Chin. Ann. of Math.
1 (1) 1980

ON THE STABILITY CONJECTURE

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§ 1. Introduction and statements of the main results.

Let M^n be an n-dimensional compact C^∞ Riemann manifold, and $\mathrm{Diff}^1(M^n)$ the space of C^1 diffeomorphisms of M^n endowed with the C^1 topology. A given $f\in\mathrm{Diff}^1(M^n)$ is called structurally stable if every g in $\mathrm{Diff}^1(M^n)$ nearby f is equivalent to f, and f is called Ω-stable if for every g nearby f, $g|\Omega(g)$ is equivalent to $f|\Omega(f)$, where $\Omega(f)$ denotes the nonwandering set of f. The main problem in the theory of differentiable discrete dynamics is to find characterizations of structurally stable and Ω-stable diffeomorphisms. This problem appears difficult in general. Smale conjectured many years ago that, a necessary and sufficient condition for $f\in\mathrm{Diff}^1(M^n)$ to be Ω-stable is that f satisfies Axiom A and the no cycle condition, and a necessary and sufficient condition for f structurally stable is that f satisfies Axiom A and the strong transversality condition. Later on, both of the sufficiency parts were proved respectively by Smale [19] and Robbin-Robinson ([14], [15]). However, the necessity parts in case $\dim M^n\geqq 2$, known as the *stabilty conjecture*, remain open; only quite recently, Mañé [8] proved the necessity for the case $\dim M^n=2$ under the additional assumption that the nonwandering set $\Omega(f)$ is exactly $=M^2$.

One aim of the present paper is to prove the stability conjecture for the case $\dim M^n=2$ (without the assumption that $\Omega(f)=M^2$). Our main results are as follows.

Theorem I. *Let $f\in\mathrm{Diff}^1(M^2)$. Then, a necessary condition for f to be Ω-stable is that f satisfies Axiom A and the no cycle condition, and a necessary condition for f structurally stable is that f satisfies Axiom A and the strong transversality condition.*

Theorem II. *In order that $f\in\mathrm{Diff}^1(M^2)$ is Ω-stable, it is necessary and sufficient that $f\in\mathscr{F}^*(M^2)$.*

Here, $\mathscr{F}^*(M^n)$ denotes the set of all diffeomorphisms $g\in\mathrm{Diff}^1(M^n)$ such that there is a neighbourhood G of g such that all periodic points of $h\in G$ are hyperbolic (or equivalently that each $h\in G$ has at most a countable number of periodic points). If $f\in\mathscr{F}^*(M^n)$, then the periodic points of f are dense in $\Omega(f)$ [8].

Theorem II remains true, if the diffeomorphism $f\in\mathrm{Diff}^1(M^2)$ is replaced by one

Manuscript received Dec. 21, 1979.

$\in \mathrm{Diff}^1(M^1)$. In fact, for $f \in \mathrm{Diff}^1(M^1)$, with the aid of arguments in [11, Chap. I] it is a trivial matter to show that $f \in \mathscr{F}^*(M^1)$ if and only if f is structurally stable, and also, if and only if f is Ω-stable. Thus, the problem mentioned in [8, p. 383] has a positive solution in case $\dim M^n \leqq 2$, and the similar version for diffeomorphisms of a conjecture mentioned in [5, p. 318] is also verified.

Theorems I and II will be proved in § 6 and partially deduced via the usual technique of suspension from diffeomorphisms to vector fields. Actually, we wish to establish in this paper some analogous results for vector fields but under more restrictions.

§ 2. The class $\mathscr{X}^*(M^n)$ of vector fields.

Let us recall first a result in [5] for later convenience. Let

$$\mathscr{X} = \mathscr{X}(M^n)$$

be the set of all C^1 tangent vector fields (i. e., differential systems) X on M^n for a given $n \geqq 2$, endowed with the C^1 norm $\|X\|_1$, and

$$\mathscr{X}^* = \mathscr{X}^*(M^n)$$

the set of all $X \in \mathscr{X}$ such that there is a neighbourhood U of X such that all singularities and all periodic orbits (different from singularities) of each $Y \in U$ are hyperbolic (or equivalently, each Y in U has only a finite number of singularities and at most a countable number of periodic orbits). We have made a study on the class $\mathscr{X}^*$ in [5].

Consider an arbitrarily given $S \in \mathscr{X}$. S induces then a C^1 one-parameter transformation group ϕ_t: $M^n \to M^n (-\infty < t < \infty)$, and induces therefore a one-parameter transformation group on the tangent bundle $\mathscr{C}$ of M^n,

$$\Phi_t = d\phi_t : \mathscr{C} \to \mathscr{C} \quad (-\infty < t < \infty).$$

Denote respectively by $\mathscr{P}_0$ and $\mathscr{P}$ the set of all singularites of S and that of all points on periodic orbits of S. Write

$$M = M^n - \mathscr{P}_0.$$

Consider the conjugate bundle

$$\mathscr{D} = \bigcup_{x \in M} \mathscr{D}_x$$

of S, which is the bundle with base space M and with fiber $\mathscr{D}_x$ over $x \in M$ consisting of all tangent vectors at x orthogonal to $S(x)$. For any $(t, u) \in (-\infty, \infty) \times \mathscr{D}_x$, take $\Psi_t(u)$ as the orthogonal projection of $\Phi_t(u)$ on $\mathscr{D}_{\phi_t(x)}$. This gives a one-parameter transformation group

$$\Psi_t : \mathscr{D} \to \mathscr{D} \quad (-\infty < t < \infty).$$

Ψ_t maps $\mathscr{D}_x$ linearly onto $\mathscr{D}_{\phi_t(x)}$ [3, § 1].

For any given $x \in \mathscr{P}$ write

$$D_-(x) = \{u \in \mathscr{D}_x \mid \lim_{t\to\infty} \|\Psi_t(u)\| = 0\},$$
$$D_+(x) = \{u \in \mathscr{D}_x \mid \lim_{t\to-\infty} \|\Psi_t(u)\| = 0\}. \tag{2.1}$$

These are linear subspaces of $\mathscr{D}_x$, and

$$\Psi_t(D_-(x))=D_-(\phi_t(x)), \quad \Psi_t(D_+(x))=D_+(\phi_t(x)).$$

Also, for any given $x\in\mathscr{P}$ and $0\leqq t<\infty$ write

$$\eta_-(t, x)=\begin{cases}\sup\limits_{u\in D_-(x),\|u\|=1} \log\|\Psi_t(u)\| & \text{if } \dim D_-(x)\geqq 1,\\ -\infty \quad \text{if} \quad \dim D_-(x)=0;\end{cases}$$

$$\eta_+(t, x)=\begin{cases}\inf\limits_{u\in D_+(x),\|u\|=1} \log\|\Psi_t(u)\| & \text{if } \dim D_+(x)\geqq 1,\\ \infty \quad \text{if} \quad \dim D_+(x)=0.\end{cases}$$

Clearly, $\mathscr{D}_x=D_-(x)\oplus D_+(x)$ if $S\in\mathscr{X}^*$.

The following can be found in [5].

