Chin. Ann. of Math.
19B: 4(1998),499-510.

ESTIMATING EXTREME-VALUE INDEX FROM RECORDS**

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Abstract

This paper constructs from the record values an estimator of the extreme-value index. It is proved that the estimator is consistent in the domain of attraction of extreme-value distributions, and that under very mild conditions the estimator is asymptotically normal.

Keywords Record, Extreme value distribution, Estimator, Consistency, Asymptotic normality
1991 MR Subject Classification 62F12, 60F05
Chinese Library Classification O212.1, O211.4

§1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of iid rv's with a nondegenerate distribution function $F(x)$. Suppose there exist some constants $a_n > 0$, $b_n \in R$ and some $\gamma \in R$ such that

$$\lim_{n\to\infty} P\Big(\frac{\max\limits_{1\leq i\leq n} X_i - b_n}{a_n} \leq x\Big) = G_\gamma(x), \quad x \in R, \tag{1.1}$$

where G_γ stands for one of the extreme value distributions:

$$G_\gamma(x) = \exp\{-(1+\gamma x)^{-1/\gamma}\} \quad \text{for } x \text{ satisfying } 1+\gamma x > 0.$$

Here the index $\gamma \in R$ is a real parameter (interpret $(1+\gamma x)^{-1/\gamma}$ as e^{-x} for $\gamma = 0$).

The estimation of the extreme-value index γ is very important both in the extreme value theory and in practice. Many statistics, such as Hill estimator (for case $\gamma > 0$), Pickands estimator and Dekkers-Einmahl-de Haan's moment estimator which are based on a finite sample, have been proposed to estimate γ. The studies of asymptotic behavior of these estimators have also attracted much attention, e.g., [2–7, 10, 11], and [13–15].

Recently, Berred[1] constructed from record values two estimators of γ in case $\gamma > 0$. From now on we always assume that F is continuous. Define the sequences of record times and record values, $\tau(n)$ and $X(n)$, by

$$\tau(1) = 1,\ \tau(n+1) = \min\{j : X_j > X_{\tau(n)}\}, \quad n \geq 1$$

and

$$X(n) = X_{\tau(n)}, \quad n \geq 1.$$

Manuscript received February 14, 1996. Revised November 14, 1996.
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**Project supported by the National Natural Science Foundation of China (19601001).

Then Berred's estimators can be written as

$$R^1_{k,n} = \frac{1}{k}(\log X(n) - \log X(n-k)),$$

$$R^2_{k,n} = \frac{1}{nk - k(k-1)/2}\sum_{j=1}^{k} \log X(n-i+1),$$

where the integers $k = k(n)$ involved in $R^1_{k,n}$ satisfy $k(n) \to \infty$ and $\frac{k(n)}{n} \to 0$ as $n \to \infty$ and in $R^2_{k,n}$, $1 \le k < n$ is fixed. Berred[1] proved that both $R^1_{k,n}$ and $R^2_{k,n}$ are consistent and under very mild conditions they are asymtotically normal.

Berred's estimators are convenient for dealing with censored data containing only record values. The purpose of this paper is to consider the problem of estimation in the general case (1.1) with $\gamma \in R$. We use the statistic

$$Q(k,n) = \frac{1}{k}\log\frac{X(n) - X(n-k)}{X(n-k) - X(n-2k)}$$

as an estimator of γ. Here $k = k(n)$ satisfy

$$k(n) \to \infty \text{ as } n \to \infty \quad \text{ and } \limsup_{n\to\infty}\frac{k(n)}{n} < \frac{1}{2}. \tag{1.2}$$

The paper is organized as follows. Section 2 gives out its main results, including the consistency and the asymptotic normality of $Q(k,n)$, and the proofs appear in Section 3. Finally, some examples and numerical results are given in Section 4.

§2. Main Results

Theorem 2.1. *Assume that* (1.1) *and* (1.2) *hold. Then* $Q(k,n)$ *converges in probability to* γ. *If, additionally,* $\frac{k(n)}{\log n} \to \infty$, *then* $Q(k,n)$ *converges to* γ *almost surely.*

Denote ω_F as the endpoint of F by $\omega_F = \sup\{x : F(x) < 1\}$ and define the inverse function of F by $F^-(u) = \inf\{x : F(x) \ge u\}$, $0 < u < 1$. And set $U(x) = F^-(1 - \frac{1}{x})$, $x > 1$. It is well known from extreme value theory (c.f., [8]) that (1.1) holds if and only if one of the following conditions holds:

$$1 - F(x) = x^{-\frac{1}{\gamma}} L_1(x), \quad \gamma > 0, \tag{2.1}$$

$$\omega_F < \infty \text{ and } 1 - F(\omega_F - \frac{1}{x}) = x^{\frac{1}{\gamma}} L_2(x), \quad \gamma < 0, \tag{2.2}$$

$$\lim_{t\to\infty}\frac{U(tx) - U(t)}{U(t) - \frac{1}{t}\int_1^t U(s)ds} = \log x, \text{ for } x > 0. \tag{2.3}$$

Here $L_1(x)$, $L_2(x)$ and $U(x) - \frac{1}{x}\int_1^x U(s)ds$ are slowly varying functions.

