Chin. Ann. of Math.
1 (1) 1980

ON THE CHARACTERISTIC FUNCTION OF SEMI-HYPONORMAL OPERATOR

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1. In the previous paper[3], we considered the spectral properties of the class of quasi-hyponormal operators and in particular semi-hyponormal operators. In this paper, we shall investigate the characteristic function of these operators, using the same notations as in the previous paper.

We shall give the relation between the unitary equivalence of the characteristic functions and the unitary equivalence of two semi-hyponormal operators or quasi-hyponormal operators and consider the Riemann-Hilbert's problem related to the characteristic function and the determinant of the value of characteristic function.

2. Let $A=U|A|_r$ be a quasi-hyponormal operator, U be a unitary operator and $B=|A|_+-|A|_-$ be the polar difference operator of A. We consider the principal domain $\mathscr{B}(A)=\overline{\mathscr{R}(B)}$ of A. The operator-valued function

$$W(\lambda)=W(\lambda;A)=I-B^{\frac{1}{2}}(\lambda I-A_-)^{-1}UB^{\frac{1}{2}},\quad \lambda\in\rho(A_-) \tag{1}$$

is called the characteristic function of A. It is obvious that

$$W(\lambda)B^{\frac{1}{2}}=B^{\frac{1}{2}}(\lambda I-A_-)^{-1}(\lambda I-A_+),\quad B^{\frac{1}{2}}W(\lambda)=(\lambda I-A_+)(\lambda I-A_-)^{-1}B^{\frac{1}{2}}. \tag{2}$$

We can easily prove that $W(\lambda)^{-1}$ exists and

$$W(\lambda)^{-1}=I+B^{\frac{1}{2}}(\lambda I-A_+)^{-1}B^{\frac{1}{2}} \tag{3}$$

for $\lambda\overline{\in}\sigma(A_+)\cup\sigma(A_-)$. The operator A is called simple, if there does not exist a non-trivial subspace which contains $\mathscr{B}(A)$ and is invariant with respect to $|A|_+$, $|A|_-$, U and U^{-1}. Let φ be a scale function.

Theorem 1. *Let $A=U|A|_r$ and $A'=U'|A'|_r$ be the φ-quasi-hyponormal operators and the operators U and U' be unitary operators.*

(i) *If there is an unitary operator T from H onto H' such that*

$$|A'|_\pm=T|A|_\pm T^{-1},\quad U'=TUT^{-1}, \tag{4}$$

then there is a unitary operator S from the principle domain $\mathscr{B}(A)$ onto $\mathscr{B}(A')$ such that

$$W(\lambda;A')=SW(\lambda;A)S^{-1}\quad\text{for}\quad \lambda\overline{\in}\sigma(A_-)=\sigma(A'_-). \tag{5}$$

(ii) *Conversely, if A and A' are simple and there is a unitary operator S from $\mathscr{B}(A)$ onto $\mathscr{B}(A')$ such that (5) holds, then there is a unitary operator T from H onto H' such that (4) holds.*

Manuscript received Oct. 17, 1979.

Proof The first part of this theorem can be easily proved by taking $S=T|_{\mathscr{B}(A)}$ and we shall omit the details.

We have to prove (ii). The subspace $\mathscr{B}(A)$ reduces the operator U, since the operator U and the operator B are commutative. From (5), it is easy to prove that

$$S(U|_{\mathscr{B}(A)})S^{-1}=U'|_{\mathscr{B}(A')} \tag{6}$$

We can also prove that

$$S(B^{\frac{1}{2}}(\lambda_1 I-A_+)^{-1}(\mu_1 I-A_-)^{-1}\cdots(\lambda_n I-A_+)^{-1}(\mu_n I-A_-)^{-1}UB^{\frac{1}{2}})S^{-1}$$
$$=B'^{\frac{1}{2}}(\lambda_1 I-A'_+)^{-1}(\mu_1 I-A'_-)^{-1}\cdots(\lambda_n I-A'_+)^{-1}(\mu_n I-A'_-)^{-1}U'B'^{\frac{1}{2}}, \tag{7}$$

