2, 10.7.6].

Corollary 2.1. *Let $\mathbf{A} = (A(x), \Lambda)$ be a continuous field of elementary C^*-algebras; for every $x \in X$, $(A(x), G, \alpha(x))$ be a C^*-dynamic system such that $\alpha(\cdot)_t(f(\cdot)) \in \Lambda$, $\forall t \in G$, $f(\cdot) \in \Lambda$. Then the following are equivalent:*

(i) *For every $x_0 \in X$, there exist a neighbourhood V of x_0, a continuous field of Hilbert spaces $\mathbf{H} = (H(x), \Gamma)$ over V with a continuous vector field $\xi(\cdot)$ over V with $\|\xi(x)\| = 1(\forall x \in V)$, and for every $x \in V$ a unitary representation $U(x)$ of G on $H(x)$ satisfying $U(x)_t(\xi(x)) = \xi(x)$, $U(\cdot)_t(\eta(\cdot)) \in \Gamma$ for every $t \in G$, $\eta(\cdot) \in \Gamma$, such that $\mathbf{A}|_V$ is isomorphic to $\mathbf{A}(\mathbf{H}) = \gamma(\mathbf{H})$ and the isomorphism maps α to $AdU(\cdot)$.*

(ii) *$\mathbf{A}$ satisfies the $\alpha(\cdot)$-invariant Fell's condition.*

Definition 2.2. *Let A be a liminal C^*-algebra with Hausdoff spectrum, (A, G, α) be a C^*-dynamic system. A is called α-invariant continuous-trace C^*-algebra if for every $x_0 \in \hat{A}$, there is a neighbourhood N of x_0 and a $p \in A^\alpha$, $p \geq 0$, such that $\forall x \in N$, $x(p)$ is a projection of rank 1. It is clear that an α-invariant continuous-trace C^*-algebra is a continuous-trace C^*-algebra.*

Lemma 2.3. *Let A be a liminal C^*-algebra with Hausdorff spectrum, (A, G, α) be a C^*-dynamic system such that for every $t \in G$, $\alpha_t \in \mathrm{Aut}_{C_b(\hat{A})}(A)$. Let $\mathbf{A}(A) = (A(x), \Lambda)$ be the continuous field of elementary C^*-algebras over $\hat{A}$ defined by A, and for every $x \in \hat{A}$, $a \in A$, let $\alpha(x)_t(a(x)) = x(\alpha_t(a))$. Then for every $f(\cdot) \in \Lambda, t \in G$, $\alpha(\cdot)_t(f(\cdot)) \in \Lambda$ and A is an α-invariant continuous-trace C^*-algebra if and only if $\mathbf{A}(A)$ satisfies the $\alpha(\cdot)$-invariant Fell's condition.*

Proof. Λ is (closed) generated by $\{a(\cdot)|a \in A\}$ (see [2]), where $a(x) = x(a)$, $\forall x \in \hat{A}$. So $\alpha(\cdot)_t(f(\cdot)) \in \Lambda$, $\forall f(\cdot) \in \Lambda, t \in G$. It is also clear that $(A(x), G, \alpha(x))$ is a C^*-dynamic system by [16, Lemma 1.4], and $A \cong \Gamma_0(\mathbf{A})$ by [2, Lemma 10.5.4]. Denoting this isomorphism by φ, we see $\varphi\alpha = \alpha(\cdot)\varphi$. If A is an α-invariant continuous-trace C^*-algebra, for every $x_0 \in \hat{A}$ there exists a neighbourhood N of x_0 and a $p \in A^\alpha$ such that for every $x \in N$, $x(p)$ is a projection of rank 1, then $\varphi(p)$ is invariant under $\alpha(\cdot)$ and $\varphi(p)(x) = x(p)$ for every $x \in N$. So $\mathbf{A}(A)$ satisfies the $\alpha(\cdot)$-invariant Fell's condition. Conversely $\forall x_0 \in \hat{A}$, there exists a neighbourhood N_1 of x_0 and a continuous field of projections $p_1(\cdot)$ of rank 1 over N_1 such that $\forall x \in N_1$, $t \in G$, $\alpha(x)_t(p_1(x)) = p_1(x)$ and $\overline{N_1}$ is compact. Taking a neighbourhood N of x_0 such that $N \subset \overline{N} \subset N_1$, we can choose a continuous function f on $\hat{A}$ such that $f(x) = 1$, if $x \in N$, and $f(x) = 0$, if $x \in N_1^c$. Let $p(\cdot)$ be the continuous field of projections over $\hat{A}$, $p(x) = f(x)p_1(x)$ if $x \in N_1$, and $p(x) = 0$ if $x \in N_1^c$. Then for every $t \in G$, $x \in \hat{A}$, $\alpha(x)_t(p(x)) = p(x)$, and for every $x \in N$, $p(x)$ is a projection of rank 1. By φ, there is a unique $p \in A^\alpha$ such that $x(p) = p(x)$. So A is an α-invariant continuous-trace C^*-algebra.