A slowly varying function $h(x)$ is said to be (SRi) (i=1,2, or 3) with remainder term g if one of the following conditions holds:

(SR1) $\forall\lambda > 1, \quad \frac{h(\lambda t)}{h(t)} - 1 = O(g(t)), \qquad$ as $t \to \infty$,

(SR2) $\forall\lambda > 1, \quad \frac{h(\lambda t)}{h(t)} - 1 \sim K(\lambda)g(t), \qquad$ as $t \to \infty$,

(SR3) $\forall\lambda > 1, \quad \frac{h(\lambda t)}{h(t)} - 1 = o(g(t)), \qquad$ as $t \to \infty$,

where $g(x) \downarrow 0$ as $x \to \infty$.

Assume that $\{E_n, n \ge 1\}$ is a sequence of iid rv's with unit exponential distribution. Set $\Gamma_n = \sum\limits_{j=1}^{n} E_j$ for each $n \ge 1$.

Theorem 2.2. *Assume that* (1.1) *holds with* $\gamma > 0$ *and* (1.2) *holds, and* $L_1(x)$ *is defined in* (2.1). *Then*

$$\frac{\sqrt{k}}{\gamma}(Q(k,n)-\gamma) \xrightarrow{d} N(0,1) \tag{2.4}$$

provided that one of the following conditions holds:

(a) $\lim\limits_{y\to\infty}\sup\limits_{x>c}\frac{|\log\frac{L_1(xy)}{L_1(x)}|}{\sqrt{\log y}} = 0$ *for some* $c > 0$.

(b) $L_1(x)$ *is* (SRi) ($i = 1, 2,$ *or* 3) *with a remainder term* g *such that*

$$\sqrt{k}g(U(\exp(\Gamma_{n-k}))) \to d_i \text{ in probability,}$$

where, if $i = 1$ *or* 2 *then* $d_i = 0$ *and if* $i = 3$ *then* $d_i \in [0,\infty)$.

(c) $L_1(x) = L_1^{(1)}(x)L_1^{(2)}(x)$ *for large* x *with* $L_1^{(1)}(x)$ *satisfying* (a) *and* $L_1^{(2)}(x)$ *satisfying* (b).

Theorem 2.3. *Assume that* (1.1) *holds with* $\gamma < 0$ *and* (1.2) *holds, and* $L_2(x)$ *is defined in* (2.2). *Then*

$$\frac{\sqrt{k}}{-\gamma}(Q(k,n)-\gamma) \xrightarrow{d} N(0,1) \tag{2.5}$$

provided that one of the following conditions holds:

(d) $\lim\limits_{y\to\infty}\sup\limits_{x>c}\frac{|\log\frac{L_2(xy)}{L_2(y)}|}{\sqrt{\log y}} = 0$ *for some* $c > 0$.

(e) $L_2(x)$ *is* (SRi)($i = 1, 2,$ *or* 3) *with a remainder term* g *such that*

$$\sqrt{k}g(U(\exp(\Gamma_{n-2k}))) \to d_i \text{ in probability,}$$

where, if $i = 1$ *or* 2 *then* $d_i = 0$ *and if* $i = 3$ *then* $d_i \in [0,\infty)$.

(f) $L_2(x) = L_2^{(1)}(x)L_2^{(2)}(x)$ *for large* x *with* $L_2^{(1)}(x)$ *satisfying* (d) *and* $L_2^{(2)}(x)$ *satisfying* (e).

In the case $\gamma = 0$, the estimator $Q(k,n)$ converges in probability to γ at a very fast rate under a mild condition.

Theorem 2.4. *Assume that* (1.1) *holds with* $\gamma = 0$ *and* (1.2) *holds, and*

$$L(x) = U(x) - \frac{1}{x}\int_1^x U(s)ds.$$

(g) *If* $\lim\limits_{x\to\infty} L(x) = c \in (0,\infty)$, *then*

$$k(Q(k,n)-\gamma) \to 0 \quad \text{in probability as } n\to\infty. \tag{2.6}$$

(h) *If* $\lim\limits_{y\to\infty}\sup\limits_{x>c}\frac{|\log\frac{L(xy)}{L(y)}|}{\sqrt{\log y}} = 0$ *for some* $c > 0$, *then*

$$\sqrt{k}(Q(k,n)-\gamma) \to 0 \quad \text{in probability as } n\to\infty. \tag{2.7}$$

(i) *If* $L(x)$ *is* (SRi)($i = 1, 2,$ *or* 3) *with a remainder term* g *such that*

$$\sqrt{k}g(\exp(\Gamma_{n-k})) \to d_i \text{ in probability,}$$

where $d_i = 0$ *for* $i = 1$ *or* 2 *and* $d_i \in [0,\infty)$ *for* $i = 3$, *then* (2.7) *holds.*

(j) *If* $L(x) = L^{(1)}(x)L^{(2)}(x)$ *for large* x *with* $L^{(1)}(x)$ *satisfying* (h) *and* $L^{(2)}(x)$ *satisfying* (i), *then* (2.7) *remains true.*

Remark 2.1. In Theorems 2.2 and 2.3 we have used the condition

$$\sqrt{k}g(U(\exp(\Gamma_{n-2k}))) \to d_i \text{ in probability,}$$

which, in case that $g(U(e^x))$ is regularly varying, is implied by

$$\lim_{n\to\infty}\sqrt{k}g(U(e^{n-2k})))=d_i.$$

Similarly, if we assume $g(e^x)$ is regularly varying, then we can use the condition

$$\lim_{n\to\infty}\sqrt{k}g(e^{n-k})=d_i$$

to replace the condition of Theorem 2.4: $\sqrt{k}g(\exp(\Gamma_{n-k}))\to d_i$ in probability.