Theorem 2.1. *There exists an open covering $\mathscr{B}$ of $\mathscr{X}^*$, and corresponding to each $H\in\mathscr{B}$ there exist numbers $\eta_H>0$ and $T_H>0$ such that: if $V\in\mathscr{B}$ and $S\in V$, then* (i_*) *Whenever x is a point on a periodic orbit of S and $T_V\leqq t<\infty$, we have*

$$\frac{1}{t}(\eta_+(t, x)-\eta_-(t, x))\geqq 2\eta_V; \tag{2.2}$$

(ii_*) *Whenever P is a periodic orbit of S with period T_0, $x\in P$, and*

$$0=t_0<t_1<\cdots<t_l=T_0$$

is a division of $\langle 0, T\rangle$ satisfying

$$t_k-t_{k-1}\geqq T_V, \qquad k=1, 2, \cdots, l,$$

we have

$$\frac{1}{T_0}\sum_{k=1}^{l}\eta_-(t_k-t_{k-1}, \phi_{t_{k-1}}(x))\leqq -\eta_V \tag{2.3}$$

and

$$\frac{1}{T_0}\sum_{k=1}^{l}\eta_+(t_k-t_{k-1}, \phi_{t_{k-1}}(x))\geqq \eta_V. \tag{2.4}$$

Corollary 2.2. *Let $V\in\mathscr{B}$ and $S\in V$. Then there is a positive number $\tilde{\eta}\leqq\eta_V$ such that if P is a periodic orbit of S with period T, the absolute value of each characteristic exponent of P is $\leqq \exp(-\tilde{\eta}T)$ or $\geqq \exp(\tilde{\eta}T)$.*

§3. Contractible periodic orbits.

Write simply Ω for the nonwandering set of $\phi_t(-\infty<t<\infty)$, and write

$$\Omega_0=\Omega-\mathscr{P}_0.$$

We want to prove in this section the following

Theorem 3.1. *Suppose $S\in\mathscr{X}^*$ and suppose that Ω_0 is closed in M^n. Then, there are only a finite number of periodic orbits of S which are contractible positively or negatively.*

In case $S\in\mathscr{X}^*$, any such contractible periodic orbit P is characterized by the property that $\dim D_+(x)=0$ or $\dim D_-(x)=0$ for $x\in P$, and then P is asymptotically stable positively or negatively in the usual sense.

We shall employ a theorem in [2] (with slight modifications) to prove the theorem above. We recall first from [3, § 1] the one-parameter transformation groups Φ_t, χ_t, $\chi_t^{\#}$ $(-\infty<t<\infty)$ induced by S on the bundles $\mathscr{U}_n$, $\mathscr{F}_n$, $\mathscr{F}_n^{\#}$ of n-frames, orthogonal n-frames, orthonormal n-frames of M^n respectively.

In fact, for any $(t, \gamma)=(t, u_1, \cdots, u_n)\in(-\infty, \infty)\times\mathfrak{U}_n$, put

$$\Phi_t(\gamma)=(\Phi_t(u_1), \Phi_t(u_2), \cdots, \Phi_t(u_n))\in\mathscr{U}_n.$$

This gives a one-parameter transformation group

$$\Phi_t:\mathfrak{U}_n\to\mathfrak{U}_n \quad (-\infty<t<\infty).$$

Let π: $\mathfrak{U}_n\to\mathscr{F}_n$ be the mapping obtained by the usual Gram-Schmidt orthogonization process, and for any $(t, \gamma)\in(-\infty, \infty)\times\mathscr{F}_n$ put $\chi_t(\gamma)=\pi\Phi_t(\gamma)\in\mathscr{F}_n$. This gives a one-parameter transformation group

$$\chi_t^{\#}:\mathscr{F}_n\to\mathscr{F}_n \quad (-\infty<t<\infty).$$

Let $\pi^{\#}$: $\mathscr{F}_n\to\mathscr{F}_n^{\#}$ be the mapping obtained by

$$\pi^{\#}(u_1, u_2, \cdots, u_n)=\left(\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}\bigg|_0, \cdots, \frac{u_n}{\|u_n\|}\right),$$

and for any $(t, \gamma)\in(-\infty, \infty)\times\mathscr{F}_n^{\#}$ put $\chi_t^{\#}(\gamma)=\pi^{\#}\chi_t(\gamma)$. This gives a one-parameter transformation group

$$\chi_t^{\#}:\mathscr{F}_n^{\#}\to\mathscr{F}_n^{\#} \quad (-\infty<t<\infty).$$

For $\gamma=(u_1, u_2, \cdots, u_n)\in\mathfrak{U}_n$, write $\mathrm{pro}\, j_k\gamma=u_k$, $k=1, 2, \cdots, n$.

For any $\gamma\in\mathscr{F}_n$, $\|\mathrm{pro}\, j_k\chi_t(\gamma)\|$ is continuously differentiable in t, and we define

$$\omega_k(\gamma)=\frac{d\|\mathrm{pro}\, j_k(\chi_t(\gamma))\|}{dt}\bigg|_0, \quad k=1, 2, \cdots, n.$$

[3, § 1]. These are continuous functions on $\mathscr{F}_n$. Moreover, for any

$$(t, \gamma)\in(-\infty, \infty)\times\mathscr{F}_n^{\#}$$

we have

$$\log\|\mathrm{pro}\, j_k\chi_t(\gamma)\|=\int_0^t\omega_k(\chi_s^{\#}(\gamma))ds, \quad (k=1, 2, \cdots, n). \tag{3.1}$$

We consider also the bundle

$$\mathscr{E}=\bigcup_{x\in M}\mathscr{E}_x$$

with base M and with fiber $\mathscr{E}_x$ over x consisting of all orthonormal $(n-1)$-frames $\gamma=(u_1, u_2, \cdots, u_{n-1})$ of M^n at x, satisfying $\langle S(x), u_k\rangle=0$ for $k=1, 2, \cdots, n-1$. For $\gamma\in\mathscr{E}_x$, $x\in M$, let $\iota^*(\gamma)=(S(x)/\|S(x)\|, \gamma)$. This gives an imbedding

$$\iota^*:\mathscr{E}\to\mathscr{F}_n^{\#}.$$

As $\Phi_t(S(x))=S(\phi_t(x))$ [3, § 1], therefore the relation $\iota^*\Theta_t(\gamma)=\chi_t^{\#}\iota^*(\gamma)$ defines a one-parameter trnsformation group

$$\Theta_t:\mathscr{E}\to\mathscr{E} \quad (-\infty<t<\infty).$$

For any $T\in(0, \infty)$, let us write

$$\left.\begin{aligned}\xi_T(x)&=\frac{1}{T}\sup_{u\in\mathscr{D}_x, \|u\|=1}\log\|\psi_T(u)\|,\\ \xi_{-T}(x)&=\frac{1}{T}\sup_{u\in\mathscr{D}_x, \|u\|=1}\log\|\psi_{-T}(u)\|,\end{aligned}\right\} \quad x\in M.$$

These are continuous functions on M. From (3.1) and the construction of Θ_t $(-\infty<t<\infty)$, we verify easily

$$\frac{1}{T}\int_0^T \omega_{k+1}(\iota^*\Theta_t(\gamma))dt \leqq \xi_T(x) \text{ for } \gamma\in\mathscr{E}_x,\ x\in M,\quad (k=1,\ 2,\ \cdots,\ n-1). \quad (3.2)$$