Lemma 2.4. *Let (A, G, α) be a C^*-dynamic system with $\alpha_t \in \mathrm{Aut}_{C_b(\hat{A})}(A)$,$A$ α-invariant continuous-trace. Then α is pointwise unitary.*

Proof. By Lemma 2.3 and Corollory 2.1 for every $x_0 \in \hat{A}$, there is a neighbourhood N of x_0 such that $\mathbf{A}(A)|_N \cong \mathbf{A}(\mathbf{H})$, where $\mathbf{H}$ is a continuous field of Hilbert spaces. Let $\varphi(\cdot)$ be this isomorphism. We also have $\varphi(x)\alpha(x)_t = AdU(x)_t\varphi(x)$,$\forall x \in N$, $t \in G$. Specially let $x = x_0$. For every $a \in A$,

$$\varphi(x_0)x_0(\alpha_t(a)) = \varphi(x_0)\alpha(x_0)_t(x_0(a)) = AdU(x_0)_t(\varphi(x_0)x_0(a)).$$

So $(\varphi(x_0)x_0, U(x_0))$ is a covariant pair of representation of (A, G, α). Since $\varphi(x_0)$ is an isomorphism from elementary C^*-algebra $x_0(A)$ onto $K(H(x_0))$, $\varphi(x_0)x_0 \cong x_0$. So A is pointwise unitary.

Theorem 2.1. *Let (A, G, α) be a C^*-dynamic system with G compact and abelian, $\alpha_t \in \mathrm{Aut}_{C_b(\hat{A})}(A)(\forall t \in G)$. Then the following are equivalent:*

(i) *The closed ideal generated by $m(A) \cap A^\alpha$ is A,and A is pointwise unitary,where $m(A)$ are the continuous-trace ideal of A.*

(ii) *For every $x_0 \in \hat{A}$, there exists $a \in m(A) \cap A^\alpha$ such that $x_0(a) \neq 0$, and A is pointwise unitary.*

(iii) *A is an α-invariant continuous-trace C^*-algebra.*

(iv) *A is a continuous-trace C^*-algebra, and α is pointwise unitary.*

(v) *(A, G, α) is locally equivalent to $(C_0(\hat{A}), G, id)$ in the sense of* [25].

Proof. The equivalence of (i) and (ii) is an easy consequence of the fact that for every closed ideal I of A, $I = \bigcap\limits_{\hat{A}\setminus\hat{I}} \ker x$. And the equivalence of (iv) and (v) is proved in Propsition

4.3 of [25].

(iii)⇒(iv). Lemma 2.4.

(iii)⇒(ii). we only need to prove the first part. Since A is of continuous-trace, $\hat{A}$ is Hausdorff and A is liminal. $\forall x_0 \in \hat{A}$, there is a neighbourhood N of x_0 and a $p \in A^\alpha$ such that $\forall x \in N$, $x(p)$ is a projection of rank 1. Choosing an open set W such that $x_0 \in W \subset \overline{W} \subset N$, we let $I = \bigcap\limits_{x \in \hat{A} \setminus W} \ker x$, so $\hat{I} \cong W$. Since $x_0 \in W$, by the theorem on transitity, we can choose $a \in I^+$ such that $x_0(a)x_0(p) = x_0(p)$. Replacing a by $\int \alpha_t(a)dt$ and using $p \in A^\alpha$, we can assume $a \in I^+ \cap A^\alpha$. Put $b = p^{\frac{1}{2}}ap^{\frac{1}{2}}$. Then $b \in m(A) \cap A^\alpha$, and $x_0(b) = x_0(p)x_0(a)x_0(p) = x_0(p) \neq 0$.