Under somewhat stronger constraint, we show that $Q(k,n)$ is also asymptotically normal. To this end, set $G(x)=U(e^x)$ for $x>0$. Assume that G is a regularly varying function at infinity with index $\beta\in R$, and G satisfies a condition of second order variation, i.e., for some function $a(t)>0$ and $g(t)>0$ with $\lim\limits_{t\to\infty}g(t)=0$,

$$\frac{G(tx)-G(t)}{a(t)}-\frac{x^\beta-1}{\beta}=O(g(t))\qquad\text{as } t\to\infty \tag{2.8}$$

holds locally uniformly on $x>0$.

Theorem 2.5. *Assume that* (2.8) *holds, and*

$$k_n\to\infty \textit{ as } n\to\infty \textit{ and } \lim_{n\to\infty}\frac{k_n}{n^{2/3}}=0. \tag{2.9}$$

If

$$\frac{n}{\sqrt{k}}g(\Gamma_{n-k})\to 0\qquad\textit{in probability,} \tag{2.10}$$

then

$$\frac{k^{3/2}}{\sqrt{2}}Q(k,n)\xrightarrow{d}N(0,1). \tag{2.11}$$

Remark 2.2. The properties of second order variationial functions like G in (2.8) can be found in [9]. An equivalence to (2.8) can be expressed in terms of F. If g in (2.8) is assumed to be regularly varying, then (2.10) can be replaced by $\lim\limits_{n\to\infty}\frac{ng(n)}{\sqrt{k}}=0$, which is satisfied if $\limsup\limits_{n\to\infty}ng(n)<\infty$.

Remark 2.3. In order to construct a confidence interval for γ, we write (2.4), (2.5) and (2.11) into a unified form

$$\frac{\sqrt{k}(1-e^{-k|\gamma|/\sqrt{2}})}{|\gamma|}(Q(k,n)-\gamma)\xrightarrow{d}N(0,1), \tag{2.12}$$

where $\frac{1-e^{-k|\gamma|/\sqrt{2}}}{|\gamma|}$ for $\gamma=0$ is defined as $\lim\limits_{\gamma\to 0}\frac{1-e^{-k|\gamma|/\sqrt{2}}}{|\gamma|}=\frac{k}{\sqrt{2}}$. Furthermore, we use $\frac{\sqrt{k}(1-e^{-k|Q(k,n)|/\sqrt{2}})}{|Q(k,n)|}$ to replace $\frac{\sqrt{k}(1-e^{-k|\gamma|/\sqrt{2}})}{|\gamma|}$ in the left-hand side of (2.12). Note that $P(Q(k,n)=0)=0$ because of the continuity of F. In case $\gamma\neq 0$, it is obvious that

$$\frac{(1-e^{-k|Q(k,n)|/\sqrt{2}})/|Q(k,n)|}{(1-e^{-k|\gamma|/\sqrt{2}})/|\gamma|}\to 1\quad\text{ in probability.} \tag{2.13}$$

For $\gamma=0$, (2.11) implies $kQ(k,n)=o(1)$ in probability, which yields (2.13). Thus, from (2.12) we see

$$\frac{\sqrt{k}(1-e^{-k|Q(k,n)|/\sqrt{2}})}{|Q(k,n)|}(Q(k,n)-\gamma)\xrightarrow{d}N(0,1).$$

Via this result one can construct a confidence interval for γ.

§3. Proofs

Keep in mind that $\{E_n, n \geq 1\}$ is a sequence of i.i.d. random variables with unit exponential distribution and $\Gamma_n = \sum_{j=1}^{n} E_j$ for each $n \geq 1$. Then from [1] or [10], $\{X(n),\ n \geq 1\}$ is distributed the same as $\{U(\exp(\Gamma_n)),\ n \geq 1\}$. For simplicity write $X(n) = U(\exp(\Gamma_n))$ for all $n \geq 1$. Thus, $Q(k, n)$ can be rewritten as

$$Q(k,n) = \frac{1}{k} \log \frac{U(\exp(\Gamma_n)) - U(\exp(\Gamma_{n-k}))}{U(\exp(\Gamma_{n-k})) - U(\exp(\Gamma_{n-2k}))}.$$

Lemma 3.1. *Suppose that* (1.1) *holds with* $\gamma > 0$. *If* (1.2) *holds, then*

$$Q(k,n) = \frac{1}{k} \log \frac{U(\exp(\Gamma_n))}{U(\exp(\Gamma_{n-k}))} + o(\frac{1}{k}) \quad \textit{in probability,} \tag{3.1}$$

and additionally, if $\lim_{n\to\infty} \frac{k_n}{\log n} = \infty$, *then*

$$Q(k,n) = \frac{1}{k} \log \frac{U(\exp(\Gamma_n))}{U(\exp(\Gamma_{n-k}))} + o(\frac{1}{k}) \quad \textit{almost surely.} \tag{3.2}$$