Lemma 3.2. *Let F be a closed subset of M^n, invariant under $\phi_t(-\infty<t<\infty)$, andcontained in M. Suppose that for a certain $T\in(0,\ \infty)$, there is a normalized measure μon F, invariant under $\phi_t(-\infty<t<\infty)$, such that*

$$\int_F \xi_T(x)d\mu<0 \quad or \quad \int_F \xi_{-T}(x)d\mu<0 \quad (3.3)$$

Then, F contains a periodic orbit of S, contractible positively or negatively according as the first inequality or the second in (3.3) *holds.*

Proof We consider the case when the first inequality in (3.3) holds (the second one is reduced to the first by considering $-S$ instead of S). Furthermore, since

$$\int_F \xi_T(x)d\mu=\int_{U_T}\mu(dx)\int_F \xi_T(y)\mu_x(dy)$$

where U_T denotes the set of all quasi-regular points $x\in F$ such that its individual measure μ_x is transitive with respect to $\phi_t(-\infty<t<\infty)$ [9, Chap. VI], and hence there exists at least one point $x_0\in U_T$ with

$$\int_F \xi_T(y)\mu_{x_0}(dy)<0,$$

we may assume therefore without loss of generality that μ is itself transitive under $\phi_t(-\infty<t<\infty)$. Let$\tau:\mathscr{E}\to M$ be the bundle projection. Clearly, $\tau^{-1}(F)$is compact and invariant under $\Theta_t(-\infty<t<\infty)$. Then, according to [2, p. 14], there is a normalized measure μ' on $\tau^{-1}(F)$, invariant and transitive under $\Theta_t(-\infty<t<\infty)$, and such that

$$\int_{\tau^{-1}(F)}\xi_T(\tau(\gamma))d\mu'=\int_F \xi_T(x)d\mu<0.$$

It follows then from (3.2) and the invariance property of μ' that

$$0>\int_{\tau^{-1}(F)}\frac{1}{T}\int_0^T \omega_{k+1}(\iota^*\Theta_t(\gamma))dt\,d\mu'=\frac{1}{T}\int_0^T dt\int_{\tau^{-1}(F)}\omega_{k+1}(\iota^*\Theta_t(\gamma))d\mu'$$

$$=\int_{\tau^{-1}(F)}\omega_{k+1}(\iota^*(\gamma))d\mu',\quad k=1,\ 2,\ \cdots,\ n-1.$$

Thus, by a theorem in [2, p. 2] modified to the case of C^1 vector fields (see a footnote in [3, p. 192]), we can conclude that there is a periodic orbit P of S contained in F. Moreover, by arguments in the proof of this theorem [2, p. 10] and using a corollary in [2, p. 7], this periodic orbit can actually be one which is contractible positively. This completes the proof.

Proof of Theorem 3.1. Suppose on the contrary that there are an infinite number of distinct contractible orbits $P_1,\ P_2,\ P_3,\ \cdots$ of S. By choosing subsequences and considering $-S$ instead of S if necessary, we may assume that all the P_i's are contractible positively, and that the sequence $\{P_i\}$ converges to a closed subset F of M^n(i. e., for any given $\varepsilon>0$, there is an i_0 such that for all $i>i_0$, the ε-neighbourhood of F (with respect to a given topological metric of M^n) contains P_i and the ε-neighbourhood of P_i contains F*). Since each P_i is invariant under $\phi_t(-\infty<t<\infty)$, F is invariant under $\phi_t(-\infty<t<\infty)$.

For each i, we can take an individual measure μ_i corresponding to a point a_i in P_i [9, Chap. VI]. By choosing subsequences if necessary, we may assume that in the space of all normalized measures on M^n (which is compact and metrizable in a natural sense [9, Chap. VI]), the sequence $\{\mu_i\}$ converges to a normalized measure μ on M^n. Since each μ_i is invariant under $\phi_t(-\infty<t<\infty)$, μ is invariant under $\phi_t(-\infty<t<\infty)$. Also, since $\{P_i\}$ converges to F and each $P_i\subset\Omega_0$ which is closed in M^n by hypotheses of the theorem, therefore F is a closed subset of $\Omega_0\subset M$, μ is a measure on F, and

$$\lim_{i\to\infty}\int_{P_i}\xi_T(x)\,d\mu_i=\int_F\xi_T(x)\,d\mu \tag{3.4}$$

for any $T\in(0,\infty)$.

But each μ_i is the individual measure corresponding to $a_i\in P_i$, so that we have

$$\int_{P_i}\xi_T(x)\,d\mu_i=\lim_{t\to\infty}\frac{1}{t}\int_0^t\xi_T(\phi_s(a_i))\,ds. \tag{3.5}$$

We shall show that for sufficiently large i, the left hand of (3.5) is $\leqq-\tilde{\eta}/2$ for certain numbers $\tilde{\eta}>0$ and $T=\tilde{T}>0$. In fact, let T_i be the period of P_i. As $S\in\mathscr{X}^*$ by hypotheses of the theorem, for each number $T'>0$ there are only a finite number of periodic orbits of S of period $\leqq T'$ [5], and hence

$$\lim_{i\to\infty}T_i=\infty. \tag{3.6}$$

Also, by Theorem 2.1 there are numbers $\tilde{T}>0$ and $\tilde{\eta}>0$ such that (2.3) holds. But now each P_i is contractible positively. So

$$\frac{\tilde{T}}{T_i}\Big(\sum_{k=1}^{m_i}\xi_{\tilde{T}}(\phi_{(k-1)\tilde{T}}(x))+\xi_{\tilde{T}}(\phi_{m_i\tilde{T}}(x))\Big)\leqq-\tilde{\eta}\ \text{ for any } x\in P_i,$$

where m_i is the greatest positive integer with $T_i-m_i\tilde{T}\geqq\tilde{T}$, which certainly exists by (3.6) for sufficiently large i. Therefore from (3.6), the continuity of the transformation group $\phi_t(-\infty<t<\infty)$, the compactness of $\Omega_0\subset M$ and the fact $P_i\subset\Omega_0$, we have

$$\frac{\tilde{T}}{m_i\tilde{T}}\Big(\sum_{k=1}^{m_i}\xi_{\tilde{T}}(\phi_{(k-1)\tilde{T}}(x))\Big)<-\frac{\tilde{\eta}}{2}\ \text{ and hence}$$

$$\frac{1}{pm_i}\Big(\sum_{k=1}^{pm_i}\xi_{\tilde{T}}(\phi_{(k-1)\tilde{T}}(x))\Big)<-\frac{\tilde{\eta}}{2},\quad p=1,2,3,\cdots$$

for any $x\in P_i$ and sufficiently large i. It follows that

$$\frac{1}{pm_i\tilde{T}}\int_0^{pm_i\tilde{T}}\xi_{\tilde{T}}(\phi_s(a_i))\,ds=\frac{1}{pm_i\tilde{T}}\sum_{k=0}^{pm_i-1}\int_{k\tilde{T}}^{(k+1)\tilde{T}}\xi_{\tilde{T}}(\phi_s(a_i))\,ds$$

$$=\frac{1}{\tilde{T}}\int_0^{\tilde{T}}\frac{1}{pm_i}\sum_{k=0}^{pm_i-1}\xi_{\tilde{T}}(\phi_{k\tilde{T}}(\phi_t(a_i)))\,dt<-\frac{\tilde{\eta}}{2},$$

$$p=1,2,3,\cdots.$$

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* This is actually the convergence in the sense of Hausdorff [1, § 28].