(ii)⇒(iii). For every $x_0 \in \hat{A}$, there exists $a_0 \in m(A) \cap A^\alpha$, $x_0(a_0) \neq 0$. We can assume that $a_0 \in A^+$, and $\|x_0(a_0)\| = 1$. For $x_0(A) = K(H_0)$, where H_0 is the Hilbert space of representation x_0, the eigenspace H_1 of $x_0(a_0)$ corresponding to eigenvalue 1 is not zero. Let $\alpha(x_0)_t(x_0(a)) = x_0(\alpha_t(a))$ $(\forall t \in G, a \in A)$. By the pointwise unitary of A, there is a unitary representation $U(x_0)$ of G on H_0 such that $x_0(\alpha_t(a)) = AdU(x_0)_t(x_0(a))$. So

$$\alpha(x_0)_t(x_0(a)) = AdU(x_0)_t(x_0(a)).$$

For every $\xi \in H_1$, $x_0(a_0)\xi = \xi$. So

$$U(x_0)_t(x_0(a_0))U(x_0)_t^*\xi = x_0(\alpha_t(a_0))\xi = x_0(a_0)\xi = \xi,$$

i.e., $x_0(a_0)U(x_0)_t^*\xi = U(x_0)_t^*\xi(\forall t \in G)$. Therefore $U(x_0)_tH_1 = H_1(\forall t \in G)$. By the compactness of $x_0(a_0)$, we assume that the dimension of H_1 is $n < \infty$. We assert that for every $t \in G$, there is a $\mu(t) \in \mathbf{C}$, $|\mu(t)| = 1$, such that $\{\overline{\mu(t)}U(x_0)_t | t \in G\}$ have a common nonzero fixed point in H_1. In fact $t \in G$, the spectrum of $U(x_0)_t|_{H_1}$ is a point spectrum and contained in the unit circle of $\mathbf{C}$. Let $\mu(t)$ be a spectral point. Then $\overline{\mu(t)}U(x_0)_t|_{H_1}$ has a point spectrum 1. So $\overline{\mu(t)}U(x_0)_t$ has fixed points in H_1. We let $H_{t,\mu(t)} = \{\xi \in H_1, \overline{\mu(t)}U(x_0)_t\xi = \xi\}$. Since G is abelian, $U(x_0)_s$ commutes with $\overline{\mu(t)}U(x_0)_t$ for every $t \in G$. Choosing $t_1 \in G$ and $\mu(t_1)$ as above, we see that $H_{t_1,\mu(t_1)}$ is a finite dimension invariant subspace of $U(x_0)_s(\forall s \in G)$. Similiarly if there is $s \in G$ such that every eigenspace of $U(x_0)_s|_{H_{t_1,\mu(t_1)}}$ corresponding to one eigenvalue is not $H_{t_1,\mu(t_1)}$, we can choose $t_2 \in G$, $\mu(t_2) \in \mathbf{C}$, $|\mu(t_2)| = 1$, such that

$$\dim(H_{t_1,\mu(t_1)}) > \dim(H_{t_1,\mu(t_1)}) \bigcap H_{t_2,\mu(t_2)}) \neq 0.$$

Continuing this process, we have two cases:

(1) There are $t_1, t_2, \cdots, t_m \in G$,$\mu(t_1)$, $\mu(t_2)$,$\cdots$,$\mu(t_m) \in \mathbf{C}$, $|\mu(t_i)| = 1$ $(1 \leq i \leq m)$ such that for $1 \leq i \leq m-1$

$$\dim\Big(\bigcap_{j=1}^{i} H_{t_j,\mu(t_j)}\Big) > \dim\Big(\bigcap_{j=1}^{i+1} H_{t_j,\mu(t_j)}\Big),\ \dim\Big(\bigcap_{j=1}^{m} H_{t_j,\mu(t_j)}\Big) = 1.$$