Proof. From [8], $U(x)$ is regularly varying at infinity with index $\gamma > 0$. From properties of regular variation we see that if $\lim_{x\to\infty} \frac{h(x)}{x} = \infty$,

$$\frac{U(h(x))}{U(x)} \to \infty \qquad \text{as } n \to \infty. \tag{3.3}$$

By the law of large numbers, if (1.2) holds, then

$$\frac{\Gamma_n - \Gamma_{n-k}}{k} \to 1 \text{ and } \frac{\Gamma_{n-k} - \Gamma_{n-2k}}{k} \to 1 \qquad \text{in probability.} \tag{3.4}$$

Additionally, if $\frac{k_n}{\log n} \to \infty$ as $n \to \infty$, then by [12],

$$\frac{\Gamma_n - \Gamma_{n-k}}{k} \to 1 \text{ and } \frac{\Gamma_{n-k} - \Gamma_{n-2k}}{k} \to 1 \qquad \text{almost surely.} \tag{3.5}$$

In view of (3.3) and (3.4), if (1.1) and (1.2) hold, then

$$\frac{U(\exp(\Gamma_{n-k}))}{U(\exp(\Gamma_n))} \to 0 \text{ and } \frac{U(\exp(\Gamma_{n-2k}))}{U(\exp(\Gamma_{n-k}))} \to 0 \text{ in probability.}$$

Since

$$Q(k,n) = \frac{1}{k} \log \frac{U(\exp(\Gamma_n))}{U(\exp(\Gamma_{n-k}))} + \frac{1}{k} \log \Big(1 - \frac{U(\exp(\Gamma_{n-k}))}{U(\exp(\Gamma_n))}\Big) - \frac{1}{k} \log \Big(1 - \frac{U(\exp(\Gamma_{n-2k}))}{U(\exp(\Gamma_{n-k}))}\Big),$$

(3.1) follows from the Taylor's expansion. Likewise, if $\frac{k_n}{\log n} \to \infty$ as $n \to \infty$, one can conclude (3.2) from (3.3) and (3.5). That completes the proof of the lemma.

Lemma 3.2. *If* (1.1) *holds with* $\gamma < 0$ *and* (1.2) *holds, then*

$$Q(k,n) = \frac{1}{k} \log \frac{U(\infty) - U(\exp(\Gamma_{n-k}))}{U(\infty) - U(\exp(\Gamma_{n-2k}))} + o\Big(\frac{1}{k}\Big) \quad \textit{in probability.}$$

Furthermore, if $\lim_{n\to\infty} \frac{k_n}{\log n} = \infty$, *then*

$$Q(k,n) = \frac{1}{k} \log \frac{U(\infty) - U(\exp(\Gamma_{n-k}))}{U(\infty) - U(\exp(\Gamma_{n-2k}))} + o\Big(\frac{1}{k}\Big) \quad \textit{almost surely.}$$

Proof. It is easily seen that $U(\infty) = \omega_F < \infty$. Set

$$\widetilde{U}(x) = \frac{1}{U(\infty) - U(x)}. \tag{3.6}$$

Note that (2.2) holds in the case. Thus $\widetilde{U}(x)$ is regularly varying with index $-\gamma > 0$. According to the proof of Lemma 3.1 we see under (1.2) that

$$\frac{\widetilde{U}(\exp(\Gamma_{n-k}))}{\widetilde{U}(\exp(\Gamma_n))} \to 0 \text{ and } \frac{\widetilde{U}(\exp(\Gamma_{n-2k}))}{\widetilde{U}(\exp(\Gamma_{n-k}))} \to 0 \text{ in probability,}$$

and under the additional condition $\lim\limits_{n\to\infty} \frac{k_n}{\log n} = \infty$ the wording "in probability" can be replaced by the wording "almost surely".

Rewrite

$$Q(k,n) = \frac{1}{k}\log\frac{\widetilde{U}(\exp(\Gamma_{n-2k}))}{\widetilde{U}(\exp(\Gamma_{n-k}))} + \frac{1}{k}\log\Big(1 - \frac{\widetilde{U}(\exp(\Gamma_{n-k}))}{\widetilde{U}(\exp(\Gamma_n))}\Big) - \frac{1}{k}\log\Big(1 - \frac{\widetilde{U}(\exp(\Gamma_{n-2k}))}{\widetilde{U}(\exp(\Gamma_{n-k}))}\Big).$$

The lemma immediately follows from the Taylor's expansion.

Lemma 3.3. *Assume that* (1.1) *holds with* $\gamma = 0$. *Let* x_n, y_n *and* z_n *be positive constants such that*

$$\lim_{n\to\infty} x_n = \infty, \quad \lim_{n\to\infty}\frac{y_n}{x_n} = 1, \quad \liminf_{n\to\infty}\frac{z_n}{x_n} > 0.$$

Set $L(x) = U(x) - \frac{1}{x}\int_1^x U(s)ds$. *Then*

$$\begin{aligned}\frac{\min\limits_{t\in[0,x_n]} L(e^{t+y_n+z_n})}{\max\limits_{t\in[0,y_n]} L(e^{t+z_n})}(1+o(1)) &\le \frac{U(e^{x_n+y_n+z_n}) - U(e^{y_n+z_n})}{U(e^{y_n+z_n}) - U(e^{z_n})} \\ &\le \frac{\max\limits_{t\in[0,x_n]} L(e^{t+y_n+z_n})}{\min\limits_{t\in[0,y_n]} L(e^{t+z_n})}(1+o(1)) \end{aligned} \tag{3.7}$$

holds for all large n.