Now, in (3.5) with $T=\tilde{T}$ we see easily that

$$\int_{P_i} \xi_{\tilde{T}}(x)d\mu_i \leqq -\frac{\tilde{\eta}}{2} \text{ for sufficiently large } i. \tag{3.7}$$

From (3.4) and (3.7),

$$\int_F \xi_{\tilde{T}}(x)d\mu<0$$

so that by Lemma 3.2, F contains a periodic orbit P of S contractible positively and hence asymptotically stable. But this is impossible, because the sequence $\{P_i\}$ of distinct periodic orbits of S converges to $F\supset P$. We get therefore a contradiction, which completes the proof of Theorem 3.1.

§ 4. The case $S\in\mathscr{X}^*(M^3)$.

In this section and the next, we assume throughout $\dim M^n=3$. The aim of these two sections is to prove

Threorem 4.1. *Suppose $S\in\mathscr{X}^*(M^3)$. Suppose also that Ω_0 is closed in M^3 and $\mathscr{P}$ is dense in Ω_0. Then S has hyperbolic structure over Ω.*

We shall proceed the proof by a number of lemmas. Write

$$\mathscr{P}_1=\{x\in\mathscr{P}\mid \dim D_-(x)=0 \quad\text{or}\quad \dim D_+(x)=0\},\quad \Omega_1=\Omega_0-\mathscr{P}_1.$$

$\mathscr{P}_1$ and Ω_1 are clearly invariant under $\phi_t(-\infty<t<\infty)$. The conjugate bundle $\mathscr{D}$ of S is metrizable. Write simply

$$\overline{\mathscr{D}}=\{u\in\mathscr{D}\mid \|u\|=1\},\ \overline{\mathscr{D}}_1=\overline{\mathscr{D}}\cap(\mathscr{D}|\Omega_1).$$

Lemma 4.2. *Under the same hypotheses of Theorem 4.1, Ω_1 is closed in Ω_0 (and hence $\overline{\mathscr{D}}_1$ is compact), and for each $x\in\Omega_1$, $\mathscr{D}_x$ has a unique direct decomposition*

$$\mathscr{D}_x=D_-(x)\oplus D_+(x),\quad \dim D_-(x)=1=\dim D_+(x) \tag{4.1}$$

such that:

$$\frac{1}{t}(\log\|\psi_t(u)\|-\log\|\psi_t(v)\|)\geqq 2\tilde{\eta} \quad \textit{and}$$

$$\frac{1}{t}(\log\|\psi_{-t}(v)\|-\log\|\psi_{-t}(u)\|)\geqq 2\tilde{\eta},\quad \tilde{T}\leqq t<\infty, \tag{4.2}$$

for $u\in D_+(x)\cap\overline{\mathscr{D}}$, $v\in D_-(x)\cap\overline{\mathscr{D}}$, where $\tilde{\eta}>0$ and $\tilde{T}=T_v>0$ are given as in Corollary 2.2 and Theorem 2.1 with $V\in\mathscr{B}$ and $S\in V$. Moreover, such a decomposition satisfies:

(1) $\psi_t(\mathscr{D}_x)=D_-(\phi_t(x))\oplus D_+(\phi_t(x))$.

(2) *In case $x\in\mathscr{P}\cap\Omega_1$, $D_-(x)$ and $D_+(x)$ are the same as given in* (2.1).

(3) *The decomposition is continuous in x in the following sense, namely, in case when $\{x_i\}$ is a sequence in Ω_1 converges to $x_0\in\Omega_1$, the sequences $\{D_-(x_i)\cap\overline{\mathscr{D}}\}$ and $\{D_+(x_i)\cap\overline{\mathscr{D}}\}$ of sets also converge respectively to $D_-(x_0)\cap\overline{\mathscr{D}}$ and $D_+(x_0)\cap\overline{\mathscr{D}}$.*

Proof If $x_0\in\mathscr{P}_1$ is a closure point of Ω_1, and lies on a periodic orbit P of S, then since $\mathscr{P}$ is dense in Ω_0, there are points arbitrarily near to x_0 and lying on periodic orbits of S different from P. But this is impossible, because P is contractible by hypotheses of the lemma, and hence is asymptotically stable. Therefore Ω_1 is closed in Ω_0. It follows easily that $\overline{\mathscr{D}}_1$ is compact.

Consider any given $x_0 \in \Omega_1$. Since $\mathscr{P}$ is dense in Ω_0, we can choose a sequence $\{x_i\}$ of points in $\mathscr{P}$, converging to x_0. But from Theorem 3.1 we see easily that $\mathscr{P}_1$ is closed in Ω_0. Therefore since $S \in \mathscr{X}^*(M^3)$, we may assume $\dim D_-(x_i) = 1 = \dim D_+(x_i)$, $i = 1, 2, 3, \cdots$. Hence, for $u_i \in D_+(x_i) \cap \overline{\mathscr{D}}$, $v_i \in D_-(x_i) \cap \overline{\mathscr{D}}$, We have from Theorem 2.1

$$\frac{1}{t}(\log\|\psi_t(u_i)\| - \log\|\psi_t(v_i)\|) \geqq 2\tilde{\eta} \quad \text{and}$$

$$\frac{1}{t}(\log\|\psi_{-t}(v_i)\| - \log\|\psi_{-t}(u_i)\|) \geqq 2\tilde{\eta} \tag{4.3}$$

for $\tilde{T} \leqq t < \infty$. But $\overline{\mathscr{D}}_1$ is compact. Therefore, by choosing subsequences if necessary, we may assume that the sequences $\{u_i\}$ and $\{v_i\}$ converge respectively to u_0 and $v_0 \in \mathscr{D}_{x_0} \cap \overline{\mathscr{D}}$, and hence if we put $D_-(x_0)$ and $D_+(x_0)$ to be the linear subspaces of $\mathscr{D}_{x_0}$ generated by u_0 and v_0 respectively, then from (4.3) and the continuity of the transformation group $\psi_t(-\infty < t < \infty)$, we get a direct decomposition (4.1) for $x = x_0$ satisfying (4.2).

We assert that such a decomposition is unique. In fact, from (4.2) we verify easily

$$\lim_{t\to\infty} \frac{\|\psi_t(v_0)\|}{\|\psi_t(u_0)\|} = 0 = \lim_{t\to\infty} \frac{\|\psi_{-t}(u_0)\|}{\|\psi_{-t}(v_0)\|},$$

so that for $w = \lambda u_0 + \mu v_0 = 0$ with $\lambda, \mu \in (-\infty, \infty)$, we have

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{\|\psi_t(w)\|}{\|\psi_t(u_0)\|} = \lim_{t\to\infty} \frac{1}{t}\left(\log|\lambda| + \log\left\|\frac{\psi_t(u_0)}{\|\psi_t(u_0)\|} + \frac{\mu}{|\lambda|}\frac{\psi_t(v_0)}{\|\psi_t(u_0)\|}\right\|\right) = 0$$

$$\text{if} \quad \lambda \neq 0, \tag{4.4}$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{\|\psi_{-t}(w)\|}{\|\psi_{-t}(v_0)\|} = 0 \quad \text{if} \quad \mu \neq 0. \tag{4.5}$$

Thus, if there are given $\bar{u}$ and $\bar{v} \in \mathscr{D}_{x_0} \cap \overline{\mathscr{D}}$ satisfying (4.2) with $u = \bar{u}$ and $v = \bar{v}$, then clearly $(\bar{u}, \bar{v}) \neq (\pm v_0, \pm u_0)$, and we must have $(\bar{u}, \bar{v}) = (\pm u_0, \pm v_0)$, for otherwise, using (4.4) and (4.5) we could see that at least one of the two equalities

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{\|\psi_t(\bar{v})\|}{\|\psi_t(\bar{u})\|} = 0, \quad \lim_{t\to\infty} \frac{1}{t} \log \frac{\|\psi_{-t}(\bar{v})\|}{\|\psi_{-t}(\bar{u})\|} = 0$$

holds, which contradicts to (4.2). This proves our assertion.