(2) There is an $H_G \subset H_1$, $\dim H_G \geq 2$, and for every $t \in G$, there is $\mu(t)$ such that $H_{t,\mu(t)} \supset H_G$. In case (2), the assertion is clear. In case (1), since $H_{t_j,\mu(t_j)}$ is an invariant subspace of $U(x_0)_t$,so is $\bigcap\limits_{j=1}^{m} H_{t_j,\mu(t_j)}$. But its dimension is 1, so there is

$$\xi \in \bigcap_{j=1}^{m} H_{t_j,\mu(t_j)},\quad \|\xi\| = 1,\quad U(x_0)_t\xi = \mu(t)\xi \quad (t \in G).$$

Therefore $\overline{\mu(t)}U(x_0)_t$ has a fixed point ξ. This completes the assertion. Let $\xi_0 \neq 0$ be a common fixed point of $\{\overline{\mu(t)}U(x_0)_t | t \in G\}$ in H_1, and p_0 be the projection on $\mathbf{C}\xi_0$. Then $\forall \eta \in H_0$,

$$\begin{aligned}\alpha(x_0)_t(p_0)\eta &= AdU(x_0)_t(p_0)\eta \\ &= U(x_0)_t p_0 U(x_0)_t^* \eta \\ &= \overline{\mu(t)}U(x_0)_t(\langle \mu(t)U(x_0)_t^*\eta, \xi_0\rangle \xi_0) \\ &= \langle \eta, \overline{\mu(t)}U(x_0)_t\xi_0\rangle (\overline{\mu(t)}U(x_0)_t\xi_0) \\ &= \langle \eta, \xi_0\rangle \xi_0 = p_0\eta.\end{aligned}$$

So $p_0 \in x_0(A)^{\alpha(x_0)} = (K(H_0))^{\alpha(x_0)}$ is a projection of rank 1. Let $b \in A^+$ such that $x_0(b) = p_0$. Replacing b by $\int \alpha_t(b)dt$, we can assume $b \in A^\alpha$. Let $b_1 = f(b)$, where $f(t) = t$, for $0 \leq t \leq 1$; $f(t) = 1$, for $t \geq 1$. Then $0 \leq b_1 \leq 1$, $b_1 \in A^\alpha$, and $x_0(b_1) = f(x_0(b)) = f(p_0) = p_0$. Let $b_2 = a_0^{\frac{1}{2}} b_1 a_0^{\frac{1}{2}} \in A^\alpha$. Then

$$x_0(b_2) = x_0(a_0)^{\frac{1}{2}} p_0 x_0(a_0)^{\frac{1}{2}} = p_0.$$

Since $b_2 \leq a_0$ (so $b_2 \in A^\alpha \cap m(A)^+$) and $\mathrm{tr}(x_0(b_2)) = 1, \|x_0(b_2)\| = 1$, there exists a neighbourhood N of x_0 such that for every $x \in N$, the largest eigenvalue of $x(b_2)$ is large than $\frac{3}{4}$, while the others are less than $\frac{1}{2}$, and $\mathrm{tr}(x(b_2)) < 2$. Let $p = g(b_2)$, where $g(t) \in C(\mathbf{R})$, $g(t) = 1$,if $t \geq \frac{3}{4}$; $g(t) = 0$ if $t \leq \frac{1}{2}$; $g(t)$ is linear if $\frac{1}{2} \leq t \leq \frac{3}{4}$. Then $p \in m(A) \cap A^\alpha$, and for every $x \in N$, $x(p)$ is a projection of rank 1.

(iv)$\Rightarrow$(ii). If A is a continuous-trace C^*-algebra, for every $x_0 \in \hat{A}$, there is $a \in m(A)^+$ such that $x_0(a) \neq 0$. Because (A, G, α) is pointwise unitary, for every $x \in \hat{A}$, there is a unitary representation $U(x)$ of G such that $\alpha(x) = AdU(x)$. Let $b = \int \alpha_t(a)dt \in A^\alpha$. Then

$$\begin{aligned}\mathrm{tr}(x(b)) &= \mathrm{tr}\Big(\int x(\alpha_t(a))dt\Big) = \mathrm{tr}\Big(\int \alpha(x)_t(x(a))dt\Big) \\ &= \int \mathrm{tr}(AdU(x)_t(x(a)))dt = \int \mathrm{tr}(x(a))dt = \mathrm{tr}(x(a)).\end{aligned}$$

So $b \in A^\alpha \cap m(A)$, and clearly $x_0(b) \neq 0$.

