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PERIODIC BOUNDARY VALUE PROBLEMS FOR IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS OF MIXED TYPE IN BANACH SPACES***

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Abstract

This paper investigates periodic boundary value problem for first order nonlinear impulsive integro-differential equations of mixed type in a Banach space. By establishing a comparison result, criteria on the existence of maximal and minimal solutions are obtained.

Keywords Periodic boundary value problem, Impulsive integro-differential equation, Ordered Banach space, Maximal solution, Minimal solution

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§1. Introduction

In this paper, we investigate the periodic boundary value problem (PBVP) for first order nonlinear impulsive integro-differential equations of mixed type in a Banach space E:

$$\begin{cases} x' = f(t, x, Tx, Sx), \quad \forall\ 0 \le t \le 2\pi, \quad t \ne t_k \quad (k = 1, 2, \cdots, m), \\ \triangle x|_{t=t_k} = I_k(x(t_k)) \quad (k = 1, 2, \cdots, m), \\ x(0) = x(2\pi), \end{cases} \tag{1.1}$$

where $f \in C[J \times E \times E \times E, E]$, $J = [0, 2\pi]$, $I_k \in C[E, E]$ $(k = 1, 2, \cdots, m)$,

$$(Tx)(t) = \int_0^t K(t, s)x(s)ds, \quad (Sx)(t) = \int_0^{2\pi} H(t, s)x(s)ds, \tag{1.2}$$

$K \in C[D, R_+]$, $D = \{(t, s) \in J \times J :\ t \ge s\}, H \in C[J \times J, R_+]$, R_+ denotes the set of all nonnegative numbers, and $0 < t_1 < \cdots < t_k < \cdots < t_m < 2\pi$. $\triangle x|_{t=t_k}$ represents the jump of $x(t)$ at $t = t_k$, i.e., $\triangle x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of $x(t)$ at $t = t_k$, respectively. A special case of PBVP (1.1) has been considered in Euclidean space recently in [2], where by developing a comparison result the monotonicity condition normally imposed on the right-hand side relative to the integral term is removed successfully and the existence of extremal solutions is established.

In Section 2 we establish a comparison result, and then we state and prove the main theorem in Section 3. Finally, to illustrate our result, Section 4 offers two examples in both finite and infinite dimensional spaces.

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§2. Comparison Result

Let $PC[J,E]=\{x:\ x$ is a map from J into E such that $x(t)$ is continuous at $t\neq t_k$, and left continuous at $t=t_k$, and the right limit $x(t_k^+)$ exists for $k=1,2,\cdots,m\}$. Evidently, $PC[J,E]$ is a Banach space with norm $\|x\|_{PC}=\sup\limits_{t\in J}\|x(t)\|$. Let $J'=J\backslash\{t_1,\cdots,t_m\}$. $x\in PC[J,E]\cap C^1[J',E]$ is called a solution of PBVP (1.1) if it satisfies (1.1).

Let E be partially ordered by a cone P of E, i.e., $x\leq y$ if and only if $y-x\in P$. P is said to be normal if there exists a positive constant N such that $\theta\leq x\leq y$ implies $||x||\leq N||y||$, where θ denotes the zero element of E, and P is said to be regular if $x_1\leq x_2\leq\cdots\leq x_n\leq\cdots\leq y$ implies $||x_n-x||\to 0$ as $n\to\infty$ for some $x\in E$. It is well known that the regularity of P implies the normality of P (see [1, Theorem 1.2.1]). Let $Q=\{x\in PC[J,E]:x(t)\geq\theta$ for $t\in J\}$. Then Q is a cone in space $PC[J,E]$, and so, $PC[J,E]$ is partial ordered by $Q: u\leq v$ if and only if $v-u\in Q$, i.e., $u(t)\leq v(t)$ for $t\in J$.

In the following, let $J_0=[0,t_1],\ J_1=(t_1,t_2],\cdots,J_{m-1}=(t_{m-1},t_m],\ J_m=(t_m,2\pi],\ \delta=\max\{t_k-t_{k-1}:\ k=1,2,\cdots,m+1\}$ (where $t_0=0,\ t_{m+1}=2\pi$) and $k_0=\max\{K(t,s):(t,s)\in D\}$. $h_0=\max\{H(t,s):\ (t,s)\in J\times J\}$.

Lemma 2.1 (Comparison result). *Assume that $p\in PC[J,E]\cap C^1[J',E]$ satisfies*

$$\begin{cases} p'\leq -Mp-NTp-N_1Sp, & \forall t\in J,\quad t\neq t_k\quad (k=1,2,\cdots,m),\\ \triangle p|_{t=t_k}\leq -L_kp(t_k) & (k=1,2,\cdots,m),\\ p(0)\leq p(2\pi), \end{cases}\tag{2.1}$$

where constants $M>0,\ N\geq 0,\ N_1\geq 0,\ 0\leq L_k\leq 1\ \ (k=1,2,\cdots,m)$, and

$$M^{-1}(Nk_0+N_1h_0)(e^{4\pi M}-1)\delta\leq\frac{\Big\{\prod\limits_{k=1}^{m}(1-L_k)\Big\}^2}{1+\sum\limits_{n=1}^{m}\prod\limits_{k=n}^{m}(1-L_k)}.\tag{2.2}$$

Then $p(t)\leq\theta$ for $t\in J$.