From Theorem 2.1, (4.3) remains true if u_i and v_i are replaced respectively by $\psi_s(u_i)$ and $\psi_s(v_i)$ for any given $s \in (-\infty, \infty)$. Thus, by the continuity of the transformation group $\psi_t(-\infty < t < \infty)$ and the uniqueness of the decomposition (4.1), we find that (i) holds. Again by Theorem 2.1 and the uniqueness of this decomposition, (ii) is true. Using some similar arguments and the technique of choosing suitable convergent sequences in the compact set $\overline{\mathscr{D}}_1$, we can verify easily (iii). This completes the proof of Lemma 4.2.

For any $(t, u) \in \overline{\mathscr{D}}$, let us put $\psi_t^{\#}(u) = \psi_t(u)/\|\psi_t(u)\|$. Clearly, this gives a one-parameter transformation group

$$\psi_t^{\#}: \overline{\mathscr{D}} \to \overline{\mathscr{D}} \quad (-\infty < t < \infty).$$

We assert that for any $u \in \mathscr{D}$, $\|\psi_t(u)\|$ is continuously differentiable in t. This is clear if $u=0$, because $\psi_t(0)=0$. If $u \neq 0 \in \mathscr{D}_x$, our assertion follows easily from arguments in § 3 by considering an orthogonal 3-frame γ of M^3 with $\operatorname{pro} j_1\gamma = S(x)$, $\operatorname{pro} j_2\gamma = u$, so that $\operatorname{pro} j_2\chi_t(\gamma) = \psi_t(u)$. Write simply

$$\bar{\omega}(u) = \left.\frac{d\|\psi_t(u)\|}{dt}\right|_{t=0}.$$

Then as in (3.1) we have

$$\log\|\psi_t(u)\| = \int_0^t \bar{\omega}(\psi_s^{\#}(u))ds \quad \text{for} \quad u \in \overline{\mathscr{D}}, \tag{4.6}$$

and from § 2, we see that $\bar{\omega}(u)$ is continuous in $u \in \mathscr{D}$. Also, since ψ_t is linear from $\mathscr{D}_x$ to $\mathscr{D}_{\phi_t(x)}$, we have $\bar{\omega}(u) = \bar{\omega}(-u)$.

Now, let us consider the direct decomposition of $\mathscr{D}_x$, $x \in \Omega_1$, given in Lemma 4.2 (which is obtained under the same hypotheses of Theorem 4.1). By the property (i) of the decomposition, we see easily

$$\psi_t^{\#}(D_-(x) \cap \overline{\mathscr{D}}) = D_-(\phi_t(x)) \cap \overline{\mathscr{D}}, \qquad \psi_t^{\#}(D_+(x) \cap \overline{\mathscr{D}}) = D_+(\phi_t(x)) \cap \overline{\mathscr{D}}$$

for $(t, x) \in (-\infty, \infty) \times \Omega_1$. Also, if for each $x \in \Omega_1$ put $\zeta(x) = \bar{\omega}(u)$ where $u \in \overline{\mathscr{D}} \cap D_-(x)$, then since $\bar{\omega}(-u) = \bar{\omega}(u)$ and $\dim D_-(x) = 1$, $\zeta(x)$ is uniquely determined by x. Clearly, $\zeta(\phi_t(x)) = \bar{\omega}(\psi_t^{\#}(u))$ and

$$\log\|\psi_t(u)\| = \int_0^t \zeta(\phi_s(x))ds$$

for $u \in \overline{\mathscr{D}} \cap D_-(x)$. By the property (iii) of the decomposition (4.1), $\zeta(x)$ is a continuous function over Ω_1.

We shall consider the set

$$\check{E} = \{x \in \Omega_1 \mid \int_0^t \zeta(\phi_s(x))ds \geqq 0 \quad \text{for all} \quad t \in (-\infty, \infty)\},$$

which is clearly a closed subset of Ω_1.

Lemma 4.3. *Under the same hypotheses of Theorem* 4.1, *let* $\check{E}$ *and* $\zeta(x)$ *be given above. Let* Π *be a closed subset of* Ω_1, *invariant under* $\phi_t(-\infty < t < \infty)$, *such that* $\Pi \cap \check{E} = 0$. *Then there are numbers* $\eta_* > 0$ *and* $d_* > 0$ *such that*

$$\int_0^T \zeta(\phi_t(x))dt < -T\eta_* \quad \textit{for all} \quad x \in \Pi \quad \textit{and} \quad T \geqq d_*.$$

Proof First of all, we can apply a theorem in [4] to conclude the following*:

Π is the u nion of three mutually exclusive subsets Π_-, Π_+ and $\hat{\Pi}$ of Π where the first two are closed in Π, each of which is invariant under $\phi_t(-\infty < t < \infty)$, and such that:

(1) There is a closed subset Π_* of Π, contained in $\hat{\Pi}$, which is a section of $\hat{\Pi}$ with respect to $\phi_t(-\infty < t < \infty)$. This means that,

$$\phi_*(t, x) = \phi_t(x) \quad \text{for} \quad (t, x) \in (-\infty, \infty) \times \Pi_*$$

* See also a paper of the present author, entitled "Obstruction sets. I" (to appear).

defines a topological mapping from $(-\infty, \infty)\times\Pi_*$ onto $\hat{\Pi}$.

(2) There are constants $\eta_*>0$ and $d_*>0$ such that

$$\int_0^T \zeta(\phi_t(x)dt<\begin{cases} -T\eta_* \text{ for all } x\in\Pi_- \text{ and } T\geqq d_*, \\ T\eta_* \text{ for all } x\in\Pi_+ \text{ and } T\leqq -d_*, \end{cases} \tag{4.7}$$

$$\int_0^T \zeta(\phi_{s+t}(x))dt<\begin{cases} -T\eta_* \text{ for all } x\in\Pi_*,\ s\geqq 0 \text{ and } T\geqq d_*, \\ T\eta_* \text{ for all } x\in\Pi_*,\ s\leqq 0 \text{ and } T\leqq -d_*. \end{cases} \tag{4.8}$$

We assert that $\Pi_+=0$. In fact, from (4.6) and (4.7),

$$\log\|\psi_t(v)\|>t\eta_* \quad\text{for}\quad v\in D_-(x)\cap\overline{\mathscr{D}},\ x\in\Pi_+,\ t\geqq d_*.$$

This together with (4.2) gives also

$$\log\|\psi_t(u)\|>t\eta_* \quad\text{for}\quad u\in D_+(x)\cap\overline{\mathscr{D}},\ x\in\Pi_+,\ t\geqq \operatorname{Max}\{\tilde{T}, d_*\}.$$

