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REMARKS ON THE EXISTENCE OF WEAK SOLUTIONS TO 2-D INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract

The authors prove the global existence of weak solutions to 2-D incompressible Navier-Stokes equations (in vorticity-stream formulation) with initial vorticity in $L^{\frac{4}{3}}$.It may be the best result that can be obtained for initial vorticity in L^p form. Moreover,the uniqueness is to be proved here.

Keywords Navier-Stokes equations, Vorticity-stream form, Riesz potential theory, Weak solution

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§1. Introduction

Consider the following homogeneous 2-D incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t V + (V\cdot\nabla)V = -\nabla P + \nu\triangle V, \\ \mathrm{div}V = 0, \\ V|_{t=0} = V_0, \quad \mathrm{div}V_0 = 0, \end{cases} \tag{1.1}$$

where $V = (V_1(t,x), V_2(t,x))$ is the velocity of the fluid, P is the scalar pressure, $V\cdot\nabla = V_1\partial_1 + V_2\partial_2$, and the constant $\nu > 0$ is the kinetic viscosity of the fluid (when $\nu = 0$, this system is called Euler equations, and we denote it by (1.1)). By acting the operation of curl on (1.1), we can obtain the evolution equation for vorticity $\omega = (\partial_1 V_2 - \partial_2 V_1)$:

$$\begin{cases} \partial_t\omega + (V\cdot\nabla)\omega = \nu\triangle\omega, \\ V = K*\omega, \quad K(x) = \dfrac{1}{2\pi}\dfrac{\begin{pmatrix} -x_2 \\ \mathbf{x}_1 \end{pmatrix}}{|x|^2}, \\ \omega(t,x)|_{t=0} = \omega_0. \end{cases} \tag{1.2}$$

It is the vorticity-stream form of (1.1) (when $\nu = 0$, we denote it by (1.2)). For (1.1), in [4], R. Diperna and A. Majda proved the global existence of weak solutions for initial velocity with vorticity $\omega_0 \in L^p \cap L^1, 1 < p < +\infty$; in [3], D. H. Chae obtained the global existence of weak solutions for initial velocity with vorticity $\omega_0 \in L(\log L)^{\frac{1}{2}}(\mathbb{R}^2)$, and having compact support (his proof may be simplified by directly applying div-curl Lemma). Obviously, when V and

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ω are smooth enough, (1.1) and (1.2) are equivalent. However, in the sense of weak solutions, the vorticity-stream formulation places a more strinent requirement on the regularity of the velocity field than the primitive-variable weak formulation since the former requires that the product $V \cdot \omega$ define a distribution , while ω is a first derivative of V. In [9], A. Majda claimed that (1.2) has a global weak solution for $\omega_0 \in L^p$ with $p > \frac{4}{3}$, but its uniqueness is not known. We have not seen his proof about the above existence.Nevertheless, it can be obtained easily by constructing approximate solution sequence and a straight-forward use of Sobolev inequality. Using the same approach, we can also obtain the same result for (1.2). As for the critical case, i.e., $\omega_0 \in L^{\frac{4}{3}}$, to the author's knowledge, there is not any result even for (1.2). We will deal with such a case in this paper. Moreover, inspired by the proof of this paper, we can also prove that (1.2) has a unique global weak solutions for $\omega_0 \in L^p,\ p > \frac{4}{3}$. Since it is more cumbersome, we omit it here.

Before we give the main result of this paper, let us recall the definitions of weak solution to (1.2). By definition the vorticity $\omega(t,x)$ and velocity $V(t,x)$ is a weak solution of (1.2) with initial vorticity ω_0, provided that $V(t,x) = K * \omega, \quad K(x) = \frac{1}{2\pi|x|^2}\begin{pmatrix} -\mathbf{x_2} \\ \mathbf{x_1} \end{pmatrix}$, and for any smooth scalar test function ϕ which is rapidly decreasing in x and vanishing for large t,

$$\int_0^\infty \int_{\mathbb{R}^2} \omega \cdot \phi_t + \omega V \cdot \nabla\phi \, dx\, dt + \nu \int_0^\infty \int_{\mathbb{R}^2} \triangle\phi \cdot \omega \, dx\, dt + \int \phi(0,x)\omega_0(x)\, dx = 0, \qquad (1.3)$$

where we must require that ωV have meaning as a distribution.

The main result of this paper is as follows.