Proof. Note that (2.3) holds under (1.1) with $\gamma = 0$ and the convergence in (2.3) is locally uniform, i.e., for any $T > 1$ and the sequence $\{t_n\}$ with $\lim\limits_{n\to\infty} t_n = \infty$, there exists a sequence $\{\varepsilon_n\}$, $\varepsilon_n \downarrow 0$ such that, for all large n

$$\Big|\frac{U(xt) - U(t)}{L(t)} - \log x\Big| \le \varepsilon_n \qquad \text{for all } x \in [T^{-1}, T], \quad t \ge t_n. \tag{3.8}$$

Set $t_n = e^{z_n}$. By choosing $x = e^{\frac{y_n}{[y_n]}}$ and $x = e^{\frac{x_n}{[x_n]}}$ in (3.8) respectively, where $[x]$ denotes the integer part of x, we get

$$\begin{aligned}\Big(\frac{y_n}{[y_n]} - \varepsilon_n\Big)L(e^{(i-1)\frac{y_n}{[y_n]}+z_n}) &\le U(e^{i\frac{y_n}{[y_n]}+z_n}) - U(e^{(i-1)\frac{y_n}{[y_n]}+z_n}) \\ &\le \Big(\frac{y_n}{[y_n]} + \varepsilon_n\Big)L(e^{(i-1)\frac{y_n}{[y_n]}+z_n})\end{aligned}$$

for $i = 1, 2, \cdots, [y_n]$ and

$$\begin{aligned}\Big(\frac{x_n}{[x_n]} - \varepsilon_n\Big)L(e^{(j-1)\frac{x_n}{[x_n]}+y_n+z_n}) &\le U(e^{j\frac{x_n}{[x_n]}+y_n+z_n}) - U(e^{(j-1)\frac{x_n}{[x_n]}+y_n+z_n}) \\ &\le \Big(\frac{x_n}{[x_n]} + \varepsilon_n\Big)L(e^{(j-1)\frac{x_n}{[x_n]}+y_n+z_n})\end{aligned}$$

for $j = 1, 2, \cdots, [x_n]$. By taking simply summations on each side of the above inequalities one can easily get the lemma.

Proof of Theorem 2.1 If $\gamma > 0$, then the conclusions of Theorem 2.1 immediately follow from Lemma 3.1 and Theorem 4 of [1].

Assume $\gamma < 0$ and define $\widetilde{U}$ as in (3.6). As we have known, $\widetilde{U}(x)$ is regularly varying with index $-\gamma > 0$. Hence $\widetilde{U}(x)$ can be written as $\widetilde{U}(x) = x^{-\gamma} h(x)$, where h is slowly varying. From [17, p.18],

$$\lim_{x\to\infty} \frac{\log h(x)}{\log x} = 0.$$

Therefore, it is easily proved from (3.4) and (3.5) that

$$\log \frac{\widetilde{U}(\exp(\Gamma_{n-k}))}{\widetilde{U}(\exp(\Gamma_{n-2k}))} \to -\gamma \qquad \text{in probability}$$

and under the additional condition $\frac{k_n}{\log n} \to \infty$ as $n \to \infty$

$$\log \frac{\widetilde{U}(\exp(\Gamma_{n-k}))}{\widetilde{U}(\exp(\Gamma_{n-2k}))} \to -\gamma \qquad \text{almost surely,}$$

which, coupled with Lemma 3.2, yields the theorem.

Assume now $\gamma = 0$. Note that $L(x)$ is a slowly varying function. By Karamata's representation theorem (see e.g. [8] or [17])

$$L(x) = c(x) \exp\Big\{ \int_1^x \frac{b(u)}{u} du \Big\} \quad \text{with } \lim_{x\to\infty} c(x) = c > 0 \quad \text{and } \lim_{x\to\infty} b(x) = 0,$$

from which one can easily show that

$$\lim_{y\to\infty} \sup_{x \ge ye} \Big| \frac{\log \frac{L(x)}{L(y)}}{\frac{x}{y}} \Big| = 0.$$

Then under the conditions of Lemma 3.3 we conclude from (3.7) that

$$\log \frac{U(e^{x_n+y_n+z_n}) - U(e^{y_n+z_n})}{U(e^{y_n+z_n}) - U(e^{z_n})} = o(x_n). \tag{3.9}$$

Now set $x_n = \Gamma_n - \Gamma_{n-k}$, $y_n = \Gamma_{n-k} - \Gamma_{n-2k}$ and $z_n = \Gamma_{n-2k}$. Then

$$Q(k,n) = \frac{1}{k} \log \frac{U(e^{x_n+y_n+z_n}) - U(e^{y_n+z_n})}{U(e^{y_n+z_n}) - U(e^{z_n})}. \tag{3.10}$$

The conclusions of Theorem 2.1 immediately follow from (3.4), (3.5) and (3.9). That completes the proof of Theorem 2.1.