Proof. For any $g\in P^*$ (P^* denotes the dual cone of P (see [1])), let $u(t)=g(p(t))$. Then $u\in PC[J,R]\cap C^1[J',R]$ and

$$u'(t)=g(p'(t)),\quad g((Tp)(t))=(Tu)(t),\quad g((Sp)(t))=(Su)(t),$$

where R denotes the set of all real numbers. By (2.1), we have

$$\begin{cases} u'\leq -Mu-NTu-N_1Su, & \forall\ t\in J,\quad t\neq t_k\quad (k=1,2,\cdots,m),\\ \triangle u|_{t=t_k}\leq -L_ku(t_k) & (k=1,2,\cdots,m),\\ u(0)\leq u(2\pi). \end{cases}\tag{2.3}$$

Let $v(t)=u(t)e^{Mt},\ \forall t\in J$. Then $v\in PC[J,R]\cap C^1[J',R]$ and (2.3) implies

$$\begin{cases} v'(t)\leq -N\int_0^t k^*(t,s)v(s)ds-N_1\int_0^{2\pi}h^*(t,s)v(s)ds,\\ \qquad\qquad \forall t\in J,\quad t\neq t_k\quad (k=1,2,\cdots,m),\\ \triangle v|_{t=t_k}\leq -L_kv(t_k)\qquad (k=1,2,\cdots,m),\\ v(0)\leq v(2\pi)e^{-2\pi M}, \end{cases}\tag{2.4}$$

where $k^*(t,s)=K(t,s)e^{M(t-s)},\ \ h^*(t,s)=H(t,s)e^{M(t-s)}$. We now prove

$$v(t)\leq 0,\quad \forall t\in J.\tag{2.5}$$

Suppose that (2.5) not true. Then, there are two cases: (a) there exists $t_1^*\in J$ such that $v(t_1^*)>0$, and $v(t)\geq 0$ for $t\in J$; (b) there exist $t_1^*,t_2^*\in J$ such that $v(t_1^*)>0$ and $v(t_2^*)<0$.

If case (a) holds, then (2.4) implies $v'(t) \le 0,\ \forall t \in J,\ \ t \ne t_k\ (k = 1, 2, \cdots, m)$, and

$$v(t_k^+) = v(t_k) + \triangle v|_{t=t_k} \le (1 - L_k)v(t_k) \le v(t_k) \quad (k = 1, 2, \cdots, m).$$

This means that $v(t)$ is nonincreasing in J, and therefore

$$v(0) \ge v(t_1^*) > 0, \tag{2.6}$$

$$v(0) \ge v(2\pi). \tag{2.7}$$

It follows from (2.7) and the last inequality in (2.4) that $v(0) \ge v(0)e^{2\pi M}$, which contradicts (2.6).

In case (b), let $\inf\limits_{t\in J} v(t) = -\lambda$. Then $\lambda > 0$, and there exists $t_i < t_0^* \le t_{i+1}$ for some i such that $v(t_0^*) = -\lambda$ or $v(t_i^+) = -\lambda$. We may assume that $v(t_0^*) = -\lambda$ since, in case of $v(t_i^+) = -\lambda$, the proof is similar. From (2.4), it is easy to see that

$$\begin{aligned} v'(t) &\le \lambda N k_0 \int_0^t e^{M(t-s)} ds + \lambda N_1 h_0 \int_0^{2\pi} e^{M(t-s)} ds \\ &\le \lambda M_0, \qquad \forall t \in J, \quad t \ne t_k \quad (k = 1, 2, \cdots, m), \end{aligned} \tag{2.8}$$

where $M_0 = M^{-1}(Nk_0 + N_1 h_0)(e^{2\pi M} - 1)$. We have

$$\begin{cases} v(2\pi) - v(t_m^+) = v'(\xi_m)(2\pi - t_m) & (t_m < \xi_m < 2\pi), \\ v(t_m) - v(t_{m-1}^+) = v'(\xi_{m-1})(t_m - t_{m-1}) & (t_{m-1} < \xi_{m-1} < t_m), \\ \cdots\cdots\cdots\cdots & \\ v(t_{i+2}) - v(t_{i+1}^+) = v'(\xi_{i+1})(t_{i+2} - t_{i+1}) & (t_{i+1} < \xi_{i+1} < t_{i+2}), \\ v(t_{i+1}) - v(t_0^*) = v'(\xi_i)(t_{i+1} - t_0^*) & (t_0^* < \xi_i < t_{i+1}), \end{cases} \tag{2.9}$$

and so, by (2.4) and (2.8),

$$\begin{cases} v(2\pi) - (1 - L_m)v(t_m) \le \lambda M_0 \delta, \\ v(t_m) - (1 - L_{m-1})v(t_{m-1}) \le \lambda M_0 \delta, \\ \cdots\cdots\cdots\cdots \\ v(t_{i+2}) - (1 - L_{i+1})v(t_{i+1}) \le \lambda M_0 \delta, \\ v(t_{i+1}) + \lambda \le \lambda M_0 \delta, \end{cases} \tag{2.10}$$

which implies

$$v(2\pi) \le -\lambda \prod_{k=i+1}^{m} (1 - L_k) + \lambda M_0 \delta \Big\{ 1 + \sum_{n=i+1}^{m} \prod_{k=n}^{m} (1 - L_k) \Big\}. \tag{2.11}$$

If $v(2\pi) > 0$, then (2.11) gives

$$M_0 \delta > \frac{\prod\limits_{k=i+1}^{m} (1 - L_k)}{1 + \sum\limits_{n=i+1}^{m} \prod\limits_{k=n}^{m} (1 - L_k)} \ge \frac{\prod\limits_{k=1}^{m} (1 - L_k)}{1 + \sum\limits_{n=1}^{m} \prod\limits_{k=n}^{m} (1 - L_k)},$$

which contradicts (2.2). So, we have $v(2\pi) \le 0$, and by (2.4), $v(0) \le v(2\pi)e^{-2\pi M} \le 0$. Hence $0 < t_1^* < 2\pi$. Let $t_j < t_1^* \le t_{j+1}$ for some j .