Theorem 1.1. *Let* $\omega_0 \in L^{\frac{4}{3}}(\mathbb{R}^2)$. *Then,* (1.2) *has a unique global weak solution* $(V(t,x), \omega(t,x))$ *in the sense of* (1.3). *Moreover,*

$$V(t,x) \in C(\mathbb{R}^+, W^{1,\frac{4}{3}}_{\text{loc}}(\mathbb{R}^2)) \cap C(\mathbb{R}^+, L^4) \cap C^1(\mathbb{R}^+, W^{-1,\frac{4}{3}}_{\text{loc}}(\mathbb{R}^2)),$$
$$\omega(t,x) \in C(\mathbb{R}^+, L^{\frac{4}{3}}(\mathbb{R}^2)) \cap C^1(\mathbb{R}^+, W^{-2,\frac{4}{3}}_{\text{loc}}(\mathbb{R}^2)).$$

Remark 1.1. If $\omega_0(x) \in L^p, p < \frac{4}{3}$, then by Riesz potential theory the corresponding velocity $V_0(x) \in L^q, \frac{1}{q} = \frac{1}{p} - \frac{1}{2}$, but $\frac{1}{p} + \frac{1}{q} > 1$. Thus $V_0(x) \cdot \omega_0(x)$ has no meaning as a distribution in general, and, in this sense the result in this paper is optimal.

In our following estimate, C is used as a generic constant and may change from line to line, C_ν is also a generic constant depending only on ν.

§2. The Existence of Weak Solutions

Let $\rho_\epsilon(x)$ be the standard molifier in $\mathbb{R}^2$, i.e.

$$\rho_\epsilon(x) = \frac{1}{\epsilon^2}\rho\Big(\frac{|x|}{\epsilon}\Big), \quad \rho(x) \in C_0^\infty(\mathbb{R}^+), \quad \rho \geq 0, \quad \text{supp}\rho \subset \{x||x| \leq 1\}, \quad \int_{\mathbb{R}^2} \rho\, dx = 1.$$

We also define a cutoff function ξ_ϵ by $\xi_\epsilon(x) = \xi(\frac{x}{\epsilon})$, where $\xi(x) \in C_0^\infty(\mathbb{R}^2), 0 \leq \xi \leq 1$, $\xi(x) = 1$ on $\{|x| \leq 1\}$, and $\xi(x) = 0$ on $\{|x| \geq 2\}$.

Then, for the initial vorticity $\omega_0(x)$ given in Theorem 1.1, we construct

$$\omega_0^\epsilon = \xi_\epsilon(x) \cdot (\rho * \omega_0)(x). \qquad (2.1)$$

Obviously, $\omega_0^\epsilon \in C_0^\infty$ and tends to $\omega_0(x)$ in $L^{\frac{4}{3}}$.It should be noted that $|V_0^\epsilon(x)| \leq \int \frac{1}{|x-y|} \cdot \omega_0^\epsilon(y)\, dy$, and $V_0^\epsilon(x) = \nabla^\perp \triangle^{-1}\omega_0(x)$, where $V_0^\epsilon(x)$ is the velocity corresponding to ω_0^ϵ by

Biot-Sarvart law. Hence according to Riesz potential theory and singular integral operator theory,we can know that $V_0^\epsilon \in H^\infty$ (see p. 75 of [11] for more details.)

Following the proof of Theorem 1 in [1] and the results in [13], there exists a smooth solution sequence $\{V^\epsilon(t,x)\}$ corresponding to the initial velocity $\{V_0^\epsilon\}$,such that

$$\int_{\mathbb{R}^2} |\omega^\epsilon(t,x)|^{\frac43}\,dx \le \int_{\mathbb{R}^2} |\omega_0^\epsilon(x)|^{\frac43}\,dx \le \int_{\mathbb{R}^2} |\omega_0(x)|^{\frac43}\,dx \equiv M, \tag{2.2}$$

and by Riesz potential theory,

$$\|V^\epsilon(t,\cdot)\|_{L^4} \le C\|\omega^\epsilon(t,\cdot)\|_{L^{\frac43}} \le C\|\omega_0(\cdot)\|_{L^{\frac43}}. \tag{2.3}$$

Since
$$\begin{cases} \partial_t\omega^\epsilon(t,x) - \nu\triangle\omega^\epsilon(t,x) = -V^\epsilon\cdot\nabla\omega^\epsilon(t,x),\\ \omega^\epsilon(t,x)|_{t=0} = \omega_0^\epsilon(x), \end{cases}$$

using the foundamental solution $F(t,x) = (2\pi\nu t)^{-1}H(t)\exp(-\frac{|x|^2}{4\nu t})$ of two dimensional heat conductive operator $\partial_t - \nu\triangle$, where $H(t)$ is the Heaviside function, we see that

$$\begin{aligned}\omega^\epsilon(t,x) = &\frac1\pi\sum_{i=1}^2\int_0^t\int_{\mathbb{R}^2}(2(t-s)\nu)^{-2}(x_i-y_i)\exp\Big(-\frac{|x-y|^2}{4\nu(t-s)}\Big)(V_i^\epsilon\omega^\epsilon)(s,y)\,ds\,dy\\ &+\int_{\mathbb{R}^2}(2\pi\nu t)^{-1}\exp\Big(-\frac{|x-y|^2}{4\nu t}\Big)\omega_0^\epsilon(y)\,dy \equiv \omega^{1,\epsilon}(t,x)+\omega^{2,\epsilon}(t,x).\end{aligned} \tag{2.4}$$