Proof of Theorem 2.2 According to Lemma 3.1, to prove Theorem 2.2 it suffices to show

$$\frac{\sqrt{k}}{\gamma} \Big(\frac{1}{k} \log \frac{U(\exp(\Gamma_n)))}{U(\exp(\Gamma_{n-k}))} - \gamma \Big) \xrightarrow{d} N(0,1). \tag{3.11}$$

Note that $U(x)$ is a regularly varying function with index γ. Write $U(x) = x^{\gamma} J(x)$, where $J(x)$ is slowly varying. Since

$$U(x) = \inf\Big\{ y : \frac{1}{1-F(y)} \ge x \Big\} = \Big(\frac{1}{1-F} \Big)^{-}(x),$$

we have $\frac{1}{1-F(U(x))} \sim x$ as $x \to \infty$, i.e.,

$$J^{\gamma}(x) \sim L_1(x) \qquad \text{as } x \to \infty. \tag{3.12}$$

Assume that (a) holds. It is obvious that for some $c_1 > 0$,

$$\lim_{y\to\infty} \sup_{x > c_1} \Big| \frac{\log \frac{L_1(xy)}{L_1(y)}}{\sqrt{\log U(x)}} \Big| = 0.$$

Thus from (3.12) and the fact

$$\lim_{x\to\infty}\frac{U(x)}{\log x}=\gamma$$

we have

$$\lim_{y\to\infty}\sup_{x>c_1}\Big|\frac{\log\frac{J(xy)}{J(y)}}{\sqrt{\log x}}\Big|=0. \tag{3.13}$$

It is clear from the central limit theorem that

$$\frac{1}{\sqrt{k}}(\Gamma_n-\Gamma_{n-k}-k)\xrightarrow{d}N(0,1), \tag{3.14}$$

and from (3.13) and (3.4) that

$$\frac{1}{\gamma\sqrt{k}}\log\frac{J(\exp(\Gamma_n))}{J(\exp(\Gamma_{n-k}))}\to 0 \text{ in probability.}$$

Then (3.11) follows from the identity

$$\frac{\sqrt{k}}{\gamma}\Big(\frac{1}{k}\log\frac{U(\exp(\Gamma_n)))}{U(\exp(\Gamma_{n-k}))}-\gamma\Big)=\frac{\Gamma_n-\Gamma_{n-k}-k}{\sqrt{k}}+\frac{1}{\gamma\sqrt{k}}\log\frac{J(\exp(\Gamma_n))}{J(\exp(\Gamma_{n-k}))}. \tag{3.15}$$

If (b) holds, (3.11) can be proved along the lines of the proof of Theorem 5 of [1]. The detail is omitted.

Suppose now (c) holds. Then from (3.12)

$$\frac{J^{\gamma}(x)}{L_1^{(1)}(x)L_1^{(2)}(x)}\sim 1 \qquad \text{as } x\to\infty.$$

Then

$$\begin{aligned}&\frac{1}{\sqrt{k}}\log\frac{J(\exp(\Gamma_n))}{J(\exp(\Gamma_{n-k}))}\\ &=\frac{1}{\sqrt{k}}\log\frac{L_1^{(1)}(\exp(\Gamma_n))}{L_1^{(1)}(\exp(\Gamma_{n-k}))}+\frac{1}{\sqrt{k}}\log\frac{L_1^{(2)}(\exp(\Gamma_n))}{L_1^{(2)}(\exp(\Gamma_{n-k}))}+o_p\Big(\frac{1}{\sqrt{k}}\Big).\end{aligned} \tag{3.16}$$

The first term on the right-hand side of (3.16) is obviously of the same order as $o_p(1)$, and by Lemma 6 of [1], the second term is also of the order $o_p(1)$. Consequently, (3.11) follows from (3.14)–(3.16).

Proof of Theorem 2.3. In view of Lemma 3.2 it suffices to show

$$\frac{\sqrt{k}}{-\gamma}\Big(\frac{1}{k}\log\frac{\widetilde{U}(\exp(\Gamma_{n-2k})))}{\widetilde{U}(\exp(\Gamma_{n-k}))}-\gamma\Big)\xrightarrow{d}N(0,1),$$

which can be proved along the the lines of the proof of Theorem 2.2.

Proof of Theorem 2.4. Like the proof of Theorem 2.1, set

$$x_n=\Gamma_n-\Gamma_{n-k},\quad y_n=\Gamma_{n-k}-\Gamma_{n-2k},\quad z_n=\Gamma_{n-2k}.$$

Then (3.10) holds. If (g) holds, then by applying Lemma 3.3, (2.6) is valid; and if (h) holds, then (2.7) holds.

Suppose now that (i) holds. Due to Theorem 2.2.2 of [7],

(1) L is SR1 if and only if

$$L(x)=\exp\Big\{C+O(g(x))+\int_1^x O(g(t))t^{-1}dt\Big\}; \tag{3.17}$$

(2) L is SR2 if and only if

$$L(x) = \exp\{C + o(g(x)) + \int_1^x (C + o(1))g(t)t^{-1}dt\};$$

(3) L is SR3 if and only if

$$L(x) = \exp\{C + o(g(x)) + \int_1^x o(g(t))t^{-1}dt\}.$$

We only prove (2.7) under (3.17). From (3.7) and (3.10), for some $D > 0$,

$$\frac{1}{k}\Big(o_p(1) - D\int_{\exp(\Gamma_{n-2k})}^{\exp(\Gamma_n)} g(t)t^{-1}dt\Big) \le Q(k,n) \le \frac{1}{k}\Big(o_p(1) + D\int_{\exp(\Gamma_{n-2k})}^{\exp(\Gamma_n)} g(t)t^{-1}dt\Big).$$

Since

$$\int_{\exp(\Gamma_{n-2k})}^{\exp(\Gamma_n)} g(t)t^{-1}dt \le g(\exp(\Gamma_{n-2k}))(\Gamma_n - \Gamma_{n-2k}) = O_p(k)g(\exp(\Gamma_{n-2k}))$$

and from (2.6),

$$\sqrt{k}g(\exp(\Gamma_{n-2k})) \to 0 \quad \text{in probability},$$

we get

$$\sqrt{k}Q(k,n) = o_p(1),$$

which proves (2.7).