for $\lambda_1, \cdots, \lambda_n\in\rho(A_+)$, $\mu_1, \cdots, \mu_n\in\rho(A_-)$. In fact, we have

$$B^{\frac{1}{2}}(\lambda I-A_+)^{-1}(\mu\cdot I-A_-)^{-1}UB^{\frac{1}{2}}=(W(\lambda)^{-1}W(\mu)-I)(\mu-\lambda)^{-1} \tag{8}$$

for λ, $\mu\in\rho(A_-)$ and $\lambda\neq\mu$. From (6), we obtain (7) for $n=1$. By the similar calculation, we can prove the formula (7) for all n.

Let $\mathscr{M}(A)$ be the linear manifold spanned by all the vectors

$$|A|_+^{l_1}|A|_-^{m_1}\cdots|A|_+^{l_n}|A|_-^{m_n}U^kB^{\frac{1}{2}}y, \quad y\in H$$

for non-negative integers l_1, m_1, $\cdots$, l_n, m_n and integer k. We construct the operator T from $\mathscr{M}(A)$ to $\mathscr{M}(A')$ by the following formula

$$T(|A|_+^{l_1}|A|_-^{m_1}\cdots|A|_+^{l_n}|A|_-^{m_n}U^kB^{\frac{1}{2}}y)=|A'|_+^{l_1}|A'|_-^{m_1}\cdots|A'|_+^{l_n}|A'|_-^{m_n}U'^kSB^{\frac{1}{2}}y.$$

From (6—7), we know that the operator T is isometric. But the manifolds $\mathscr{M}(A)$ and $\mathscr{M}(A')$ are dense in the spaces H and H' respectively. Thus the operator T can be uniquely extended to a unitary operator from H onto H' and then (2.4) holds.

In the following, we shall only consider the case when the operator A in the Hilbert space H is semi-hyponormal. Let $\mathscr{D}$ be an auxiliary separable Hilbert space, K be a linear operator from $\mathscr{D}$ to H such that the polar difference operator $Q=|A|_r-|A|_l$ satisfies $Q=KK^*$. If A is the singular integral operator in [3], then we take

$$K: a\mapsto\alpha(\cdot)a, \quad a\in\mathscr{D}.$$

In this case, $K^*f=P_0(\alpha f)$. We define an analytic function① $Y(z, \lambda)$ of Complex variables $\lambda\in\rho(A)$ and z for $|z|\neq1$

$$Y(z, \lambda)=I-zK^*(I-zU^*)^{-1}(A-\lambda I)^{-1}K. \tag{8}$$

From $[A-\lambda I, I-zU^*]=zQ$, we obtain

$$Y(z, \lambda)^{-1}=I+zK^*(A-\lambda I)^{-1}(I-zU^*)^{-1}K, \tag{9}$$

$$KY(z, \lambda)=(I-zU^*)(A-\lambda I)(I-zU^*)^{-1}(A-\lambda I)^{-1}K, \tag{10}$$

and

$$Y(z, \lambda)K^*=K^*(I-zU^*)^{-1}(A-\lambda I)^{-1}(I-zU^*)(A-\lambda I). \tag{11}$$

Theorem 2. *Let $A=U|A|_r$ and $A'=U'|A'|_r$ be the semi-hyponormal operators of Hilbert spaces H and H' respectively, U and U' be unitary operators.* (i) *If there is a*

① Here the domain of definition is not connected.