We first assume that $t_0^* < t_1^*$. So, $i \le j$. We have, similar to (2.9),

$$\begin{cases} v(t_1^*) - v(t_j^+) = v'(\xi_j)(t_1^* - t_j) & (t_j < \xi_j < t_1^*), \\ v(t_j) - v(t_{j-1}^+) = v'(\xi_{j-1})(t_j - t_{j-1}) & (t_{j-1} < \xi_{j-1} < t_j), \\ \cdots\cdots\cdots\cdots & \\ v(t_{i+2}) - v(t_{i+1}^+) = v'(\xi_{i+1})(t_{i+2} - t_{i+1}) & (t_{i+1} < \xi_{i+1} < t_{i+2}), \\ v(t_{i+1}) - v(t_0^*) = v'(\xi_i)(t_{i+1} - t_0^*) & (t_0^* < \xi_i < t_{i+1}), \end{cases} \tag{2.12}$$

and so, as in (2.10) and (2.11), we get

$$0 < v(t_1^*) \le -\lambda \prod_{k=i+1}^{j} (1-L_k) + \lambda M_0\delta\Big\{1+\sum_{n=i+1}^{j}\prod_{k=n}^{j}(1-L_k)\Big\}, \tag{2.13}$$

which implies

$$M_0\delta > \frac{\prod\limits_{k=i+1}^{j}(1-L_k)}{1+\sum\limits_{n=i+1}^{j}\prod\limits_{k=n}^{j}(1-L_k)} = \frac{\prod\limits_{k=i+1}^{m}(1-L_k)}{\prod\limits_{k=j+1}^{m}(1-L_k)+\sum\limits_{n=i+1}^{j}\prod\limits_{k=n}^{m}(1-L_k)}$$
$$\ge \frac{\prod\limits_{k=1}^{m}(1-L_k)}{1+\sum\limits_{n=1}^{m}\prod\limits_{k=n}^{m}(1-L_k)},$$

and this contradicts (2.2).

Next assume that $t_1^* < t_0^*$. So $j \le i$. Similar to (2.12) and (2.13), we have

$$0 < v(t_1^*) \le v(0)\prod_{k=1}^{j}(1-L_k) + \lambda M_0\delta\Big\{1+\sum_{n=1}^{j}\prod_{k=n}^{j}(1-L_k)\Big\},$$

which implies

$$v(0)\prod_{k=1}^{j}(1-L_k) > -\lambda M_0\delta\Big\{1+\sum_{n=1}^{j}\prod_{k=n}^{j}(1-L_k)\Big\}. \tag{2.14}$$

On the other hand, we have, by (2.4),

$$v(0) \le v(2\pi)e^{-2\pi M}. \tag{2.15}$$

It follows from (2.11), (2.14) and (2.15) that

$$\begin{aligned}&-\lambda M_0\delta\Big\{1+\sum_{n=1}^{j}\prod_{k=n}^{j}(1-L_k)\Big\}\\ &< -\lambda e^{-2\pi M}\prod_{k=1}^{j}(1-L_k)\prod_{k=i+1}^{m}(1-L_k)\\ &\quad+\lambda M_0\delta e^{-2\pi M}\prod_{k=1}^{j}(1-L_k)\Big\{1+\sum_{n=i+1}^{m}\prod_{k=n}^{m}(1-L_k)\Big\},\end{aligned}$$

or

$$\begin{aligned}&\prod_{k=1}^{j}(1-L_k)\prod_{k=i+1}^{m}(1-L_k)\\ &< M_0\delta e^{2\pi M}\Big\{1+\sum_{n=1}^{j}\prod_{k=n}^{j}(1-L_k)\Big\}\\ &\quad+M_0\delta\prod_{k=1}^{j}(1-L_k)\Big\{1+\sum_{n=i+1}^{m}\prod_{k=n}^{m}(1-L_k)\Big\}.\end{aligned}$$

Hence

$$\begin{aligned}
&\Big\{\prod_{k=1}^{m}(1-L_k)\Big\}^2\\
&\le \Big[\prod_{k=j+1}^{m}(1-L_k)\Big]\Big[\prod_{k=1}^{j}(1-L_k)\prod_{k=i+1}^{m}(1-L_k)\Big]\\
&< M_0\delta e^{2\pi M}\Big\{\prod_{k=j+1}^{m}(1-L_k)+\sum_{n=1}^{j}\prod_{k=n}^{m}(1-L_k)\Big\}\\
&\quad + M_0\delta\prod_{k=1}^{m}(1-L_k)\Big\{1+\sum_{n=i+1}^{m}\prod_{k=n}^{m}(1-L_k)\Big\}\\
&\le M_0\delta e^{2\pi M}\Big\{1+\sum_{n=1}^{m}\prod_{k=n}^{m}(1-L_k)\Big\}+M_0\delta\Big\{1+\sum_{n=1}^{m}\prod_{k=n}^{m}(1-L_k)\Big\}\\
&= M_0\delta(e^{2\pi M}+1)\Big\{1+\sum_{n=1}^{m}\prod_{k=n}^{m}(1-L_k)\Big\},
\end{aligned}$$

which contradicts (2.2). The proof is thus complete.

Remark 2.1. Lemma 2.1 develops some ideas in [2] and [3].

Lemma 2.2. *Let* $\sigma,\ \eta\in PC[J,E]$ *and* $M,\ N,\ N_1,\ L_k\ (k=1,2,\cdots,m)$ *be constants with* $M\neq 0$. *Then* $x\in PC[J,E]\cap C^1[J',E]$ *is a solution of the PBVP for linear impulsive integro-differential equation*

$$\begin{cases} x'+Mx+NTx+N_1Sx=\sigma(t), \quad \forall t\in J, \quad t\neq t_k \quad (k=1,2,\cdots,m),\\ \triangle x|_{t=t_k}=I_k(\eta(t_k))-L_k[x(t_k)-\eta(t_k)] \qquad (k=1,2,\cdots,m),\\ x(0)=x(2\pi),\end{cases} \tag{2.16}$$

if and only if $x\in PC[J,E]$ *is a solution of the following impulsive integral equation*

$$\begin{aligned}
x(t)=e^{-Mt}\Big\{&\frac{1}{e^{2\pi M}-1}\Big[\int_0^{2\pi}e^{Ms}(\sigma(s)-N(Tx)(s)-N_1(Sx)(s))ds\\
&+\sum_{k=1}^{m}e^{Mt_k}(I_k(\eta(t_k))-L_k(x(t_k)-\eta(t_k)))\Big]\\
&+\int_0^t e^{Ms}(\sigma(s)-N(Tx)(s)-N_1(Sx)(s))ds\\
&+\sum_{0<t_k<t}e^{Mt_k}(I_k(\eta(t_k))-L_k(x(t_k)-\eta(t_k)))\Big\}.
\end{aligned} \tag{2.17}$$