In the above equality, we have applied the properties that $\mathrm{div}V^\epsilon(t,\cdot)=0$, $F(t,x)$ is rapidly decreasing in x, and that $V_i^\epsilon(t,x)\cdot\omega^\epsilon(t,x)\in L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^2))$ by (2.2) and (2.3).

Now we will prove that $\{V^\epsilon(t,x)\}$ and $\{\omega^\epsilon(t,x)\}$are compact subsets of $C_{\mathrm{loc}}(\mathbb{R}^+, L^4(\mathbb{R}^2))$ and $C_{\mathrm{loc}}(\mathbb{R}^+, L^{\frac43})$ respectively. To this end ,we first establish:

Lemma 2.1. *For any $T>0$, $\{\omega^{2,\epsilon}(t,x)\}$ is convergent in $C(\mathbb{R}^+, L^{\frac43}(\mathbb{R}^2))$.*

Proof. In fact, by Young inequality,

$$\begin{aligned}&\Big(\int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}(2\pi\nu t)^{-1}\exp\Big(-\frac{|x-y|^2}{4\nu t}\Big)(\omega_0^\epsilon(\cdot)-\omega_0(\cdot))\,dy\Big|^{\frac43}dx\Big)^{\frac34}\\ &\le\Big(\Big\|(2\pi\nu t)^{-1}\exp\Big(-\frac{|\cdot|^2}{4\nu t}\Big)\Big\|_{L^1}^{\frac43}\cdot\|\omega_0^\epsilon(\cdot)-\omega_0(\cdot)\|_{L^{\frac43}}^{\frac43}\Big)^{\frac34}\\ &=\|\omega_0^\epsilon(\cdot)-\omega_0(\cdot)\|_{L^{\frac43}}\to 0,\quad \text{as}\quad \epsilon\to 0.\end{aligned} \tag{2.5}$$

Lemma 2.2. *For any $T>0$, and $t\in[0,T]$, $\{|\omega^{1,\epsilon}(t,x)|^{\frac43}\}$ is a weakly compact subset of $L^1_{\mathrm{loc}}(\mathbb{R}^2)$.*

Proof. By Dunford-Pettis Theorem[6], we only need to prove that, for any $\theta>0$, there exists some $\eta>0$ such that for any measurable subset B of $\mathbb{R}^2$, with $\mathrm{meas}B\le\eta$, we have

$$\sup_{t\in[0,T]}\int_B|\omega^{1,\epsilon}(t,x)|^{\frac43}\,dx\le\theta. \tag{2.6}$$

By (2.4),

$$\begin{aligned}&\Big(\int_B|\omega^{1,\epsilon}(t,x)|^{\frac43}\,dx\Big)^{\frac34}\\ &\le\frac1\pi\sum_{i=1}^2\Big(\int_{\mathbb{R}^2}\Big|\int_o^t\int_{\mathbb{R}^2}1_B(x)(2(t-s)\nu)^{-2}(x_i-y_i)\exp\Big(-\frac{|x-y|^2}{4\nu(t-s)}\Big)\\ &\quad\cdot(V_i^\epsilon\cdot\omega^\epsilon)(s,y)\,ds\,dy\Big|^{\frac43}dx\Big)^{\frac34},\end{aligned} \tag{2.7}$$

where $1_B(x)$ is the character function of the set B.

By applying Minkowski inequality for (2.7), we find

$$(2.7) \le \frac{1}{\pi}\sum_{i=1}^{2}\int_0^t\int_{\mathbb{R}^2} ds\,dy\Big|\int_{\mathbb{R}^2} 1_B(x)\Big|(2(t-s)\nu)^{-2}(x_i-y_i) \cdot\exp\Big(-\frac{|x-y|^2}{4\nu(t-s)}\Big)(V_i^\epsilon\cdot\omega^\epsilon)(s,y)\Big|^{\frac{4}{3}}dx\Big|^{\frac{3}{4}}$$
$$\le \frac{2M^2}{\pi}\sup_{y\in\mathbb{R}^2}\int_0^t\Big|\int_{\mathbb{R}^2} 1_B(x)\Big|(2(t-s)\nu)^{-\frac{8}{3}}(x_i-y_i)^{\frac{4}{3}}\exp\Big(-\frac{|x-y|^2}{3\nu(t-s)}\Big)\Big|dx\Big|^{\frac{3}{4}}ds.$$