But from (4.2), the continuity of the transformation group $\psi_t(-\infty<t<\infty)$ and the compactness of $\overline{\mathscr{D}}_1$ we must have

$$\sup_{\substack{u\in D_+(x)\cap\overline{\mathscr{D}},\\ v\in D_-(x)\cap\overline{\mathscr{D}},\ x\in\Omega_1}} |\langle u, v\rangle|<1.$$

Combining these togother, we verify easily that there exists a number $\overline{T}\geqq\operatorname{Max}\{\tilde{T}, d_*\}$ such that $\inf\limits_{u\in\mathscr{D}_x\cap\overline{\mathscr{D}},\ x\in\Pi_+}\log\|\psi_{\overline{T}}(u)\|\geqq\frac{1}{2}\eta_*$, or equivalently,

$$\xi_{-\overline{T}}(x)\leqq-\frac{1}{2}\eta_*,\qquad x\in\Pi_+, \tag{4.9}$$

where $\xi_{-\overline{T}}(x)$ is given as in § 3. Now, suppose that $\Pi_+\neq 0$. Then, there exists certainly at least a normalized measure μ on Π_+, invariant under $\phi_t(-\infty<t<\infty)$. Thus, in view of (4.9), we can apply Lemma 3.2 to conclude that $\Pi_+(\subset\Omega_1)$ contains a contractible periodic orbit of S. We get thus a contradiction to the definition of Ω_1, which proves our assertion.

From (4.7), (4.8) and the continuity of the function $\zeta(x)$ and the transformation group $\phi_t(-\infty<t<\infty)$, we see that the α-limit set of each orbit of S through a point in Π_* is contained in Π_+. Since Ω_1 is compact and metrizable, such α-limit sets are non-empty. But now $\Pi_+=0$. Therefore $\Pi_*=0$ also. It follows then from the property (1)of Π that $\Pi=\Pi_-$. This proves Lemma 4.3.

Consider now any $X\in\mathscr{X}(M^3)$. As the same for S, X induces also one-parameter transformation groups on M^3 and on its tangent bundle $\mathscr{C}$, which we shall denote respectively by ϕ_{Xt}: $M^3\to M^3$ and $\Phi_{Xt}=d\phi_{Xt}$: $\mathscr{C}\to\mathscr{C}$.

Let $\mathscr{X}_0$ be the set of all $X\in\mathscr{X}(M^3)$ which has the same singularities of S. As the same for S, each $X\in\mathscr{X}_0$ has its conjugate bundle, denoted by $\mathscr{D}_X$, with base space M and with fiber $\mathscr{D}_{Xx}$ over $X\in M$, and induces naturally a one-parameter transformation group ψ_{Xt}: $\mathscr{D}_X\to\mathscr{D}_X$ $(-\infty<t<\infty)$. Write

$$\Gamma=\{(X, s, w)\in\mathscr{X}_0\times(-\infty, \infty)\times\mathscr{C}\,|\,w\in\mathscr{D}_X\}$$

considered as a subspace of $\mathscr{X}\times(-\infty, \infty)\times\mathscr{C}$ (which is metrizable), and let

$$H\colon \Gamma\to\mathscr{C}$$

be defined by $H(X, s, w)=\psi_{Xs}(w)$. Then H is continuous at every $(S, t, u)\in\Gamma^*$.

If $X\in\mathscr{X}^*(M^3)$ and x is a point on a periodic orbit of X, then as the same for S, $\mathscr{D}_{Xx}$ is decomposable into

$$\mathscr{D}_{Xx}=D_{X-}(x)\oplus D_{X+}(x)$$

where

$$D_{X-}(x)=\{u\in\mathscr{D}_{Xx}\,|\lim_{t\to\infty}\|\psi_{Xt}(u)\|=0\},\quad D_{X+}(x)=\{u\in\mathscr{D}_{Xx}\,|\lim_{t\to-\infty}\|\psi_{Xt}(u)\|=0\}.$$

Lemma 4.4. *Under the same hypotheses of Theorem 4.1, let Π be a non-empty closed subset of Ω_1, invariant under $\phi_t(-\infty<t<\infty)$, and let $\tilde{\eta}>0$ be the same as in Lemma 4.2. Then, there is $v_*\in D_-(b_*)\cap\overline{\mathscr{D}}$ with $b_*\in\Pi$ such that*

$$\log\|\psi_t(v_*)\|=\int_0^t\zeta(\phi_s(b_*))ds\leqq -t\tilde{\eta}\ \ \textit{for all}\ \ t\geqq 0. \tag{4.10}$$

Proof. Let $a_*\in\Pi$ be a nonwandering point of the transformation group $\phi_t|\Pi(-\infty<t<\infty)$. Then, for each $j=1, 2, 3, \cdots$, there is $X_j\in V\cap\mathscr{X}_0$ where $V\in\mathscr{B}$ is the same as considered in Lemma 4.2 with $S\in V$, and there is an

$$\text{arc}L_j=\{\phi_t(a_j)\,|\,t\in\langle t_j, t'_j\rangle\}\subset\Pi\,(0\leqq t_j<t'_j<\infty)$$

together with a mapping f_j from L_j into a periodic orbit Q_j of X_j such that

$$\|X_j-S\|_1<1/j, \tag{4.11}$$

$$\text{dist}(x, f_j(x))<1/j\ \ \text{for all}\ \ x\in L_j,\ f_j(L_j)=Q_j, \tag{4.12}$$

$$\text{dist}(a_*, \phi_{t_j}(a_j))<1/j,\quad \text{dist}(a_*, \phi_{t'_j}(a_j))<1/j, \tag{4.13}$$

where dist (,) denotes a preassigned topological metric on M^3 (see [7]). Write simply

$$c_j=f_j\phi_{t_j}(a_j))\in Q_j,\qquad n_j=\dim D_{X_j-}(c_j).$$

By choosing subsequences if necessary, we may assume that one of the following three cases will occur:

(1) $n_j=1$ for all j,

(2) $n_j=2$ for all j,

(3) $n_j=0$ for all j.

Let T_j be the period of Q_j. Suppose $\lim_{j\to\infty}T_j\neq\infty$. Then, by (4.11)—(4.13) a subsequence of $\{Q_j\}$will converge to a periodic orbit Q of S, $\subset\Pi\subset\Omega_1$, and hence Q cannot be contractible by the definition of Ω_1, i. e., dim $D_-(x)=1$ for $x\in Q$. Thus, by corollary 2.2 and Lemma 4.2 we can find $v_*\in D_-(b_*)\cap\overline{\mathscr{D}}$ with $b_*\in Q$ satisfying (4.10). In the following, we assume $\lim_{j\to\infty}T_j=\infty$. Then, $T_j\geqq\tilde{T}$ for sufficiently large j where $\tilde{T}>0$ is the same as considered in Lemma 4.2, so that applying Theorem 2.1 to X_j we have

$$\log\|\psi_{X_jT_j}(v)\|\leqq -T_j\tilde{\eta}\ \text{for sufficiently large}\ j\ \text{and}$$
$$v\in D_{X_j-}(x),\ \|v\|=1,\ x\in Q_j. \tag{4.14}$$

In case (1), from (4.14) and $\psi_{X_jT_j}(D_{X_j-}(x))=D_{X_j-}(x)$ we can choose a point $b_j\in Q_j$ for sufficiently large j such that

* See Liao Shan-Tao, Acta Scientiarum Naturalium Universitatis Pekinensis, 12 (1966), 1—43.