Proof. Assume that $x\in PC[J,E]\cap C^1[J',E]$ is a solution of PBVP (2.16). Let $z(t)=x(t)e^{Mt}$. Then $z\in PC[J,E]\cap C^1[J',E]$ and, by (2.16),

$$z'(t)=[\sigma(t)-N(Tx)(t)-N_1(Sx)(t)]e^{Mt},\ \forall t\in J,\ t\neq t_k\ \ (k=1,2,\cdots,m), \tag{2.18}$$

$$\triangle z|_{t=t_k}=\{I_k(\eta(t_k))-L_k[x(t_k)-\eta(t_k)]\}e^{Mt_k} \quad (k=1,2,\cdots,m). \tag{2.19}$$

It is easy to establish the following formula (see [4, Lemma 1])

$$z(t)=z(0)+\int_0^t z'(t)ds+\sum_{0<t_k<t}[z(t_k^+)-z(t_k)], \quad \forall t\in J. \tag{2.20}$$

Substituting (2.18) and (2.19) into (2.20), we get

$$x(t)e^{Mt} = x(0) + \int_0^t e^{Ms}[\sigma(s) - N(Tx)(s) - N_1(Sx)(s)]ds + \sum_{0<t_k<t} e^{Mt_k}[I_k(\eta(t_k)) - L_k(x(t_k) - \eta(t_k))], \quad \forall t \in J. \tag{2.21}$$

Letting $t = 2\pi$ in (2.21), we find

$$x(2\pi)e^{M2\pi} = x(0) + \int_0^{2\pi} e^{Ms}[\sigma(s) - N(Tx)(s) - N_1(Sx)(s)]ds + \sum_{k=1}^{m} e^{Mt_k}[I_k(\eta(t_k)) - L_k(x(t_k) - \eta(t_k))]. \tag{2.22}$$

Since $x(0) = x(2\pi)$, (2.22) implies

$$x(0) = \frac{1}{e^{2\pi M} - 1}\Big\{\int_0^{2\pi} e^{Ms}[\sigma(s) - N(Tx)(s) - N_1(Sx)(s)]ds + \sum_{k=1}^{m} e^{Mt_k}[I_k(\eta(t_k)) - L_k(x(t_k) - \eta(t_k))]\Big\}. \tag{2.23}$$

Substituting (2.23) into (2.21), we see that $x(t)$ satisfies (2.17).

Conversely, if $x \in PC[J,E]$ is a solution of Equation (2.17), then, it is easy to see by direct differentiations that $x \in C^1[J',E]$ and x satisfies (2.16).

Lemma 2.3. *Let constants* $M > 0,\ N \geq 0,\ N_1 \geq 0,\ L_k \geq 0\ (k = 1, 2, \cdots, m)$ *and* $\sigma,\ \eta \in PC[J,E]$. *If*

$$2\pi M^{-1}(Nk_0 + N_1 h_0)(2 - e^{-2\pi M}) + \sum_{k=1}^{m}[1 + (e^{2\pi M} - 1)^{-1}e^{Mt_k}]L_k < 1, \tag{2.24}$$

then Equation (2.17) *has a unique solution in* $PC[J,E]$.

Proof. Define operator F by

$$(Fx)(t) = e^{-Mt}\Big\{\frac{1}{e^{2\pi M} - 1}\Big[\int_0^{2\pi} e^{Ms}(\sigma(s) - N(Tx)(s) - N_1(Sx)(s))ds + \sum_{k=1}^{m} e^{Mt_k}(I_k(\eta(t_k)) - L_k(x(t_k) - \eta(t_k)))\Big] + \int_0^t e^{Ms}(\sigma(s) - N(Tx)(s) - N_1(Sx)(s))ds + \sum_{0<t_k<t} e^{Mt_k}(I_k(\eta(t_k)) - L_k(x(t_k) - \eta(t_k)))\Big\}.$$

It is easy to see that F is an operator from $PC[J,E]$ into $PC[J,E]$ and it satisfies

$$\|Fx - Fy\|_{PC} \leq \gamma\|x - y\|_{PC}, \quad \forall x, y \in PC[J,E],$$

where

$$\gamma = 2\pi M^{-1}(Nk_0 + N_1 h_0)(2 - e^{-2\pi M}) + \sum_{k=1}^{m}[1 + (e^{2\pi M} - 1)^{-1}e^{Mt_k}]L_k < 1$$

on account of (2.24). Thus, the Banach fixed point theorem implies that F has a unique fixed point in $PC[J,E]$, and the lemma is proved.

Lemma 2.4. *Let $x_n \in PC[J,E]$ $(n = 1,2,3,\cdots)$. If functions $x_n(t)$ $(n = 1,2,3,\cdots)$ are equicontinuous on each J_k $(k = 0,1,\cdots,m)$ and*

$$\lim_{n\to\infty} x_n(t) = x(t), \quad \forall t \in J, \tag{2.25}$$

then $x \in PC[J,E]$ and

$$\|x_n - x\|_{PC} \to 0 \quad (n \to \infty). \tag{2.26}$$

Proof. Let $V = \{x_n : n = 1,2,3,\cdots\}$ and $V_k = \{x_n|_{J_k} : n = 1,2,3,\cdots\}$ $(k = 0,1,\cdots,m)$. Since $x_n(t_k^+)$ exist $(n = 1,2,3,\cdots)$, V_k may be regarded as a subset of space $C[\bar{J}_k,E]$, where $\bar{J}_k$ denotes the closure of J_k, i.e., $\bar{J}_k = [t_k, t_{k+1}]$. Hence, by hypotheses and the Ascoli-Arzela theorem, V_k is relatively compact in $C[\bar{J}_k,E]$ $(k = 0,1,\cdots,m)$. Consequently, V is relatively compact in $PC[J,E]$.

Assume that (2.26) is not true. Then, there exists an $\varepsilon_0 > 0$ and a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that

$$\|x_{n_i} - x\|_{PC} \geq \varepsilon_0 \ \ (i = 1,2,3,\cdots). \tag{2.27}$$

Since V is relatively compact in $PC[J,E]$, $\{x_{n_i}\}$ contains a subsequence which converges uniformly on J to some $y \in PC[J,E]$. Without loss of generality, we may assume that $\{x_{n_i}\}$ itself converges uniformly on J to y, i.e.

$$\|x_{n_i} - y\|_{PC} \to 0 \quad (i \to \infty). \tag{2.28}$$

Now, (2.28) and (2.25) imply that $y(t) = x(t)$ for $t \in J$, i.e. $y = x$, and (2.28) becomes

$$\|x_{n_i} - x\|_{PC} \to 0 \quad (i \to \infty). \tag{2.29}$$

Evidently, (2.29) contradicts (2.27), and therefore, (2.26) holds.