Remark 2.1. The equivalence of (i) and (ii) does not require that G should be compact abelian and $\alpha_t \in \mathrm{Aut}_{C_b(\hat{A})}(A)$; (iii)$\Rightarrow$(vi) does not require that G should be compact abelian; (iii)$\Rightarrow$(ii) does not require that G should be abelian.

§3. α-invariant Continuous-Trace Property of A and Continuous-Trace Property of A^α

In this part, we discuss the continuous-trace relation between A^α and A.

Lemma 3.1. *Let $A \subset K(H)$ be a nondegenerate C^*-algebra. Then $H = \bigoplus_i \bigoplus_{j=1}^{n_i} H_{ij}$ ($n_i < \infty$) such that (id, H_{ij}) is an irreducible representation of A,and (id, H_{ij}) is equivalent to (id, H_{il}), (id, H_{ij}) is disjoint with $(id, H_{kl})(i \neq k)$, and for every $\pi \in \hat{A}$, π is equivalent to some (id, H_{ij}).*

Proof. By [2, 5.4.13], (id, H) is equivalent to $\bigoplus_i \bigoplus_{j=1}^{n_i} (\pi_{ij}, K_{ij})$, where $n_i < \infty$, and (π_{ij}, K_{ij}) is an irreducible representation of A, π_{ij} is equivalent to π_{il}, π_{ij} is disjoint with

π_{kl} $(i \neq k)$. So (id, H) is equal to $\bigoplus_i \bigoplus_{j=1}^{n_i} (id, H_{\pi_{ij}})$. Denoting $H_{\pi_{ij}}$ by H_{ij}, we see that $H = \oplus H_{ij}$, and (id, H_{ij}) is equivalent to (id, H_{il}), (id, H_{ij}) is disjoint with (id, H_{kl}) $(i \neq k)$. Let $E_{ij} \in A^{'}$ be the projection corresponding to (id, H_{ij}). Then $E_{ij}H = H_{ij}$, and for fixed i the central projections of $E_{ij}(j = 1, 2, \cdots, n_i)$ in $A^{'} \cap A^{''}$ are the same, notated by F_i. By irreducibility, if $i \neq k$,F_i is orthogonal to F_k. Since

$$\sum_i F_i \geq \sum_{i,j} E_{ij} = 1, \quad \sum_i F_i = 1.$$

Let (π, H_0) be an irreducible representation of A. There is an irreducible representation $(\widetilde{\pi}, \widetilde{H}_0)$ of $K(H)$ such that $\widetilde{\pi}|_{H_0} = \pi$. By irreducibility, $(\widetilde{\pi}, \widetilde{H}_0)$ is equivalent to (id, H), so there is an isometric isomorphism U from H onto $\widetilde{H}_0$ such that $\widetilde{\pi} = (AdU)id$. Let $H_\pi = U^*H_0$. Then

$$AU^*H_0 = U^*UAU^*H_0 = U^*\widetilde{\pi}(A)H_0 = U^*\pi(A)H_0 \subset U^*H_0.$$

So (id, H_π) is equivalent to (π, H_0) as the irreducible representation of A. Let E_π be the projection corresponding to (id, H_π), F_π be the central projection of E_π. By the irreducibility of (id, H_π) and (id, H_{ij}), there is an i_0 such that $F_{i_0} = F_\pi$. So (id, H_π) is equivalent to $(id, H_{i_0 j})$, i.e., (π, K) is equivalent to some (id, H_{ij}).

The following lemma is well known (for example [24, Theorem 3.2] is essentially one of it is generalized forms). For clearness we give it a simple proof.

Lemma 3.2. *Let A be a C^*-algebra with Hausdorff spectrum, (A, G, α) be a C^*-dynamic system with G compact,$\alpha_t \in \mathrm{Aut}_{C_b(\hat{A})}(A)$. Then $X_\alpha = \{x \in \hat{A} | x(A^\alpha) \neq 0\} = \hat{I}$ is an open set of $\hat{A}$, where I is the closed ideal generated by A^α, and there exists a continuous open mapping θ from $Prim(A^\alpha)$ onto X_α.*

Proof. Let $\mathbf{A} = (A(x), \Lambda)$ be a continuous field of C^*-algebras defined by A. For every $x \in X_\alpha$, let $E_\alpha(x) = A(x)^{\alpha(x)}$ where $\alpha(x)_t(x(a)) = x(\alpha_t(a))$ for every $a \in A$, and let Λ_0 be the restriction of Λ on X_α in the sense of [2], and let

$$\Lambda_\alpha = \{a(\cdot) \in \Lambda_0 | a(x) \in A(x)^{\alpha(x)}, \forall x \in X_\alpha\}.$$

We assert that $\mathbf{A}^\alpha = (E_\alpha(x), \Lambda_\alpha)$ is a continuous field of C^*-algebras over X_α.