unitary operator $T: H \to H'$ *such that*

$$A' = TAT^{-1}, \quad U' = TUT^{-1}, \tag{12}$$

then there is a unitary operator S *from* $\mathscr{D}$ *onto* $\mathscr{D}'$ *such that*

$$Y'(z, \lambda)^{①} = SY(z, \lambda)S^{-1}, \tag{13}$$

for $\lambda \in \rho(A)$ *and* $|z| \neq 1$. (ii) *Conversely, if the operators* A *and* A' *are simple and there is a unitary operator* S *from* $\mathscr{D}$ *onto* $\mathscr{D}'$ *such that* (13) *holds for* λ *in a neighborhood of* ∞ *and* z *in the neighborhood of* $z=0$ *or* $z=\infty$, *then there is a unitary operator* T *from* H *onto* H' *such that* (12) *holds.*

Proof We only have to prove (ii) From (8—11), we obtain

$$K^*(I-zU^*)^{-1}(A-\lambda I)^{-1}(I-z'U^*)^{-1}K = (I-Y(z, \lambda)Y(z', \lambda)^{-1})/(z-z')$$

By (13), we can prove that

$$K'^*U'^{*m}A'^lU'^nK' = SK^*U^{*m}A^lU^nKS^{-1}$$

for natural number l and integers m and n. Now we construct the operator T:

$$U^nKa \to U'^nK'S_a, \qquad a \in \mathscr{D}$$

and then extend its domain of definition so that T becomes a unitary operator from H onto H' and satisfies (12).

This theorem is similar to some results in [2].

Theorem 3. *If the operator* K *is a Hilbert-Schmidt operator then the determinant of* $Y(z, \lambda)$ *exists for* $\lambda \in \rho(A)$ *and*

$$|z| \neq 1 \text{ and } \det(Y(z, \lambda)) = \det((I-zU^*)(A-\lambda I)(I-zU^*)^{-1}(A-\lambda I)^{-1}). \tag{14}$$

Proof We denote $X = (I-zU^*)(A-\lambda I)(I-zU^*)^{-1}(A-\lambda I)^{-1}$. Then

$$X = I + KK^*C \quad \text{and} \quad Y(z, \lambda) = I + K^*C'K,$$

where C and C' are bounded linear operators. We take projections P_n and Q_n with rank n such that $KP_n = Q_nK = Q_nKP_n$ and that $\|KP_n - K\|_2 \to 0$ as $n \to \infty$. Let

$$X_n = I + Q_nKK^*CQ_n \quad \text{and} \quad Y_n = I + P_nK^*C'KP_n.$$

From (10), it is obvious that

$$Q_nKP_nY_n = X_nQ_nKP_n.$$

Thus $\det(Y_n) = \det(X_n)$ for sufficiently large n. Then (14) holds, since

$$\lim_{n\to\infty} \|X_n - X\|_1 = \lim_{n\to\infty} \|Y_n - Y(z, \lambda)\|_1 = 0.$$

3. Now we shall consider the singular integral model of the semihyponormal operator A. We suppose that A is completely nonnormal. In this case, from Corollary 5 in [3], we may suppose that the measure μ in the function model is the Lebesgue measure m. In this case, $W(\lambda, A)$ is a multiplicative operator in $L^2(\mathscr{D})$ and

$$(W(\lambda, A)f)(e^{i\theta}) = W(e^{i\theta}, \lambda)f(e^{i\theta}), \quad f \in L^2(\mathscr{D}),$$

where $W(e^{i\theta}, \lambda) = I - \alpha(e^{i\theta})(\lambda e^{-i\theta}I - \beta(e^{i\theta}))^{-1}\alpha(e^{i\theta})$. The function $W(e^{i\theta}, \lambda)$ is also called the characteristic function of A. Let $P_- = I - P_+$.

① Here $\mathscr{D}'$ and $Y'(z,\lambda)$ are respectively the auxiliary domain and the function corresponding to the operator A'.

Lemma 1. *Let $f, g \in L^2(\mathscr{D})$ and $\lambda \in \rho(A_-)$. If*

$$(A-\lambda I)f = B^{\frac{1}{2}}g. \tag{15}$$

then

$$W(e^{i\theta}, \lambda)(g - e^{i\theta}P_+(\alpha f)) = g + e^{i\theta}P_-(\alpha f), \tag{16}$$

Conversely, if the null space of the operator B is $\{0\}$ then (16) implies (15).