§3. Main Theorem

We shall state and prove our main theorem in this section. For convenience let us list some conditions for later use.

(H$_1$) There exist $u_0,\ v_0 \in PC[J,E] \cap C^1[J',E]$ satisfying $u_0(t) \leq v_0(t)$ $(\forall t \in J)$ and

$$\begin{cases} u_0' \leq f(t,u_0,Tu_0,Su_0), & \forall t \in J,\ t \neq t_k \ \ (k = 1,2,\cdots,m), \\ \triangle u_0|_{t=t_k} \leq I_k(u_0(t_k)) & (k = 1,2,\cdots,m), \\ u_0(0) \leq u_0(2\pi), \end{cases}$$

$$\begin{cases} v_0' \geq f(t,v_0,Tv_0,Sv_0), & \forall t \in J,\ t \neq t_k \ \ (k = 1,2,\cdots,m), \\ \triangle v_0|_{t=t_k} \geq I_k(v_0(t_k)) & (k = 1,2,\cdots,m), \\ v_0(0) \geq v_0(2\pi), \end{cases}$$

i.e. $u_0(t)$ and $v_0(t)$ are lower and upper solutions of PBVP (1.1) respectively.

(H$_2$) There exist constants $M > 0,\ N \geq 0$ and $N_1 \geq 0$ such that

$$f(t,x,y,z) - f(t,\overline{x},\overline{y},\overline{z}) \geq -M(x-\overline{x}) - N(y-\overline{y}) - N_1(z-\overline{z}),$$

whenever $t \in J,\ u_0(t) \leq \overline{x} \leq x \leq v_0(t)$, $(Tu_0)(t) \leq \overline{y} \leq y \leq (Tv_0)(t)$, and $(Su_0)(t) \leq \overline{z} \leq z \leq (Sv_0)(t)$.

(H$_3$) There exist constants $0 \leq L_k \leq 1$ $(k = 1,2,\cdots,m)$ such that

$$I_k(x) - I_k(\overline{x}) \geq -L_k(x-\overline{x}),$$

whenever $u_0(t_k) \le \overline{x} \le x \le v_0(t_k)$, $(k = 1, 2, \cdots, m)$. As usual, $[u_0, v_0] = \{x \in PC[J, E] : u_0 \le x \le v_0\}$ denotes an ordered interval in $PC[J, E]$.

Theorem 3.1. *Let cone P be regular, f be bounded on $J \times B_r \times B_r \times B_r$ and I_k be bounded on B_r $(k = 1, 2, \cdots, m)$ for any $r > 0$, where $B_r = \{x \in E : ||x|| \le r\}$. Let conditions* $(H_1)-(H_3)$ *be satisfied. Assume that inequlities* (2.2) *and* (2.24) *hold. Then there exist monotone sequences $\{u_n\}$, $\{v_n\} \subset PC[J, E] \cap C^1[J', E]$ which converge uniformly and monotonically on J to the minimal and maximal solutions $\overline{x}$, $x^* \in PC[J, E] \cap C^1[J', E]$ of* PBVP (1.1) *in $[u_0, v_0]$ respectively. That is, if $x \in PC[J, E] \cap C^1[J', E]$ is any solution of* PBVP (1.1) *satisfying $x \in [u_0, v_0]$, then*

$$\begin{aligned} u_0(t) \le u_1(t) \le \cdots \le u_n(t) \le \overline{x}(t) \le x(t) \le x^*(t) \\ \le v_n(t) \le \cdots \le v_1(t) \le v_0(t), \quad \forall t \in J, \quad \forall n \ge 0. \end{aligned} \tag{3.1}$$

Proof. For any $\eta \in [u_0, v_0]$, consider the linear PBVP (2.16) with

$$\sigma(t) = f(t, \eta(t), (T\eta)(t), (S\eta)(t)) + M\eta(t) + N(T\eta)(t) + N_1(S\eta)(t). \tag{3.2}$$

By Lemma 2.2 and Lemma 2.3, PBVP (2.16) has a unique solution $x \in PC[J, E] \cap C^1[J', E]$. Let $x = A\eta$. Then A is an operator from $[u_0, v_0]$ into $PC[J, E] \cap C^1[J', E]$. We now show that (a) $u_0 \le Au_0$, $Av_0 \le v_0$ and (b) A is nondecreasing in $[u_0, v_0]$. To prove (a), we set $u_1 = Au_0$ and $p = u_0 - u_1$. By (2.16) and (3.2), we have

$$\begin{cases} u_1' + Mu_1 + NTu_1 + N_1Su_1 = f(t, u_0, Tu_0, Su_0) + Mu_0 + NTu_0 + N_1u_0, \\ \qquad \forall t \in J,\ t \ne t_k \quad (k = 1, 2, \cdots, m), \\ u_1|_{t=t_k} = I_k(u_0(t_k)) - L_k[u_1(t_k) - u_0(t_k)] \quad (k = 1, 2, \cdots, m), \\ u_1(0) = u_1(2\pi), \end{cases}$$

and so, by (H_1),

$$\begin{cases} p' = u_0' - u_1' \le -Mp - NTp - N_1Sp, & \forall t \in J,\ t \ne t_k \quad (k = 1, 2, \cdots, m), \\ \triangle p|_{t=t_k} = \triangle u_0|_{t=t_k} - \triangle u_1|_{t=t_k} \le -L_kp(t_k) & (k = 1, 2, \cdots, m), \\ p(0) = u_0(0) - u_1(0) \le u_0(2\pi) - u_1(2\pi) = p(2\pi), \end{cases}$$