Let φ be the homomorphism from A^α to $\Gamma_0(\mathbf{A}^\alpha)$(the C^*-algebra defined by $\mathbf{A}^\alpha$) defined by $\varphi(a)(x) = x(a)$ $(\forall x \in X_\alpha)$. By Dans-Hoffmann theorem and [2, 11.5.3.], $A^\alpha = \Gamma_0(\mathbf{A}^\alpha)$ and this completes the proof by [11, Theorem 4].

Remark 3.1. We have proved that A^α is isomorphic to the C^*-algebra defined by $\mathbf{A}^\alpha$.

Theorem 3.1. *Let (A, G, α) be a C^*-dynamic system with A liminal, $\hat{A}$ Hausdorff, A^α continuous-trace, G compact and $\alpha_t \in \mathrm{Aut}_{C_b(\hat{A})}(A)$. Let I be the (closed) ideal of A generated by A^α. If $\theta : \widehat{A^\alpha} \to \hat{I}$ (defined in* Lemma 3.2) *is a sheaf and for every $x \in \hat{I}$, every minimal projection in $(A(x)^{\alpha(x)}|_{\overline{A(x)^{\alpha(x)}H(x)}})^{'} \subset B(\overline{A(x)^{\alpha(x)}H(x)})$ is central, then I is an α-invariant continuous-trace C^*-algebra.*

Proof. Because $A^\alpha = I^\alpha$, we assume $I = A$. For $x \in \widehat{A}$, there are an open set $N \subset \widehat{A^\alpha}$ and $p \in (A^\alpha)^+$ such that $\theta(N)$ is a neighbourhood of x, $\theta|_N$ is a homeomorphism, and for every $y \in N$,$y(p)$ is a projection of rank 1, and for every $y_1 \in N^c$ such that

$$\theta(y_1) \in \theta(N), \quad y_1(p) = 0.$$

By Remark 3.1, and the construction in Theorem 4 of [11], $y(p) = \pi(\theta(y)(p))$, where π is an irreducible representation of $A(\theta(y))^{\alpha(\theta(y))} \triangleq A^y$. But A is liminal, so $A(\theta(y)) = K(H(\theta(y)))$. By Lemma 3.1, $\pi \cong (id, H_\pi)$, where $H_\pi \subset \overline{A^y(H(\theta(y)))} \triangleq H_y$. So $\theta(y)(p)|_{H_\pi}$ is a projection of rank 1. Let E_π be the projection of H_y on H_π. Then E_π is a minimal projection in $(A^y|_{H_y})' \subset B(H_y)$ by the irreducibility of π. So E_π is central. By the same reason and Lemma 3.1, $H_y = \oplus H_j$, and letting E_j be the projection on H_j, we see that E_j is orthogonal to $E_i (i \neq j)$, and E_j is central.

If there is a j_0 such that $E_{j_0} \neq E_\pi$ (so $(id, H_{j_0}) \ncong (id, H_\pi)$) and $\theta(y)(p)E_{j_0} \neq 0$, let $y_{j_0} = (id, H_{j_0})\theta(y)$ be an irreducible representation of A^α, then

$$\theta(y_{j_0}) = \theta(y) \in \theta(N), \quad y_{j_0} \notin N, \quad \text{and} \quad y_{j_0}(p) = \theta(y)(p)E_{j_0} \neq 0.$$

This is a contradiction, so $\theta(y)(p)$ is a projection in $H(\theta(y))$ of rank 1, since $\theta(y)(p)|_{H_y^\perp} = 0$.