Hence, we only need to prove that for any $\theta>0$ there exists some $\eta>0$ such that when $\text{meas}\,B\le\eta$,

$$\sup_{\substack{t\in[0,T]\\ y\in\mathbb{R}^2}}\int_0^t\Big|\int_{\mathbb{R}^2} 1_B(x)\Big|(2(t-s)\nu)^{-\frac{8}{3}}(x_i-y_i)^{\frac{4}{3}}\exp\Big(-\frac{|x-y|^2}{3\nu(t-s)}\Big)\Big|dx\Big|^{\frac{3}{4}}ds\le\frac{\pi\theta}{2M^2}. \tag{2.8}$$

In fact

$$\begin{aligned}&\int_0^t\Big|\int_{\mathbb{R}^2} 1_B(x)\Big|(2(t-s)\nu)^{-\frac{8}{3}}(x_i-y_i)^{\frac{4}{3}}\exp\Big(-\frac{|x-y|^2}{3\nu(t-s)}\Big)\Big|dx\Big|^{\frac{3}{4}}ds\\ &=\Big(\int_0^{\delta\wedge t}+\int_{\delta\wedge t}^t\Big)\Big|\int_{\mathbb{R}^2}1_B(x)(2s\nu)^{-\frac{8}{3}}(x_i-y_i)^{\frac{4}{3}}\cdot\exp\Big(-\frac{|x-y|^2}{3\nu s}\Big)dx\Big|^{\frac{3}{4}}ds\\ &\le C_\nu\delta^{\frac{1}{4}}+C(\nu\delta)^{-\frac{3}{2}}T\text{meas}\,B^{\frac{3}{4}},\end{aligned} \tag{2.9}$$

where we have applied the property that $(x)^{\frac{4}{3}}\exp(-x^2)\le C$, and $\delta\wedge t=\min(\delta,t)$. Then (2.6) is satisfied by taking $\eta=(\frac{\pi\theta}{4CM^2T})^{\frac{4}{3}}(\nu\delta)^2$, and $\delta=(\frac{\pi\theta}{4C_\nu M^2})^4$. In view of (2.7),(2.8) and (2.9), we obtain (2.6). Thus Lemma 2.2 is verified.

Remark 2.1. Following the proof of Lemma 2.2 , we may not take η independent of ν, such that (2.6) holds. Thus the proof of this paper can not be used to prove the corresponding case for (1.2) by viscosity vanishing method. In addition, we can see this by the proof of the following lemma, where we may not take a positive number T_1 independent of ν, such that (2.14) holds.

Lemma 2.3. *For any $T>0$, we have*

$$\lim_{R\to\infty}\sup_{\substack{\epsilon>0\\ t\in[0,T]}}\int_{|x|\ge R}|\omega^{1,\epsilon}(t,x)|^{\frac{4}{3}}dx=0.$$

proof. Firstly, as the proof of Lemma 2.2, by Minkowski inequality,

$$\begin{aligned}&\Big(\int_{|x|\ge R}\frac{1}{\pi}\Big|\int_0^t\int_{\mathbb{R}^2}\sum_{i=1}^2(V_i^\epsilon\omega^\epsilon)(s,y)(2\nu(t-s))^{-2}(x_i-y_i)\exp\Big(-\frac{|x-y|^2}{4\nu(t-s)}\Big)ds\,dy\Big|^{\frac{4}{3}}dx\Big)^{\frac{3}{4}}\\ &\le\frac{1}{\pi}\sum_{i=1}^2\Big(\int_0^{\delta\wedge t}+\int_{\delta\wedge t}^t\Big)\int_{\mathbb{R}^2}|V_i^\epsilon\omega^\epsilon|(s,y)\Big|\int_{|x|\ge R}(2\nu s)^{-\frac{8}{3}}\\ &\quad\cdot(x_i-y_i)^{\frac{4}{3}}\exp\Big(-\frac{|x-y|^2}{3\nu s}\Big)dx\Big|^{\frac{3}{4}}ds\,dy=(\mathbb{D})+(\mathbb{E}).\end{aligned} \tag{2.10}$$