Proof By calculation, we find that (15) is equivalent to

$$B^{\frac{1}{2}}(A_- - \lambda I)^{-1}[(A-\lambda I)f - B^{\frac{1}{2}}g] = 0.$$

So we have Lemma 1.

Let $H^p_\pm(\mathscr{D})$, $1 \leqslant p < \infty$ be the Banach space of all analytic functions of the form

$$f(z) = \sum_{n=0}^{\infty} f_n \frac{1}{z^{\pm n}} \quad \text{for} \quad \pm|z| > \pm 1,$$

$\{f_n\} \subset \mathscr{D}$ with

$$\|f\|_p = \sup_{0 \leqslant r < 1} \left(\int_0^{2\pi} \|f(re^{i\theta})\|^p_{\mathscr{D}} d\theta \right)^{\frac{1}{p}}.$$

It is obvious that the space $H_0(\mathscr{D}) = H^p_+(\mathscr{D}) \cap H^p_-(\mathscr{D})$ is the space of all constant functions with values in $\mathscr{D}$. The $H^p_\pm(\mathscr{D})$ is simply denoted by $H^p_\pm$, when $\mathscr{D}$ is the field of complex numbers. The boundary value of the function $f \in H^p_\pm(\mathscr{D})$ is denoted by $f(e^{i\theta}) = \lim\limits_{r \to 1 \pm 0} f(re^{i\theta})$. If $f \in H^p_\pm(\mathscr{D})$, then $\lim\limits_{r \to 1 \pm 0} \|f(r(\cdot)) - f(\cdot)\|_p = 0$.

For fixed $\lambda \in \rho(A)$ and $a \in \mathscr{D}$, we have $B^{\frac{1}{2}}(A-\lambda I)^{-1}B^{\frac{1}{2}}a \in L^2(\mathscr{D})$. Hence the function $z \mapsto Y(z, \lambda)$ as a function in $|z| < 1$ ($|z| > 1$) is a function in $H^2_-(\mathscr{D})$ ($H^2_+(\mathscr{D})$ respectively). It has boundary value function

$$Y_\pm(e^{i\theta}, \lambda)a = \lim_{r \to 1 \pm 0} Y(re^{i\theta}, \lambda)a.$$

We can prove that

$$Y_\pm(e^{i\theta}, \lambda)a = a \mp e^{i\theta}P_\pm(B^{\frac{1}{2}}(A-\lambda I)^{-1}B^{\frac{1}{2}}a).$$

We take $g = a$ and $f = (A-\lambda I)^{-1}B^{\frac{1}{2}}a$ in (3.1). By Lemma 1, we have immediately the following theorem.

Theorem 4. *For fixed $\lambda \in \rho(A)$ and $a \in \mathscr{D}$, the boundary value function $Y_\pm(e^{i\theta}, \lambda)a$ is the solution of the Riemann-Hilbert's problem*

$$Y_-(e^{i\theta}, \lambda)a = W(e^{i\theta}, \lambda)Y_+(e^{i\theta}, \lambda)a. \tag{17}$$

Let $\mathscr{L}(\mathscr{D})$ be the Banach space of all bounded linear operators in $\mathscr{D}$. Let $f(e^{i\theta})$ be an operator-valued function. If there are functions $u_\pm(\cdot) \in H^q_\pm(\mathscr{L}(\mathscr{D}))$ with $q > 2$ such that $u_-(\cdot)$ is invertible and $u_-(e^{i\theta})^{-1}u_+(e^{i\theta}) = f(e^{i\theta})$ then the function f is called factorisable. We notice that if $\mathscr{D}$ is n-dimensional and the functions α and β are in a R-ring[3], then $W(e^{i\theta}, \lambda)$ is fatorisable for $\lambda \in \rho(A)$ (cf. [3]).