Corollary 3.1. *Let (A, G, α) be a C^*-dynamic system with A liminal, $\hat{A}$ Hausdorff, G compact and $\alpha_t \in \mathrm{Aut}_{C_b(\hat{A})}(A)$, I be the closed ideal generated by A^α. If for every $x \in \hat{I}$, $x(A^\alpha) = K(\overline{A(x)^{\alpha(x)}H(x)})$, then I is an α-invariant continuous-trace C^*-algebra iff A^α is a continuous-trace C^*-algebra.*

Proof. In this case, $(A(x)^{\alpha(x)}|_{\overline{A(x)^{\alpha(x)}(H(x))}})'$ is trival, and by Lemma 3.2, [11, Theorem 4], θ is a homeomorphism.

Theorem 3.2. *Let (A, G, α) be a C^*-dynamic system with A^α continuous-trace,G compact and abelian,$\alpha_t \in \mathrm{Aut}_{C_b(\hat{A})}(A)$ $(\forall t \in G)$. If I is an α-invariant continuous-trace C^*-algebra, then $\theta : \widehat{A^\alpha} \to \hat{I}$ is a sheaf, and for every $x \in \hat{I}$, every minimal projection in $(A(x)^{\alpha(x)}|_{\overline{A(x)^{\alpha(x)}H(x)}})' \subset B(\overline{A(x)^{\alpha(x)}H(x)})$ is central.*

Proof. Since $A^\alpha = I^\alpha$, without loss of generality,we assume $A = I$. First for every $x \in \hat{A}$, we see that $\theta^{-1}(x)$ is discret in $\widehat{A^\alpha}$. Otherwise because $\widehat{A^\alpha}$ is Hausdorff, there is a net $\{z_i\} \subset \theta^{-1}(x)$, and $z \in \theta^{-1}(x)$ such that $z_i \to z$, $z_i \neq z$. By Lemma 3.1,

$$z_i \cong (id, H_i)x, \quad z \cong (id, H_0)x,$$

where $H_0,\ H_i \subset \overline{A(x)^{\alpha(x)}H(x)}$ (notated by $H(x,\alpha)$) and (id, H_i), (id, H_0) are irreducible representations of $A(x)^{\alpha(x)}$. Let F_i and F respectively be the central projections in $(A(x)^{\alpha(x)}|_{H(x,\alpha)})'$ corresponding to (id, H_i) and (id, H_0), $U(x)$ be the unitary representation of G on $H(x)$ such that $AdU(x)_t = \alpha(x)_t$. Then

$$\ker z_i = \{a \in A^\alpha | x(a)F_i = 0\},$$

and for every $a \in A^\alpha$,$\xi \in H(x)$,

$$U(x)_t(x(a)\xi) = x(a)(U(x)_t\xi).$$

So $H(x,\alpha)$ is invariant under $U(x)_t$. Let $U(x,\alpha)_t$ be the restriction of $U(x)_t$ on $H(x,\alpha)$. For every $b \in A^\alpha$,

$$U(x,\alpha)_t x(b) U(x,\alpha)_t^* = U(x)_t x(b) U(x)_t^*|_{H(x,\alpha)} = x(b)|_{H(x,\alpha)},$$

i.e., $U(x,\alpha)_t \in (A(x)^{\alpha(x)}|_{H(x,\alpha)})'$. So

$$AdU(x,\alpha)_t(F_i) = F_i,$$

and so is it for F. Since A^α is a continuous-trace C^*-algebra, there is $p \in A^\alpha$ such that $z(p)$ is a projection of rank 1. So $x(p)|_{FH(x,\alpha)}$ has eigenvalue 1, whose eigenspace is invariant under $U(x,\alpha)_t(\forall t \in G)$. By the same discussion as in the proof of Theorem 2.1. (ii)$\Rightarrow$(iii), there is a $\xi_0 \in FH(x,\alpha)$, $\|\xi_0\| = 1$ such that

$$U(x)_t\xi_0 = \lambda(x,t)\xi_0,$$

where $\lambda(x,t)$ is a complex number with module 1. Let p_0 be the projection on $\mathbf{C}\xi_0$, a be an element in A such that $x(a) = p_0$. Since $F \perp F_i$, $x(a)F_i = 0$, $x(a)F = p_0$. Let $b = \int \alpha_t(a)dt \in A^\alpha$. Then

$$x(b)F_i = \int U(x)_t x(a) U(x)^*_t F_i dt = \int U(x)_t x(a) F_i U(x,\alpha)^*_t F_i dt = 0,$$

$$x(b)F = \int U(x)_t x(a) U(x)^*_t F dt = \int U(x,\alpha)_t p_0 U(x,\alpha)^*_t dt = p_0 \neq 0.$$

So $0 \neq b \in \ker z_i \backslash \ker z$, i.e., $z \notin \overline{\{z_i\}}$, which is a contradiction.