which implies by virtue of Lemma 1.1 that $p(t) \le \theta$ for $t \in J$, i.e. $u_0 \le Au_0$. Similarly, we can show that $Av_0 \le v_0$. To prove (b), let η_1, $\eta_2 \in [u_0, v_0]$ such that $\eta_1 \le \eta_2$ and let $p = x_1 - x_2$, where $x_1 = A\eta_1$ and $x_2 = A\eta_2$. It is easy to see from (2.16), (3.2) and (H_2),(H_3) that

$$\begin{aligned} p' = x_1' - x_2' = &-[f(t, \eta_2, T\eta_2, S\eta_2) - f(t, \eta_1, T\eta_1, S\eta_1) \\ &+ M(\eta_2 - \eta_1) + NT(\eta_2 - \eta_1) + N_1S(\eta_2 - \eta_1) \\ &- Mp - NTp - N_1Sp \le -Mp - NTp - N_1Sp, \\ &\forall t \in J,\ t \ne t_k \quad (k = 1, 2, \cdots, m), \end{aligned}$$

$$\begin{aligned} \triangle p|_{t=t_k} &= \triangle x_1|_{t=t_k} - \triangle x_2|_{t=t_k} \\ &= -\{I_k(\eta_2(t_k) - I_k(\eta_1(t_k)) + L_k[\eta_2(t_k) - \eta_1(t_k)]\} - L_kp(t_k) \\ &\le -L_kp(t_k), \quad (k = 1, 2, \cdots, m), \\ p(0) &= p(2\pi). \end{aligned}$$

Hence, Lemma 2.1 implies that $p(t) \le \theta$ for $t \in J$, i.e. $A\eta_1 \le A\eta_2$, and (b) is proved.

Let $u_n = Au_{n-1}$ and $v_n = Av_{n-1}$ $(n = 1, 2, 3, \cdots)$. By (a) and (b) just proved, we have

$$u_0(t) \le u_1(t) \le \cdots \le u_n(t) \le \cdots \le v_n(t) \le \cdots \le v_1(t) \le v_0(t), \quad \forall t \in J. \tag{3.3}$$

So, the regularity of P implies

$$\lim_{n\to\infty} u_n(t) = \overline{x}(t), \quad \forall t \in J. \tag{3.4}$$

Let $V = \{u_n : n = 0, 1, 2, \cdots\}$. Since P is also normal, it follows from (3.3) that V is a bounded set in $PC[J, E]$, and so, by hypotheses, there is a positive constant β such that

$$\begin{aligned} &\|f(t, u_{n-1}(t), (Tu_{n-1})(t), (Su_{n-1})(t)) + Mu_{n-1}(t) - N(T(u_n - u_{n-1}))(t) \\ &- N_1(S(u_n - u_{n-1})(t)\| \le \beta, \quad \forall t \in J \ (n = 1, 2, 3, \cdots), \end{aligned} \tag{3.5}$$

$$\|I_k(u_{n-1})(t)\| \le \beta, \quad \forall t \in J \ (n = 1, 2, 3, \cdots). \tag{3.6}$$

On account of the definition of u_n and (2.16), (2.17), (3.2), we have

$$\begin{aligned} u_n(t) = e^{-Mt}\Big\{&\frac{1}{e^{2\pi M} - 1}\Big[\int_0^{2\pi} e^{Ms}(f(s, u_{n-1}(s), (Tu_{n-1})(s), (Su_{n-1})(s)) \\ &+ Mu_{n-1}(s) - N(T(u_n - u_{n-1}))(s) - N_1(S(u_n - u_{n-1}))(s))ds \\ &+ \sum_{k=1}^{m} e^{Mt_k}(I_k(u_{n-1}(t_k)) - L_k(u_n(t_k) - u_{n-1}(t_k)))\Big] \\ &+ \int_0^t e^{Ms}(f(s, u_{n-1}(s), (Tu_{n-1})(s), (Su_{n-1})(s)) + Mu_{n-1}(s) \\ &- N(T(u_n - u_{n-1}))(s) - N_1(S(u_n - u_{n-1}))(s))ds \\ &+ \sum_{0<t_k<t} e^{Mt_k}((I_k(u_{n-1}(t_k)) - L_k(u_n(t_k) - u_{n-1}(t_k)))\Big\}, \\ &\forall t \in J \quad (n = 1, 2, 3, \cdots). \end{aligned} \tag{3.7}$$

It follows from (3.5)–(3.7) that V is equicontinuous on each J_k $(k = 0, 1, \cdots, m)$, and consequently, (3.4) and Lemma 2.4 imply that $\overline{x} \in PC[J, E]$ and $\{u_n\}$ converges to $\overline{x}$ uniformly on J. Now, we have

$$\begin{aligned} &f(t, u_{n-1}(t), (Tu_{n-1})(t), (Su_{n-1})(t)) + Mu_{n-1}(t) - N(T(u_n - u_{n-1}))(t) \\ &- N_1(S(u_n - u_{n-1}))(t) \to f(t, \overline{x}(t), (T\overline{x})(t), (S\overline{x})(t)) + M\overline{x}(t) \\ &\qquad \text{as} \ \ n \to \infty, \quad \forall t \in J, \end{aligned} \tag{3.8}$$

and, by (3.5),

$$\begin{aligned} &\|f(t, u_{n-1}(t), (Tu_{n-1})(t), (Su_{n-1})(t)) + Mu_{n-1}(t) - N(T(u_n - u_{n-1}))(t) \\ &- N_1(S(u_n - u_{n-1}))(t) - f(t, \overline{x}(t), (T\overline{x})(t), (S\overline{x})(t)) - M\overline{x}(t)\| \le 2\beta, \\ &\qquad \forall t \in J \quad (n = 1, 2, 3, \cdots). \end{aligned} \tag{3.9}$$

Observing (3.8) and (3.9) and taking limits as $n \to \infty$ in (3.7),we get

$$\begin{aligned} \overline{x}(t) = e^{-Mt}\Big\{&\frac{1}{e^{2\pi M} - 1}\Big[\int_0^{2\pi} e^{Ms}(f(s, \overline{x}(s), (T\overline{x})(s), (S\overline{x})(s)) + M\overline{x}(s))ds \\ &+ \sum_{k=1}^{m} e^{Mt_k}(I_k(\overline{x}(t_k))\Big] \\ &+ \int_0^t e^{Ms}(f(s, \overline{x}(s), (T\overline{x})(s), (S\overline{x})(s)) + M\overline{x}(s))ds \\ &+ \sum_{0<t_k<t} e^{Mt_k} I_k(\overline{x}(t_k))\Big\}, \end{aligned}$$

which implies by virtue of Lemma 2.2 that $\overline{x} \in PC[J,E] \cap C^1[J',E]$ and $\overline{x}(t)$ is a solution of PBVP (1.1).