Theorem 5. *If the characteristic function $W(e^{i\theta}, \lambda_0)$ of A is factorisable for $\lambda_0 \in \rho(A)$, then $Y(z, \lambda_0)$ is uniquely determined by $W(e^{i\theta}, \lambda_0)$. If there is a neighborhood $\mathscr{N}_\infty$ of infinity such that $W(e^{i\theta}, \lambda)$ is factorisable for every $\lambda \in \mathscr{N}_\infty \cap \rho(A)$, then operator A is uniquely determined in the sense of unitary equivalence by the characteristic* *function of* A.

Proof There are functions $u_\pm \in H^q_\pm(\mathscr{L}(\mathscr{D}))$ such that $W(e^{i\theta}, \lambda_0) = u_-(e^{i\theta})^{-1} u_+(e^{i\theta})$, since $W(\cdot, \lambda_0)$ is factorisable. From (17), we have

$$u_-(e^{i\theta})Y_-(e^{i\theta}, \lambda_0)a = u_+(e^{i\theta})Y_+(e^{i\theta}, \lambda_0)a. \tag{18}$$

However, $(u_\pm(e^{i\theta})Y_\pm(e^{i\theta})a, b)$ is the boundary value of a function in $H^k_\pm$ for every $b \in \mathscr{D}$ and $k = 2q/(2+q) > 1$. From (18), we have $u_\pm(\cdot)Y_\pm(\cdot)a \in H_0(\mathscr{D})$. There is an operator $L \in \mathscr{L}(\mathscr{D})$ such that $u_\pm(z)Y(z, \lambda) = L$. Hence $L = u_-(0)$, since $Y(0, \lambda) = I$. We notice that $Y(z, \lambda)^{-1}$ exists for $|z| \neq 1$, so $u_\pm(z)^{-1}$ also exists and

$$Y(z, \lambda) = \begin{cases} u_+(z)^{-1}u_-(0), & \text{for } |z| > 1, \\ u_-(z)^{-1}u_-(0), & \text{for } |z| < 1. \end{cases}$$

Thus the function $Y(z, \lambda)$ is determined by $W(e^{i\theta}, \lambda)$.

The second part of this theorem is obtained by the first part of this theorem and Theorem 2.

4. In the following, we suppose further that in the function model of the semi-hyponormal operator A, the space $\mathscr{D}$ is n-dimensional and the matrix-functions α and β are continuous. In this case, A is called an operator of C^n-type. We write

$$A_k = kA_+ + (1-k)A_-.$$

Theorem 6. *Let A be a semi-hyponormal operator of C^n-type. If $0 \leqslant k \leqslant 1$, then*

$$\sigma(kA_+ + (1-k)A_-) \subset \sigma(A). \tag{19}$$

Proof For any $n \times n$ Hermite matrix R, let $\lambda_1(R) \leqslant \cdots \leqslant \lambda_n(R)$ be the eigen-values of the matrix R. Let

$$|A|_k(e^{i\theta}) = k|A|_+(e^{i\theta}) + (1-k)|A|_-(e^{i\theta}).$$

Since $|A|_-(e^{i\theta}) \leqslant |A|_k(e^{i\theta}) \leqslant |A|_+(e^{i\theta})$, we have

$$\lambda_j(|A|_-(e^{i\theta})) \leqslant \lambda_j(|A|_k(e^{i\theta})) \leqslant \lambda_j(|A|_+(e^{i\theta})), \quad (j = 1, 2, \cdots, n),$$

From Corollary 7 of [3], we have

$$\{\rho \mid \rho e^{i\theta} \in \sigma(A), \rho \geqslant 0\} \supset \bigcup_{j=1}^{n} [\lambda_j(|A|_-(e^{i\theta})), \lambda_j(|A|_+(e^{i\theta}))].$$

Hence $\lambda_j(|A|_k(e^{i\theta}))e^{i\theta} \in \sigma(A)$. However, we have

$$\sigma(A_k) = \{\lambda_j(|A|_k(e^{i\theta}))e^{i\theta} \mid 0 \leqslant \theta \leqslant 2\pi, j = 1, \cdots, n\}.$$

Thus (19) holds.