Moreover using the notation in Lemma 3.1, by replacing H and A by $H(x,\alpha)$ and $A(x)^{\alpha(x)} = x(A^\alpha)$, the proof above says that for every $F_i = \sum\limits_{j=1}^{n_i} E_{ij}$, there is $b \in A^\alpha$ such that

$$x(b)|F_i = \bigoplus_{j=1}^{n_i} x(b)|_{H_{ij}}$$

(where $\oplus H_{ij} = H(x,\alpha)$) is a projection of rank 1. But $(id, H_{il}) \cong (id, H_{ik})$, so n_i must be 1. From this it is easy to see every minimal projection in $(x(A^\alpha)|_{H(x,\alpha)})'$ is central.

For the completion of the proof, since θ is open and continuous, we need only to check that for every $z \in \widehat{A^\alpha}$, there is a neighbourhood N of z such that $\theta|_N$ is injective. Otherwise since $\theta^{-1}(x)$ is discrete, where $x = \theta(z)$, there is a neighbourhood N_1 of z such that $\theta^{-1}(x) \cap N_1 = z$, and there are $\{z_i\}$, $\{z_i'\} \subset N_1$ such that

$$z_i \to z, \quad z_i' \to z \quad \text{and} \quad \theta(z_i) = \theta(z_i'), \quad z_i \neq z_i'.$$

Let $x_i = \theta(z_i)$. Then $x_i \to x$. Since A^α is a continuous-trace C^*-algebra, we can get a positive element $p \in A^\alpha$ and a neighbourhood N of z such that $N \subset \overline{N} \subset N_1$, and for every $z' \in N$,$z'(p)$ is a projection of rank 1; for every $z' \in N_1^c$, $z'(p) = 0$. Without loss of generality, we assume z_i, z_i', $z \in N$. Then

$$x(p), x_i(p) \in A(x)^{\alpha(x)}, \quad \mathrm{tr}(x(p)) = 1, \quad \mathrm{tr}(x_i(p)) \geq 2.$$

But A is a continuous-trace C^*-algebra, $p(\cdot)$ is a continuous field over $\hat{A}$. Since $x(p) = p(x)$ is a projection, by spectral calculus we can assume that $p \geq 0$ and $p(y)$ is a projection for every y near x. Let $\{a_i\}$ be a net converging to p with a_i positive and of continuous-trace. By spectral calculus again, we can get a positive continuous-trace element $a_0 \in A$ such that $a_0(y)$ is a projection and $|a_0(y) - p(y)| < 1$ for every y near x. So $\mathrm{tr}(p(y)) = \mathrm{tr}(a_0(y))$. But $\mathrm{tr}(a_0(y))$ is continuous, therefore $\mathrm{tr}(x_i(p)) \to \mathrm{tr}(x(p))$. This is a contradiction.

Corollary 3.2. *Let (A, G, α) be a separable C^*-dynamic system with G compact and abelian,A liminal,$\alpha_t \in \mathrm{Aut}_{C_b(\hat{A})}(A)$. Then A is of α-invariant continuous-trace iff the four conditions below hold:*

(i) *A^α is a continuous-trace C^*-algebra;*

(ii) *the closed ideal I of A generated by A^α is A;*

(iii) *$\theta : \hat{A}^\alpha \to \hat{A}$ is a sheaf;*

(iv) *for every* $x \in \hat{A}$*, every minimal projection in* $(x(A^\alpha)|_{\overline{x(A^\alpha)H(x)}})' \subset B(\overline{x(A^\alpha)H(x)})$ *is central.*

Proof. By Theorem 3.1. and Theorem 3.2., it is enough to prove that if A is an α-invariant continuous-trace C^*-algebra,then A^α is a continuous-trace C^*-algebra. In fact, since G is compact, A^α is strong Morita equivalent to a closed ideal I of $A \times_\alpha G$(for example [13, p.113]). So it is enough to prove I is a continuous-trace C^*-algebra, but it is clear.

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