By a similar proof of (2.9), we find

$$(\mathbb{D})\le C_\nu\delta^{\frac{1}{4}}. \tag{2.11}$$

A very simple calculation tells that

$$\begin{aligned}&\lim_{R\to\infty}\int_{|x|\ge R}(2\nu t)^{-\frac{8}{3}}|x_i|^{\frac{4}{3}}\exp\Big(-\frac{|x|^2}{3\nu t}\Big)\,dx=0,\quad\text{uniformly for } T\ge t\ge\delta,\\&\int_0^T\Big|\int_{\mathbb{R}^2}(2\nu t)^{-\frac{8}{3}}|x_i|^{\frac{4}{3}}\exp\Big(-\frac{|x|^2}{3\nu t}\Big)\,dx\Big|^{\frac{3}{4}}ds\le C_\nu T^{\frac{1}{4}}.\end{aligned}\tag{2.12}$$

Then

$$\begin{aligned}(\mathbb{E})\le{}&\frac{2M^2}{\pi}\int_\delta^T\Big|\int_{|x|\ge\frac{R}{2}}(2\nu s)^{-\frac{8}{3}}|x_i|^{\frac{4}{3}}\exp\Big(-\frac{|x|^2}{3\nu t}\Big)\,dx\Big|^{\frac{3}{4}}ds\\&+\frac{2M}{\pi}\sup_{s\in[0,T]}\Big(\int_{|y|\ge\frac{R}{2}}|\omega^\epsilon(s,y)|^{\frac{4}{3}}\,dy\Big)^{\frac{3}{4}}\cdot\int_0^T\Big|\int_{\mathbb{R}^2}(2\nu s)^{\frac{8}{3}}|x_i|^{\frac{4}{3}}\exp\Big(-\frac{|x|^2}{3\nu t}\Big)\,dx\Big|^{\frac{3}{4}}ds.\end{aligned}\tag{2.13}$$

Summing up (2.7)–(2.13), we find

$$\lim_{R\to\infty}\sup_{\substack{\epsilon>0\\t\in[0,T]}}\Big(\int_{|x|\ge R}|\omega^\epsilon(t,x)|^{\frac{4}{3}}\,dx\Big)^{\frac{3}{4}}\le C_\nu\delta^{\frac{1}{4}}+C_\nu T^{\frac{1}{4}}\lim_{R\to\infty}\sup_{\substack{\epsilon>0\\t\in[0,T]}}\Big(\int_{|y|\ge\frac{R}{2}}|\omega^\epsilon(s,y)|^{\frac{4}{3}}\,dy\Big)^{\frac{3}{4}}.$$

Thus, when we take $T_1=(2C_\nu)^{-\frac{1}{4}}$ and let δ tend to 0_+, we find

$$\lim_{R\to\infty}\sup_{\substack{\epsilon>0\\t\in[0,T]}}\Big(\int_0^{T_1}\int_{|y|\ge R}|\omega^\epsilon(s,y)|^{\frac{4}{3}}\,ds\,dy\Big)^{\frac{3}{4}}=0.\tag{2.14}$$

By the induction method, we conclude the proof of Lemma 2.3.

Lemma 2.4. *For any $T>0$, $\{\omega^{1,\epsilon}(t,x)\}$ is a compact subset of $C([0,T],L^{\frac{4}{3}}(\mathbb{R}^2))$.*

Proof. First, by Lemma 2.3,

$$\lim_{R\to\infty}\sup_{\substack{\epsilon>0\\t\in[0,T]}}\Big(\int_{|x|\ge R}|\omega^\epsilon(s,y)|^{\frac{4}{3}}\,dy\Big)^{\frac{3}{4}}=0.$$

We find by Cauchy-Schwartz inequality,

$$\lim_{R\to\infty}\sup_{\substack{\epsilon>0\\t\in[0,T]}}\int_{|x|\ge R}|V_i^\epsilon(t.\cdot)\omega^\epsilon(t,\cdot)|\,dx\le M\lim_{R\to\infty}\sup_{\epsilon>0\in[0,T]}\Big(\int_{|x|\ge R}|\omega^\epsilon(s,y)|^{\frac{4}{3}}\,dy\Big)^{\frac{3}{4}}=0.$$