It is conjectured that (19) holds for any quasi-hyponormal operator.

Theorem 7. *If the semi-hyponormal operator A is of C^n-type and*

$$R(\lambda) = \text{ind}\,(\det W(\cdot, \lambda)),$$

then

$$\det(Y(z, \lambda)) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty \frac{R(\rho e^{i\theta})e^{i\theta}z\,d\rho\,d\theta}{(\rho e^{i\theta} - \lambda)(e^{i\theta} - z)}\right\} \tag{20}$$

and

$$\det(W(e^{i\theta}, \lambda)) = \exp\int_0^\infty \frac{R(\rho e^{i\theta})\,d\rho}{\rho - \lambda e^{-i\theta}}, \tag{21}$$

for $\lambda \in \rho(A)$ and $|z| \neq 1$.

Proof Let $A_k(e^{i\theta})=kA_+(e^{i\theta})+(1-k)A_-(e^{i\theta})$, $\mu=\lambda e^{-i\theta}$, $\alpha=\alpha(e^{i\theta})$ and $\beta=\beta(e^{i\theta})$. By theorem 5, the operator $(\lambda I-A_k(e^{i\theta}))^{-1}$ exists for $\lambda\in\rho(A)$. On the otherhand, we have

$$(I-k\alpha(\mu-\beta)^{-1}\alpha)^{-1}=I+k\alpha(\mu-\beta-k\alpha^2)^{-1}\alpha \tag{22}$$

From the formula (1.14) of ch. IV in [4] and (22), we have

$$\ln\det W(e^{i\theta},\ \lambda)=\operatorname{tr}\int_0^1 \alpha(\beta+k\alpha^2-\mu)^{-1}\alpha dk. \tag{23}$$

Let the function of μ on the right side of the equality (23) be $f(\mu)$. We can prove that (i) this $f(\mu)$ is analytic in the whole plane except a finite interval in the real axis, (ii) $\vartheta\mu\vartheta f(\mu)\geqslant 0$, and (iii) there is a piece-wise continuous function $R(\rho e^{i\theta})$ which vanishes beyond a bounded subset of the complex plane and such that

$$f(\mu)=\int_0^\infty \frac{R(\rho e^{i\theta})d\mu}{\rho-\mu}.$$

This implies (21). From (21),we can easily prove that

$$R(\lambda)=\operatorname{ind}(\det W(\cdot,\ \lambda)).$$

Let $\varphi(z,\ \lambda)$ be the function on the right side of the equality (20). By calculation, we obtain

$$\varphi_-(e^{i\theta},\ \lambda)=\det(W(e^{i\theta},\ \lambda))\varphi_+(e^{i\theta},\ \lambda).$$

But the formula (17) implies

$$\det(Y_-(e^{i\theta},\ \lambda))=\det(W(e^{i\theta},\ \lambda))\det(Y_+(e^{i\theta},\ \lambda)).$$

Hence the functions $\varphi(z,\ \lambda)$ and $Y(z,\ \lambda)$ are the same solution of the Riemann-Hilbert problem. Thus $\varphi(z,\ \lambda)=\det(Y(z,\ \lambda))$.

Further, we can prove that $R(\lambda)$ is the principal function $G(\lambda)$ in the sense of [1].