In the same way, we can show that $\{v_n\}$ converges uniformly on J to some x^*, and $x^*(t)$ is a solution of PBVP (1.1) in $PC[J,E] \cap C^1[J',E]$.

Finally, let $x \in PC[J,E] \cap C^1[J',E]$ be any solution of PBVP (1.1) in $[u_0, v_0]$. Assume that $u_{n-1}(t) \le x(t) \le v_{n-1}(t)$ for $t \in J$, and let $p = u_n - x$. Then, as before by (2.16), (3.2) and (H_2), (H_3), it is easy to verify that p satisfies (2.1), and so, Lemma 2.1 implies that $p(t) \le \theta$ for $t \in J$, i.e. $u_n(t) \le x(t)$ for $t \in J$. Similarly , we can show that $x(t) \le v_n(t)$ for $t \in J$. Consequently , by induction,we have $u_n(t) \le x(t) \le v_n(t)$ for $t \in J$ $(n = 0, 1, 2, \cdots)$, and by taking limits, we get $\overline{x}(t) \le x(t) \le x^*(t)$ for $t \in J$. Hence, (3.1) holds, and the theorem is proved.

Remark 3.1. The condition that P is regular will be satisfied if E is weakly complete (reflexive, in particular) and P is normal (see [5, Theorem 2.2]).

Remark 3.2. In some cases, it is easy to find a lower solution and an upper solution for PBVP (1.1). For example, let $I_k(\theta) = \theta$ $(k = 1, 2, \cdots, m)$. If $f(t, \theta, \theta, \theta) \ge \theta$ for $t \in J,\ t \ne t_k$ $(k = 1, 2, \cdots, m)$, then $u_0(t) \equiv \theta$ $(t \in J)$ is a lower solution of PBVP (1.1); if $f(t, x_0, Tx_0, Sx_0) \le \theta$ for some $x_0 > \theta$ and $t \in J,\ t \ne t_k$ $(k = 1, 2, \cdots, m)$, then $v_0(t) \equiv x_0$ $(t \in J)$ is an upper solution of PBVP (1.1).

§4. Examples

Example 4.1. Consider the PBVP of finite system for scalar nonlinear impulsive integro-differential equations

$$\begin{cases} x_i' = f_i(t, x, Tx, Sx), & \forall 0 \le t \le 2\pi, \quad t \ne t_k \quad (k = 1, 2, \cdots, m), \\ \triangle x_i|_{t=t_k} = I_{ik}(x(t_k)) & (k = 1, 2, \cdots, m), \\ x_i(0) = x_i(2\pi) & (i = 1, 2, \cdots, n), \end{cases} \tag{4.1}$$

where $f_i = f_i(t, x, y, z)$, $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$,$z = (z_1, \cdots, z_n)$, $f_i \in C^1[J \times R^n \times R^n \times R^n, R]$, $J = [0, 2\pi]$, $0 < t_1 < \cdots < t_k < \cdots < t_m < 2\pi$, $I_{ik} \in C^1[R^n, R]$ $(k = 1, 2, \cdots, m; i = 1, 2, \cdots, n)$, Tx and Sx are defined by (1.2) with $K \in C[D, R_+]$ and $H \in C[J \times J, R_+]$. Let $u_0 = (u_{01}, \cdots, x_{0n})$ and $v_0 = (v_{01}, \cdots, v_{0n})$ be lower and upper solutions of (4.1) respectively with $u_0(t) \le v_0(t)$ for $t \in J$ (i.e., $u_{0i}(t) \le v_{0i}(t)$ for $t \in J,\ i = 1, 2, \cdots, n$). Let $\Omega = \{(t, x, y, z):\ t \in J,\ u_0(t) \le x \le v_0(t),\ (Tu_0)(t) \le y \le (Tv_0)(t),\ (Su_0)(t) \le z \le (Sv_0)(t)\}$ and $\Omega_k = \{x:\ u_0(t_k) \le x \le v_0(t_k)\}$ $(k = 1, 2, \cdots, m)$.

Conclusion 4.1. *If there exist constans $M > 0,\ N \ge 0,\ N_1 \ge 0$ and $0 \le L_k \le 1$ $(k = 1, 2, \cdots, m)$ such that*

$$\frac{\partial f_i}{\partial x_j} \ge \begin{cases} 0, & i \ne j; \\ -M, & i = j, \end{cases} \quad in\ \Omega \quad (i,\ j = 1, 2, \cdots, n),$$

$$\frac{\partial f_i}{\partial y_j} \ge \begin{cases} 0, & i \ne j; \\ -N, & i = j, \end{cases} \quad in\ \Omega \quad (i,\ j = 1, 2, \cdots, n),$$

$$\frac{\partial f_i}{\partial z_j} \ge \begin{cases} 0, & i \ne j; \\ -N_1, & i = j, \end{cases} \quad in\ \Omega \quad (i,\ j = 1, 2, \cdots, n),$$

$$\frac{\partial I_{ik}}{\partial x_j} \ge \begin{cases} 0, & i \ne j; \\ -L_k, & i = j, \end{cases} \quad in\ \Omega_k \quad (i,\ j = 1, 2, \cdots, n;\ k = 1, 2, \cdots, m),$$

and inequalities (2.2) *and* (2.24) *hold, then* PBVP (4.1) *has a minimal solution and a maximal solution in* $[u_0, v_0]$.