Moreover, following (2.2) and (2.3), we see that $V^\epsilon(t,x)\cdot\omega^\epsilon(t,x)$ is uniformly bounded in $L^\infty(\mathbb{R}^+,\ L^1(\mathbb{R}^2))$. Hence,there exists some sequence $\{\epsilon_j\}$ such that ϵ_j goes to 0 as j tends to ∞,and $\{(V_i^{\epsilon_j}\cdot\omega^{\epsilon_j})(t,x)\}$ converges tightly in $[0,T]\times\mathbb{R}^2$ to some Randon measure $\mu_i(t,x), i=1,2$. Thus,when we construct function $\phi_\delta(\tau)\in C_0^\infty(\mathbb{R})$ such that $\phi_\delta(\tau)=0$ for $\tau\le\delta$, $\phi_\delta(\tau)=1$ for $\tau\ge2\delta$, $0\le\phi_\delta(\tau)\le1$, and set

$$g_\delta^\epsilon=\frac{1}{\pi}\sum_{i=1}^2\int_0^t\int_{\mathbb{R}^2}\phi_\delta(t-s)(2\nu(t-s))^{-2}(x_i-y_i)\exp\Big(-\frac{|x-y|^2}{4\nu(t-s)}\Big)(V_i^\epsilon\omega^\epsilon)(s,y)\,ds\,dy,$$

we have

$$g_\delta^{\epsilon_j}\to\frac{1}{\pi}\sum_{i=1}^2\int_0^t\int_{\mathbb{R}^2}\phi_\delta(t-s)(2\nu(t-s))^{-2}(x_i-y_i)\exp\Big(-\frac{|x-y|^2}{4\nu(t-s)}\Big)\,d\mu_i(s,y)\quad\text{for every } (t,x)\in[0,T]\times\mathbb{R}^2\quad\text{as } j\to\infty.\tag{2.15}$$

Moreover,

$$\sup_{\substack{\epsilon>0 \\ t\in[0,T]}} \|(g_\delta^\epsilon - g^\epsilon)(t,\cdot)\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \to 0 \quad \text{as } \delta \to 0_+. \tag{2.16}$$

In fact, by Minkowski inequality,

$$\begin{aligned}&\Big(\int_{\mathbb{R}^2} |g_\delta^\epsilon - g^\epsilon|^{\frac{4}{3}}(t,x)\,dx\Big)^{\frac{3}{4}} \\ &\le \frac{1}{\pi}\sum_{i=1}^{2} \sup_{s\in[0,T]} \int_{\mathbb{R}^2} |V_i^\epsilon \omega^\epsilon)(s,y)|\,dy \sup_{y\in\mathbb{R}^2} \int_0^t \int_{\mathbb{R}^2} (1-\phi_\delta(t-s))^{\frac{4}{3}} (2\nu(t-s))^{-\frac{8}{3}} \\ &\quad \cdot |x_i - y_i|^{\frac{4}{3}} \cdot \exp\Big(-\frac{|x-y|^2}{3\nu(t-s)}\Big)\,dx|^{\frac{3}{4}}\,ds \le \frac{2}{\pi} M^2 2\delta^{\frac{1}{4}} \cdot C_\nu.\end{aligned}$$

Thus, (2.16) holds.

On the other hand, by the definition of $g_\delta^\epsilon(t,x)$ and Lemma 2.3, we find that for any $\theta > 0$ there exists some positive number R such that

$$\sup_{t\in[0,T]} \int_{|x|\ge R} |g_\delta^\epsilon(t,x)|^{\frac{4}{3}}\,dx \le \frac{\theta}{8}, \tag{2.17}$$

while by (2.15), Egorov Theorem and Lemma 2.1, for θ taken as above, there exist some $\eta > 0$, a positive integer N and some measurable subset B of $\{x||x| \le R\}$, such that $\operatorname{meas} B \le \eta$ and

$$\sup_{t\in[0,T]} \int_B |g_\delta^{\epsilon_j}(t,x)|^{\frac{4}{3}}\,dx \le \frac{\theta}{8} \quad \text{for } j \ge N, \tag{2.18}$$

and $g_\delta^{\epsilon_j}(t,x)$ converges uniformly in $\{x||x| \le R\} \setminus B$. And then

$$\sup_{t\in[0,T]} |g_\delta^{\epsilon_j}(t,x) - g_\delta^{\epsilon_k}(t,x)|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \le \theta \text{ for } \ j,k > N. \tag{2.19}$$

Thus, for every $t \in [0,T]$, $g_\delta^{\epsilon_j}(t,x)$is a compact subset of $L^{\frac{4}{3}}(\mathbb{R}^2)$.

On the other hand, by applying Minkowski inequality and a similar calculation as the proof of Lemma 2.3, we find

$$\|g^\epsilon(t_1,\cdot) - g^\epsilon(t_2,\cdot)\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \le |t_1^{\frac{1}{4}} - t_2^{\frac{1}{4}}|. \tag{2.20}$$

Hence by Ascoli-Arzela Theorem, (2.15) holds strongly in $C([0,T], L^{\frac{4}{3}}\mathbb{R}^2)$. Combining this and (2.16) and using triangle inquality, we conclude the proof of Lemma 2.4.