If (j) holds, then one can show (2.7) in a similar way. The dertail is omitted here.

The proof of Theorem 2.4 is completed.

Proof of Theorem 2.5. Note that

$$Q(k,n) = \frac{1}{k}\log\frac{G(\Gamma_n) - G(\Gamma_{n-k})}{G(\Gamma_{n-k}) - G(\Gamma_{n-2k})}.$$

Setting $t = \Gamma_{n-k}$ in (2.8) and mentioning that $\frac{\Gamma_{n-k}}{n-k} \to 1$ in probability, we have

$$\frac{\Gamma_n - \Gamma_{n-k}}{a(\Gamma_{n-k})} = \frac{(\frac{\Gamma_n}{\Gamma_{n-k}})^\beta - 1}{\beta} + O_p(g(\Gamma_{n-k})),$$

$$\frac{\Gamma_{n-2k} - \Gamma_{n-k}}{a(\Gamma_{n-k})} = \frac{(\frac{\Gamma_{n-2k}}{\Gamma_{n-k}})^\beta - 1}{\beta} + O_p(g(\Gamma_{n-k})).$$

Using Taylor's expansion one can easily show

$$k^{3/2}Q(k,n) = \frac{\Gamma_n - \Gamma_{n-k} - k}{\sqrt{k}} - \frac{\Gamma_{n-k} - \Gamma_{n-2k} - k}{\sqrt{k}} + o_p(1) \xrightarrow{d} N(0,2),$$

yielding (2.11).

§4. Examples and Numerical Results

First of all we give some examples satisfying the conditions of Theorems 2.2 and 2.3.

Example 4.1. Assume that

$$1 - F(x) = c(x)x^{-\frac{1}{\gamma}}(\log x)^\beta$$

for large x, where $\gamma > 0$, $\beta \in R$ and $\lim_{x\to\infty} c(x) = c > 0$. Then the condition (a) holds. Thus, if $\{k_n\}$ satisfy (1.2), then

$$\frac{\sqrt{k_n}}{\gamma}(Q(k_n,n) - \gamma) \xrightarrow{d} N(0,1).$$

Example 4.2. Set

$$1 - F(x) = cx^{-\frac{1}{\gamma}} \exp\{(\log x)^\beta\}$$

for all large x, where $\gamma > 0$, $c > 0$, $\beta \in (0,1)$. In this case,

$$L_1(x) = c \exp\{(\log x)^\beta\}.$$

It is easily seen that $L_1(x)$ is (SR2) with $g(t) = (\log t)^{-(1-\beta)}$ and $K(\lambda) = \beta \log \lambda$ (see also [1]). If (1.2) holds and

$$k_n = o(n^{2(1-\beta)}) \quad \text{as} \quad n \to \infty,$$

then the condition (b) is satisfied. Hence, we have

$$\frac{\sqrt{k_n}}{\gamma}(Q(k_n, n) - \gamma) \xrightarrow{d} N(0,1).$$

Example 4.3. Take

$$1 - F(x) = c(x) x^{-\frac{1}{\gamma}} (\log x)^{-\beta_1} \exp\{(\log x)^{\beta_2}\}$$

for all large x, where $\gamma > 0$, $\beta_1 \in R$, $\beta_2 \in (0,1)$ and $\lim\limits_{x\to\infty} c(x) = c > 0$. Put

$$L_1^{(1)}(x) = c(x)(\log x)^{\beta_1} \text{ and } L_1^{(2)}(x) = \exp\{(\log x)^{\beta_2}\}.$$

Then $L_1(x) = L_1^{(1)}(x) L_1^{(2)}(x)$ satisfies the condition (c) of Theorem 2.2 if $k_n = o(n^{2(1-\beta_2)})$. Therefore, if (1.2) holds and

$$k_n = o(n^{2(1-\beta_2)}) \text{ as } n \to \infty,$$

then

$$\frac{\sqrt{k_n}}{\gamma}(Q(k_n, n) - \gamma) \xrightarrow{d} N(0,1).$$

Example 4.4. Assume

$$1 - F(x_0 - \frac{1}{x}) = c(x) x^{\frac{1}{\gamma}} (\log x)^{-\beta_1} \exp\{(\log x)^{\beta_2}\}$$

for all large x, where $\gamma < 0$, $x_0 \in R$, $\beta_1 \in R$, $\beta_2 \in (0,1)$ and $\lim\limits_{x\to\infty} c(x) = c > 0$. Then, according to Theorem 2.3, if (1.2) holds and