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关于半亚正常算子的特征函数

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摘要

本文继[3]之后,研究拟亚正常算子和半亚正常算子的特征函数. 设 $A=U|A|_r$ 是 H 上拟亚正常算子, U 是酉算子, $B=|A|_+-|A|_-$. 作算子 A 的特征函数

$$W(\lambda, A)=I-B^{\frac{1}{2}}(\lambda I-A_-)^{-1}UB^{\frac{1}{2}}.$$

定理 1 设 $A=U|A|_r$ 及 $A'=U'|A'|_r$ 为 φ-拟亚正常算子而且都是简单的. 又设 U 与 U' 是酉算子. 如果有酉算 T 将 H 映照成 H' 而且

$$|A'|_\pm=T|A|_\pm T^{-1}, \quad U'=TUT^{-1},$$

那末必有 $\mathscr{B}(A)$ 到 $\mathscr{B}(A')$ 上的酉算子 S 使当 $\lambda\in\sigma(A_-)=\sigma(A'_-)$ 时

$$W(\lambda, A')=SW(\lambda, A)S^{-1},$$

反之亦真.

下面设 A 是半亚正常的. 又设 $\mathscr{D}$ 为一辅助的希尔伯特空间, K 为 $\mathscr{D}$ 到 H 中的线性算子使 $Q=|A|_r-|A|_l=KK^*$. 当 $\lambda\in\rho(A)$, $|z|\neq 1$ 时作

$$Y(z, \lambda)=I-zK^*(I-zU^*)^{-1}(A-\lambda I)^{-1}K.$$

定理 2 设 $A=U|A|_r$ 及 $A'=U'|A'|_r$ 分别是 H 与 H' 中的半亚正常算子, U 与 U' 是酉算子而且 A 与 A' 都是简单的. 如果存在 $\mathscr{D}\to\mathscr{D}'$ 上的酉算子 S 使

$$Y'(z, \lambda)=SY(z, \lambda)S^{-1}$$

那末必有由 H 到 H' 上的酉算子 T 使(1)成立. 反之亦真.

定理 3 若 K 是希尔伯特–许密特算子则 $Y(z, \lambda)$ 的行列式 (当 $|z|\neq 1$ 时) 存在, 且

$$\det(Y(z, \lambda))=\det((I-zU^*)(A-\lambda I)(I-zU^*)^{-1}(A-\lambda I)^{-1}).$$

下面只考虑奇型积分模型这时 $W(\lambda; A)$ 成为乘法算子,

$$(W(\lambda, A)f)(e^{i\theta})=W(e^{i\theta}, \lambda)f(e^{i\theta})$$

其中

$$W(e^{i\theta}, \lambda)=I-\alpha(e^{i\theta})(\lambda e^{-i\theta}I-\beta(e^{i\theta}))^{-1}\alpha(e^{i\theta}).$$

我们又假设 A 是完全非正常的. 记 $Y_\pm(e^{i\theta}, \lambda)a=\lim\limits_{r\to 1\pm 0} Y(e^{i\theta}, \lambda)a$.

定理 4 设 $\lambda\in\rho(A)$, $a\in\mathscr{D}$ 为固定的,那末 $Y_\pm(e^{i\theta}, \lambda)a$ 为黎曼–希尔伯特问题

$$Y_-(e^{i\theta}, \lambda)a=W(e^{i\theta}, \lambda)Y_+(e^{i\theta}, \lambda)a$$

的解.

设 $\mathscr{L}(\mathscr{D})$ 为 $\mathscr{D}$ 上线性有界算子全体所成的 Banach 空间, $H^p_\pm(\mathscr{L}(\mathscr{D}))$ 为单位圆外, 内取值于 $\mathscr{L}(\mathscr{D})$ 的某些解析函数所成的 Hardy 空间. 设 $f(e^{i\theta})$ 是单位圆周上的函数,如果有 $u_\pm\in H^p_\pm(\mathscr{L}(\mathscr{D}))$ $(p>2)$ 使 u_-^{-1} 存在 $u_-(e^{i\theta})^{-1}u_+(e^{i\theta})=f(e^{i\theta})$ 则称 f 是可分解的.