$$\log\|\psi_{X_j t}(v_j)\| \leqq -\tilde{\eta}t \quad \text{for} \quad v_j \in D_{X_j-}(b_j),\ \|v_j\|=1, \text{ and all } t\geqq 0.$$

Also, we can apply Theorem 2.1 to X_j and obtain

$$\frac{1}{t}(\log\|\psi_{X_j t}(u_j)\| - \log\|\psi_{X_j t}(v_j)\|) \geqq 2\tilde{\eta} \quad \text{and}$$

$$\frac{1}{-s}(\log\|\psi_{X_j s}(v_j)\| - \log\|\psi_{X_j s}(u_j)\|) \geqq 2\tilde{\eta} \quad \text{for some}$$

$$u_j \in D_{X_j+}(b_j),\ v_j \in D_{X_j-}(b_j),\ \|u_j\| = 1 = \|v_j\|, \quad \text{and all } t \geqq \tilde{T},\ s \leqq -\tilde{T}.$$

In view of (4.11), (4.12) and the continuity property of H mentioned above, by choosing subsequences if necessary, we may assume that the sequence $\{b_j\}$ converges to a point $b_* \in \Pi$ and $\{u_j\}$ and $\{v_j\}$ converge respectively to

$$u_* \quad \text{and} \quad v_* \in \mathscr{D}_{b_*} \cap \overline{\mathscr{D}},$$

so that we have (4.10) and also (4.2) (for $u = u_*$, $v = v_*$). Here, we notice that from Lemma 4.2, v_* must $\in D_-(b_*) \cap \overline{\mathscr{D}}$. To sum up, we get v_* which satisfies the requirement in the conclusion of Lemma 4.4.

We say that the case (2) cannot occur. In fact, suppose on the contrary that $n_j = 0$ for all j. Then, as in the proof of Theorem 3.1 we can see also now that

$$\int_{Q_j} \xi_{j\tilde{T}}(x)\,d\nu_j \leqq -\frac{\tilde{\eta}}{2}$$

where ν_j is the individual measure corresponding to $c_j \in Q_j$ with respect to the transformation group $\phi_{X_j t}(-\infty < t < \infty)$, and

$$\xi_{j\tilde{T}}(x) = \frac{1}{\tilde{T}} \sup_{\substack{u \in \mathscr{D}_{X_j x},\\ \|u\|=1}} \|\psi_{X_j \tilde{T}}(u)\|, \quad x \in M$$

Also, in view of (4.11) and (4.12) and the continuity property of H mentioned above, by choosing subsequences if necessary, we may assume that, $\{Q_j\}$ converges to a closed subset C of Π, and in the space of all normalized measures on M^3, $\{\nu_j\}$ converges to a normalized measure ν on M^3 with $\nu(C) = 1$. It is then easy to see that both C and ν are invariant under $\phi_t(-\infty < t < \infty)$, and

$$\int_C \xi_{\tilde{T}}(x)\,d\nu = \lim_{j\to\infty} \int_{Q_j} \xi_{j\tilde{T}}(x)\,d\nu_j \leqq -\frac{\tilde{\eta}}{2}$$

where $\xi_{\tilde{T}}(x)$ is the same as in § 3. Thus, as in the proof of Theorem 3.1, we shall get finally in $C(\subset \Omega_1)$ a contractible periodic orbit of S. But this contradicts to the definition of Ω_1.

The case (3) cannot occur also, because this case is reduced to the case (2) if we consider $-S$ and $-X$ instead of S and X. The proof of Lemma 4.4 is now complete.

§ 5. The "sifting" lemma and the proof of Theorem 4.1.

We shall now establish the following important "sifting" lemma, which we formulate in a more situation for some future purpose.

Lemma 5.1. *Let F be a compact metric space, on which there is given a one-parameter transformation group $\theta_t(-\infty<t<\infty)$. Suppose that on F there is a continuous function $f(x)$ and there is a number $\eta>0$ satisfying the following* (1)—(2):

(1) *There is a point $a_*\in F$ such that*

$$\int_0^t f(\theta_s(a_*))ds \geqq 0 \quad \text{for all} \quad t\geqq 0; \tag{5.1}$$

(2) *if $a\in F$ is any point such that*

$$\int_0^t f(\theta_s(a))ds \geqq \frac{-t\eta}{3} \quad \text{for all} \quad t\geqq 0, \tag{5.2}$$

then the ω-limit set of the orbit $\{\theta_s(a)\,|\,s\in(-\infty,\infty)\}$ contains a point b such that

$$\int_0^t f(\theta_s(b))ds \leqq -t\eta \quad \text{for all} \quad t\geqq 0. \tag{5.3}$$

Then for each pair (l, k) of positive integers, we can pick up $l+2$ numbers

$$0<t_l(0,k)<t_l(1,k)<\cdots<t_l(l,k)<\bar{t}_l(l,k)$$

such that the following (5.4)—(5.7) *holds:*

$$t_l(i,k)-t_l(i-1,k)\geqq k \quad \text{for} \quad i=1,2,\cdots,l; \tag{5.4}$$

$$\int_0^t f(\theta_{t_l(i-1,k)+s}(a_*))ds \leqq \frac{-t\eta}{4} \quad \text{and} \quad \int_0^{-t} f(\theta_{t_l(i,k)+s}(a_*))ds \leqq \frac{t\eta}{3}$$
$$\text{for} \quad 0\leqq t\leqq t_l(i,k)-t_l(i-1,k),\ i=1,2,\cdots,l-1; \tag{5.5}$$

$$\int_0^t f(\theta_{t_l(l-1,k)+s}(a_*))ds < \frac{-t\eta}{4} \quad \text{for} \quad 0\leqq t<\bar{t}_l(l,k)-t_l(l-1,k), \quad \text{but}$$
$$\int_0^{\bar{t}_l(l,k)-t_l(l-1,k)} f(\theta_{t_l(l-1,k)+s}(a_*))ds = \frac{-(\bar{t}_l(l,k)-t_l(l-1,k))}{4}\eta; \tag{5.6}$$

$$\int_0^t f(\theta_{t_l(l-1,k)+s}(a_*))ds \leqq \frac{-t\eta}{3} \quad \text{for} \quad 0\leqq t\leqq t_l(l,k)-t_l(l-1,k), \quad \text{but}$$
$$\int_0^{t_l(l,k)-t_l(l-1,k)} f(\theta_{t_l(l-1,k)+s}(a_*))ds = \frac{-(t_l(l,k)-t_l(l-1,k))}{3}\eta. \tag{5.7}$$

Proof (by induction on l). (5.1) implies (5.2). Thus, the ω-limit set of $\{\theta_s(a_*)\,|\,s\in(-\infty,\infty)\}$ contains a point $b=b_0$ satisfying (5.3). Then, by the continuity of $f(x)$ and the transformation group $\theta_t(-\infty<t<\infty)$, we can take a point $c_0=\phi_{t_1(0,k)}(a_*)$, $t_1(0,k)>0$, nearby b_0 such that

$$\int_0^t f(\theta_s(c_0))ds < -\frac{t\eta}{3} \quad \text{for} \quad 0\leqq t<k.$$

But from (5.1),

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t f(\theta_s(c_0))ds=0.$$

So there are $\bar{t}_1(1,k)>t_1(1,k)\geqq t_1(0,k)+k$ such that (5.6) and (5.7) holds. This gives the required numbers $t_l(i,k)$, $\bar{t}_l(l,k)$ for $l=1$ and each k.