$$k_n = o(n^{2(1-\beta)}) \text{ as } n \to \infty,$$

then

$$\frac{\sqrt{k_n}}{\gamma}(Q(k_n, n) - \gamma) \xrightarrow{d} N(0,1).$$

Example 4.5. Assume

$$1 - F(x) = e^{-x} \text{ for } x > 0.$$

Then it is obvious that the condition (g) is satisfied. Hence, if (1.2) holds, then $k_n Q(k_n, n) \to 0$ in probability. Note that $U(x) = \log x$ for $x > 0$. Thus, it is easily seen fron Section 3 that

$$k_n^{3/2} Q(k_n, n) \xrightarrow{d} N(0,2).$$

Some numerical results for the estimator $Q(k,n)$ and the Berred's estimator $R_{k,n}^1$ are listed below. The distribution functions utilized here can also be found in [1]. In the tables, $\hat{Q}(k,n)$ and $\hat{R}_{k,n}^1$ are the averages of 500 estimates of $Q(k,n)$ and $R_{k,n}^1$; $V(Q)$ and $V(R)$ are the corresponding standard errors, σ is the asymptotic standard deviation for $Q(k,n)$ (also for $R_{k,n}^1$ in Table 1).

Table 1

$1-F(x)$	γ	n	k	$\hat{Q}(k,n)$	$\hat{R}^1_{k,n}$	$V(Q)$	$V(R)$	σ
$x^{-\frac{1}{\gamma}}$	0.5	10	3	0.4924	0.4955	0.4094	0.2903	0.2886
	0.5	20	7	0.5057	0.5062	0.2046	0.1963	0.1889
	1.0	10	3	1.0283	1.0231	0.6357	0.5765	0.5773
	1.0	20	7	0.9740	0.9741	0.3733	0.3714	0.3779
	1.5	10	3	1.5195	1.5203	0.8878	0.8577	0.8660
	1.5	20	7	1.4959	1.4961	0.5974	0.5969	0.5669
$7x^{-\frac{1}{\gamma}}$	0.5	10	3	0.4894	0.4940	0.3974	0.2885	0.2886
	0.5	20	7	0.4895	0.4901	0.1911	0.1811	0.1889
	1.0	10	3	1.0375	1.0195	0.5986	0.5503	0.5773
	1.0	20	7	1.0071	1.0071	0.3908	0.3890	0.3779
	1.5	10	3	1.5190	1.5158	0.9187	0.8910	0.8660
	1.5	20	7	1.5122	1.5121	0.5626	0.5623	0.5669
$x^{-\frac{1}{\gamma}}(1+2x^{-2})$	0.5	10	3	0.5334	0.5211	0.4227	0.3142	0.2886
	0.5	20	7	0.5092	0.5088	0.1963	0.1883	0.1889
	1.0	10	3	1.0382	1.0355	0.6431	0.5905	0.5773
	1.0	20	7	1.0082	1.0080	0.3776	0.3769	0.3779
	1.5	10	3	1.5189	1.5196	0.9175	0.8818	0.8660
	1.5	20	7	1.4930	1.4929	0.5565	0.5562	0.5669

Table 2

$1-F(\omega_F-\frac{1}{x})$	γ	n	k	$\hat{Q}(k,n)$	$V(Q)$	σ
$x^{\frac{1}{\gamma}}$	−0.5	10	3	−0.49073	0.39665	0.28867
	−0.5	20	7	−0.50420	0.20005	0.18898
	−1.0	10	3	−1.01256	0.67874	0.57735
	−1.0	20	7	−0.99525	0.38733	0.37796
	−1.5	10	3	−1.48508	0.85813	0.86602
	−1.5	20	7	−1.50848	0.54982	0.56694
$7x^{\frac{1}{\gamma}}$	−0.5	10	3	−0.48963	0.38434	0.28867
	−0.5	20	7	−0.49981	0.19443	0.18898
	−1.0	10	3	−1.03351	0.60076	0.57735
	−1.0	20	7	−0.99875	0.35944	0.37796
	−1.5	10	3	−1.53061	0.88645	0.86602
	−1.5	20	7	−1.51652	0.58206	0.56694
$x^{\frac{1}{\gamma}}(1+2x^{-2})$	−0.5	10	3	−0.49609	0.37922	0.28867
	−0.5	20	7	−0.49274	0.20316	0.18898
	−1.0	10	3	−1.02106	0.66302	0.57735
	−1.0	20	7	−0.99861	0.39331	0.37796
	−1.5	10	3	−1.53080	0.88346	0.86602
	−1.5	20	7	−1.51161	0.55880	0.56694

Table 3 ($\gamma = 0$)

$1 - F(x)$	β	n	k	$\hat{Q}(k,n)$	$V(Q)$	σ
$\exp(x^{-\beta})$	0.5	10	3	0.15068	0.31420	0.27216
		10	3	+0.13306	0.32554	0.27216
		20	7	+0.07977	0.08233	0.07636
		20	7	+0.07815	0.08295	0.07636
	1	10	3	+0.00916	0.28984	0.27216
		10	3	−0.00677	0.29122	0.27216
		20	7	−0.00056	0.08052	0.07636
		20	7	+0.00015	0.08055	0.07636
	2	10	3	−0.07387	0.29031	0.27216
		10	3	−0.09409	0.28928	0.27216
		20	7	−0.04312	0.07727	0.07636
		20	7	−0.04115	0.08016	0.07636

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