In view of, Lemma 2.4, by taking $T = n$, (2.2) and a diagnoal process,there exists some $\omega(t,x) \in C(\mathbb{R}^+, L^{\frac{4}{3}}(\mathbb{R}^2))$, and some subsequence of $\{\omega^\epsilon(t,x)\}$ (without arousing ambiguity, we still denote it by $\{\omega^\epsilon(t,x)\}$), such that $\{\omega^\epsilon(t,x)\}$ tends to $\{\omega(t,x)\}$ in $C_{\text{loc}}(\mathbb{R}^+, L^{\frac{4}{3}}(\mathbb{R}^2))$. Thus, there exists some velocity field $V(t,x)$ in $\mathbb{R}^2$ corresponding to the vorticity $\omega(t,x)$. Moreover, owing to

$$\begin{aligned}|V^\epsilon(t,x) - V(t,x)| &= \Big|\int_{\mathbb{R}^2} K(x-y)(\omega^\epsilon(t,y) - \omega(t,y))\,dy\Big| \\ &\le \int_{\mathbb{R}^2} \frac{1}{|x-y|} |\omega^\epsilon(t,y) - \omega(t,y)|\,dy,\end{aligned}$$

and Riesz potential theory (Chapter V of [11]), $V^\epsilon(t,x)$ tends to $V(t,x)$ in $C_{\text{loc}}(\mathbb{R}^+, L^4)$, and $V(t,x) \in C(\mathbb{R}^+, L^4(\mathbb{R}^2))$. Rewriting $V(t,x) = \nabla^\perp \triangle^{-1}\omega(t,x)$, $\partial_i V(t,x) = \partial_i \nabla^\perp \triangle^{-1}\omega(t,x)$, and noting that $\frac{\xi_i\xi_j}{|\xi|^2}$ is a multiplier which is homogeneous of degree zero and infinitely differentiable on the unit sphere (see p.75 of [11] for details), we conclude that $\partial_i V(t,x) \in C(\mathbb{R}^+, L^{\frac{4}{3}}(\mathbb{R}^2))$, and then $V(t,x) \in C(\mathbb{R}^+, W^{1,\frac{4}{3}}_{loc}(\mathbb{R}^2))$.On the other hand, for any $T>0$, by Cauchy Schwartz inequality,

$$\int_0^T \int_{\mathbb{R}^2} |V^\epsilon(t,x)\omega^\epsilon(t,x) - V(t,x)\omega(t,x)|\, dt\, dx \to 0, \text{as } \epsilon \to 0. \tag{2.21}$$

Thus

$$\lim_{\epsilon\to 0} \iint \phi(t,x) V_i^\epsilon(t,x)\omega^\epsilon(t,x)\, dt\, dx = \iint \phi(t,x) V_i^\epsilon(t,x)\omega^\epsilon(t,x)\, dt\, dx,$$

for every $\phi(t,x)$ as the correspondence in (1.3).

It implies that $(V(t,x), \omega(t,x))$ is a weak solution of (1.2) in the sense of (1.3).

Thus, in order to finish the proof of the existence part of Theorem 1.1, we only need to prove that

$$V(t,x) \in C^1(\mathbb{R}^+, W^{-1,\frac{4}{3}}_{\text{loc}}(\mathbb{R}^2)), \text{and } \ \omega(t,x) \in C^1(\mathbb{R}^+, W^{-2,\frac{4}{3}}_{\text{loc}}(\mathbb{R}^2)). \tag{2.22}$$

By the result of [1], the first equation of (1.1) can be rewritten as

$$\partial_t V = -\nabla(V \otimes V) + \nu \triangle V + \nabla \sum_{i=1}^2 \triangle^{-1}(\partial_i \partial_j (V_i V_j)). \tag{2.23}$$

Since $V_i(t,x), V_j(t,x) \in C(\mathbb{R}^+, L^4(\mathbb{R}^2))$, $(V_iV_j)(t,x) \in C(\mathbb{R}^+, L^2(\mathbb{R}^2))$, by a similar intepretation as above (2.21), we find

$$-\nabla(V \otimes V) + \nabla \sum_{i=1}^2 \triangle^{-1}(\partial_i \partial_j (V_i V_j)) \in C(\mathbb{R}^+, W^{-1,2}).$$

Obviously, $\triangle V \in C(\mathbb{R}^+, W^{-1,\frac{4}{3}}_{\text{loc}})$, and $W^{-1,2} \hookrightarrow W^{-1,\frac{4}{3}}_{\text{loc}}$, so (2.22) is proved when we note that $\omega(t,x)$ is a combination of the first order derivatives of $V(t,x)$.