Proof. Let $E = R^n$ and $P = \{x = (x_1, \cdots, x_n) \in R^n : \ x_i \geq 0, \ i = 1, 2, 3, \cdots, n\}$. Then P is a regular cone in E and (4.1) can be regarded as a PBVP of type (1.1) in E. For $(t, x, y, z), \ (t, \overline{x}, \overline{y}, \overline{z}) \in \Omega$ satisfying $\overline{x} \leq x, \ \overline{y} \leq y$ and $\overline{z} \leq z$, we have, by hypotheses and the mean value theorem,

$$\begin{aligned} f_i(t, x, y, z) - f_i(t, \overline{x}, \overline{y}, \overline{z}) &= \sum_{j=1}^{n} \Big[(x_j - \overline{x}_j) \frac{\partial}{\partial x_j} + (y_j - \overline{y}_j) \frac{\partial}{\partial y_j} + (z_j - \overline{z}_j) \frac{\partial}{\partial z_j} \Big] \\ &\quad \cdot f_i(t, \overline{x} + \xi(x - \overline{x}), \overline{y} + \xi(y - \overline{y}), \overline{z} + \xi(z - \overline{z})) \\ &\geq -M(x_i - \overline{x}_i) - N(y_i - \overline{y}_i) - N_1(z_i - \overline{z}_i) \quad (i = 1, 2, \cdots, n), \end{aligned}$$

and, for $x, \ \overline{x} \in \Omega_k$ satisfying $\overline{x} \leq x$,

$$\begin{aligned} I_{ik}(x) - I_{ik}(\overline{x}) &= \sum_{j=1}^{n} (x_j - \overline{x}_j) \frac{\partial}{\partial x_j} I_{ik}(\overline{x} + \xi_1(x - \overline{x})) \\ &\geq -L_k(x_i - \overline{x}_i) \quad (i = 1, 2, \cdots, n; k = 1, 2, \cdots, m), \end{aligned}$$

where $0 < \xi < 1$ and $0 < \xi_1 < 1$. So, (H_2) and (H_3) are satisfied, and our conclusion follows from the main theorem.

Example 4.2. Consider the PBVP of infinite system for scalar nonlinear impulsive integro-differential equations

$$\begin{cases} x_n' = \dfrac{4}{\pi} \Big(\dfrac{1}{4n^2} - x_n + x_{2n} \Big) + \dfrac{t}{2\pi^3 n^2} \Big(\displaystyle\int_0^t e^{-ts} x_{n+1}(s) ds \Big) \\ \qquad - \dfrac{2}{10^8 \pi^2 (n+1)^2} \left(\displaystyle\int_0^t e^{-ts} x_n(s) ds \right)^2 \\ \qquad - \dfrac{1}{10^8 \pi^3 (n+2)^3} \Big(\displaystyle\int_0^{2\pi} \dfrac{x_n(s) ds}{1 + t^2 + s^2} \Big)^3, \qquad \forall 0 \leq t \leq 2\pi, \ t \neq \pi, \\ \triangle x_n |_{t=\pi} = -\dfrac{1}{2n} x_n(\pi) + x_{n+2}(\pi), \\ x_n(0) = x_n(2\pi) \qquad (n = 1, 2, 3, \cdots). \end{cases} \tag{4.2}$$

Conclusion 4.2. PBVP (4.2) *admits minimal and maximal solutions which continuously differentiable on* $[0, \pi) \cup (\pi, 2\pi]$ *and satisfy*

$$0 \leq x_n(t) \leq \begin{cases} \frac{1}{n^2}, & \forall \ \ 0 \leq t \leq \pi \\ \frac{1}{n^2}(3 - \frac{t}{\pi}), & \forall \ \ \pi < t \leq 2\pi \end{cases} \ (n = 1, 2, 3, \cdots).$$

Proof. Let $E = \ell^1 = \{x = (x_1, \cdots, x_n, \cdots) : \ \sum\limits_{n=1}^{\infty} |x_n| < \infty\}$ with norm

$$||x|| = \sum_{n=1}^{\infty} |x_n| \ \text{ and } \ P = \{x = (x_1, \cdots, x_n, \cdots) \in \ell^1 : \ x_n \geq 0, \ n = 1, 2, 3, \cdots\}.$$

Then P is a normal cone in E. Since ℓ^1 is weakly complete, we know from Remark 3.1 that P is regular. (4.2) can be regarded as a PBVP of type (1.1) in E, where

$$\begin{aligned} K(t, s) = e^{-ts}, \quad H(t, s) = (1 + t^2 + s^2)^{-1}, \quad x = (x_1, \cdots, x_n, \cdots), \\ y = (y_1, \cdots, y_n, \cdots), \quad z = (z_1, \cdots, z_n, \cdots), \quad f = (f_1, \cdots, f_n, \cdots). \end{aligned}$$

in which

$$f_n(t,x,y,z) = \frac{4}{\pi}\Big(\frac{1}{4n^2} - x_n + x_{2n}\Big) + \frac{t}{2\pi^3 n^2} y_{n+1} - \frac{2}{10^8\pi^2(n+1)^2} y_n^2 - \frac{1}{10^8\pi^3(n+2)^3} z_n^3$$

and $m = 1,\ t_1 = \pi,\ I_1 = (I_{11}, \cdots, I_{1n}, \cdots)$ with

$$I_{1n}(x) = -\frac{1}{2n} x_n + x_{n+2}.$$

Let

$$u_0(t) = (0, \cdots, 0, \cdots), \quad \forall\ \ 0 \le t \le 2\pi,$$

and

$$v_0(t) = \begin{cases} (1, \cdots, \frac{1}{n^2}, \cdots), & \forall \quad 0 \le t \le \pi; \\ (3 - \frac{t}{\pi}, \cdots, \frac{1}{n^2}(3 - \frac{t}{\pi}), \cdots), & \forall \quad \pi < t \le 2\pi. \end{cases}$$

It is not difficult to verify that u_0 and v_0 satisfy condition (H_1).

On the other hand, it is easy to see that conditions (H_2) and (H_3) are satisfied for

$$M = \frac{4}{\pi},\ N = \frac{3}{10^8\pi},\ N_1 = \frac{1}{10^8\pi}, \quad \text{and} \quad L_1 = \frac{1}{2}.$$

Evidently , $k_0 = h_0 = 1$, and it is easy to check that inequalities (2.2) and (2.24) hold. Thus , our conclusion follows from the main theorem.

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