§3. The Uniqueness of Weak Solutions

Let $\{V^j(t,x), \omega^j(t,x)\}_{j=1,2}$ be two weak solutions of (1.2) with the same initial data, and $V^j(t,x) \in C(\mathbb{R}^+,\ L^4)$, $\omega^j(t,x) \in C(\mathbb{R}^+, L^{\frac{4}{3}})$, then, if we set $W^\epsilon(t,x) = j_\epsilon(\cdot) * (\omega^1(t,\cdot) - \omega^2(t,\cdot))$, $W^\epsilon(t,x)$ must satisfy

$$\begin{cases} \partial_t W^\epsilon(t,x) - \nu \triangle W^\epsilon(t,x) = \displaystyle\sum_{i=1}^2 j_\epsilon * ((V_i^1 \partial_i \omega^1)(t,\cdot) - (V_i^2 \partial_i \omega^2)(t,\cdot)), \\ W^\epsilon(t,x)|_{t=0} = 0. \end{cases}$$

Then, by a similar reason as that in the proof of (2.4)

$$\begin{aligned} W^\epsilon(t,x) = &\frac{1}{\pi} \sum_{i=1}^2 \int_0^t \int_{\mathbb{R}^2} (2(t-s)\nu)^{-2}(x_i - y_i) \exp\Big(-\frac{|x-y|^2}{4\nu(t-s)}\Big) j_\epsilon(\cdot) \\ &* ((V_i^1 - V_i^2)\omega^1(s,\cdot) + V_i^2(\omega^1 - \omega^2)(s,\cdot))\, ds\, dy \\ = &W_1^\epsilon(t,x) + W_2^\epsilon(t,x). \end{aligned} \tag{3.1}$$

And

$$\begin{aligned}\|W_1^\epsilon(t,\cdot)\|_{L^{\frac{4}{3}}} &\le \frac{1}{\pi}\sum_{i=1}^{2}\sup_{s\in[0,T]}\int_{\mathbb{R}^2}|j_\epsilon(\cdot)*((V_i^1-V_i^2)\omega_1)(s,y)| \\ &\quad\cdot\int_0^T\Big(\int_{\mathbb{R}^2}1_{(t>s)}(2(t-s)\nu)^{-\frac{8}{3}}(x_i-y_i)^{\frac{4}{3}}\exp\Big(-\frac{|x-y|^2}{3\nu(t-s)}\Big)\,dx\Big)^{\frac{3}{4}}\,dt\,dy \\ &\le C_\nu T^{\frac{1}{4}}\sup_{s\in[0,T]}\|\omega^1(s,\cdot)-\omega^2(s,\cdot)\|_{L^{\frac{4}{3}}}\cdot\|\omega^1(s,\cdot)\|_{L^\infty(\mathbb{R}^+,L^{\frac{4}{3}})},\end{aligned}\tag{3.2}$$

where $1_{t>s}$ is the character function of the set $\{(t,s,x,y)|t>s\}$. Similarly

$$\|W_2^\epsilon(t,\cdot)\|_{L^{\frac{4}{3}}}\le C_\nu T^{\frac{1}{4}}\|V^2(t,x)\|_{L^\infty(\mathbb{R}^+,L^4)}\cdot\sup_{s\in[0,T]}\|\omega^1(s,\cdot)-\omega^2(s,\cdot)\|_{L^{\frac{4}{3}}}.\tag{3.3}$$

Thus, by (2.23),(3.1) and (3.2)

$$\begin{aligned}\|W^\epsilon(t,x)\|_{L^\infty([0,T],L^{\frac{4}{3}})}&\le C_\nu T^{\frac{1}{4}}(\|\omega^1(s,\cdot)\|_{L^\infty(\mathbb{R}^+,L^{\frac{4}{3}})}+\|V^2(t,x)\|_{L^\infty(\mathbb{R}^+,L^4)}) \\ &\quad\cdot\sup_{s\in[0,T]}\|(\omega^1-\omega^2)(t,x)\|_{L^{\frac{4}{3}}}.\end{aligned}\tag{3.4}$$

Hence, when we take $T_1=\frac{1}{(4(\|\omega^1(t,\cdot)\|_{L^\infty(\mathbb{R}^+,L^{\frac{4}{3}})}+\|V_2(t,x)\|_{L^\infty(\mathbb{R}^+,L^4)}))^4C_\nu^4}$, and let ϵ go to 0, we immediately have $\sup_{t\in[0,T_1]}\|(\omega^2-\omega^2)(t,x)\|_{L^{\frac{4}{3}}(\mathbb{R}^2))}=0$. Then, again by the induction method, we can prove the uniqueness assertion of Theorem 1.1.

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