In general, suppose that for a given l and each k, the numbers $t_l(i,k)$, $\bar{t}_l(l,k)$ have been constructed to meet all the requirements (5.4)—(5.7). Then, for each k we shall show the existence of a sufficiently large $j_k>k$ such that between $t_l(l,j_k)$ and

$\bar{t}_l(l, j_k)$ we can insert certain $t_{l+1}(l, k)<t_{l+1}(l+1, k)<\bar{t}_{l+1}(l+1, k)$ satisfying (5.6) and (5.7) with l replaced by $l+1$, and satisfying also

$$t_{l+1}(l+1, k)-t_{l+1}(l, k)\geqq k, \tag{5.8}$$

$$\int_0^{-t} f(\theta_{t_{l+1}(l,k)+s}(a_*))ds\leqq\frac{t\eta}{3} \quad \text{for} \quad 0\leqq t\leqq t_{l+1}(l, k)-t_l(l-1, j_k). \tag{5.9}$$

As soon as the existence of such j_k is proved, it is quite easy to see that the set of numbers

$$t_{l+1}(i, k)=t_l(i, j_k) \quad \text{for} \quad i=0, 1, \cdots, l-1,$$
$$t_{l+1}(l, k), \quad t_{l+1}(l+1, k), \quad \bar{t}_{l+1}(l+1, k)$$

satisfies the requirements in the conclusion of the lemma for the pair $(l+1, k)$.

To prove the existence of such j_k, let $\tilde{t}_l(l, j)\in(t_l(l, j), \bar{t}_l(l, j))$ be such that

$$\int_0^{\tilde{t}_l(l,j)-t_l(l-1,j)} f(\theta_{t_l(l-1,j)+s}(a_*))ds=-\frac{\tilde{t}_l(l, j)-t_l(l-1, j)}{3}\eta, \tag{5.10}$$

$$\int_0^t f(\theta_s(a_j))ds>-\frac{t\eta}{3} \quad \text{for} \quad 0<t\leqq s_j \tag{5.11}$$

where $a_j=\theta_{\tilde{t}_l(l,j)}(a_*)$, $s_j=\bar{t}_l(l, j)-\tilde{t}_l(l, j)$. The existence of such $\tilde{t}_l(l, j)$ follows from (5.6), (5.7) and the continuity of $f(\theta_{t_l(l-1,j)+s}(a_*))$ with respect to s. Since from (5.6), (5.10) and (5.4),

$$\int_0^{s_j} f(\theta_s(a_j))ds=-\frac{\bar{t}_l(l, j)-t_l(l-1, j)}{4}\eta+\frac{\tilde{t}_l(l, j)-t_l(l-1, j)}{3}\eta$$
$$=\frac{-3s_j+\tilde{t}_l(l, j)-t_l(l-1, j)}{12}\eta\geqq\left(\frac{j}{12}-\frac{s_j}{4}\right)\eta,$$

and the continuity of $f(x)$ and the compactness of F implies that $f(x)$ is bounded on F, we have

$$\lim_{j\to\infty} s_j=\infty.$$

Also, from the compactness of the metric space F, by choosing subsequences if necessary, we may assume without loss of generality that $\{a_j\}$ converges to a point $a\in F$. It follows then from (5.11) and the continuity of the function $f(x)$ and the transformation group $\theta_t(-\infty<t<\infty)$ that (5.2) holds, and hence by (2), the ω-limit set of $\{\theta_s(a)\,|\,s\in(-\infty, \infty)\}$ contains a point b satisfying (5.3). Then, by some similar arguments, a sufficiently large $j_k>k$ and $0<t_k<t_k+k<s_{j_k}$ can be chosen such that the arc $\{\phi_t(a_{j_k})\,|\,0\leqq t\leqq s_{j_k}\}$ contains a point $c=\phi_{t_k}(a_{j_k})$ satisfying

$$\int_0^t f(\theta_s(c))ds\leqq-\frac{t\eta}{2} \quad \text{for} \quad 0\leqq t\leqq k. \tag{5.12}$$

We see easily that

$$\bar{t}_l(l, j_k)>t_k+k+\tilde{t}_l(l, j_k)>t_k+\tilde{t}_l(l, j_k)>\tilde{t}_l(l, j_k)\geqq t_l(l, j_k)\geqq t_l(l-1, j_k)+k.$$

Clearly, we can take always a number

$$t_{l+1}(l, k)\in\langle t_l(l, j_k),\ t_k+k+\tilde{t}_l(l, j_k)\rangle$$

such that

$$\int_0^t f(\theta_{t_{l+1}(l,k)+s}(a_*))ds \leqq \frac{-t\eta}{3} \quad \text{for} \quad 0 \leqq t \leqq t_k + k + \bar{t}_l(l, j_k) - t_{l+1}(l, k), \tag{5.13}$$

$$\int_0^{-t} f(\theta_{t_{l+1}(l,k)+s}(a_*))ds \leqq \frac{t\eta}{3} \quad \text{for} \quad 0 \leqq t \leqq t_{l+1}(l, k) - t_l(l, j_k). \tag{5.14}$$

But then from (5.12), certainly we have

$$t_{l+1}(l, k) \leqq \bar{t}_l(l, j_k) + t_k. \tag{5.15}$$

From (5.6),

$$\int_0^{\bar{t}_l(l,j_k)-t_{l+1}(l,k)} f(\theta_{t_{l+1}(l,k)+s}(a_*))ds > \frac{-(\bar{t}_l(l, j_k) - t_{l+1}(l, k))}{4} \eta.$$

This together with (5.13), (5.15) and the continuity of $f(\theta_{t_{l+1}(l,k)+s}(a_*))$ with respect to s gives two numbers

$$t_{l+1}(l+1, k) < \bar{t}_{l+1}(l+1, k) \in \langle t_{l+1}(l, k) + k, \bar{t}_l(l, j_k)\rangle,$$

satisfying (5.6) and (5.7) with l replaced by $l+1$, and satisfying also (5.8). By (5.7) and (5.14) we see that (5.9) is also satisfied. Now, the induction is complete, and Lemma 5.1 is proved.

Corollary 5.2. *Under the same hypotheses of Lemma* 5.1, *for arbitrarily given numbers $\delta>0$ and $d>0$ we have a point $w \in F$ and a number $T \geqq d$ such that*

(1) *the distance of w and $\theta_T(w)$ is $<\delta$,*

(2) $\int_0^t f(\theta_s(w))ds \leqq \frac{-t\eta}{4}$ *and* $\int_0^{-t} f(\theta_{T+s}(w))ds \leqslant \frac{t\eta}{3}$ *for* $0 \leqq t \leqq T$;

in particular,

$$\int_0^T f(\theta_s(w))ds \geqq -\frac{T\eta}{3}.$$

Proof For the compact metric space F, we have an integer $l \geqq 2$ such that among any l points of F, there are at least two of them are in a distance $<\delta$. Take an integer $k \geqq d$. We construct then the numbers $t_{l+1}(i, k)$ as in Lemma 5.1. Using (5.5) we see that for certain $0 \leqq i_0 < i_1 < l$, the points

$$w = \theta_{t_{l+1}(i_0,k)}(a_*), \ \theta_T(w) \quad \text{with} \quad T = (i_1 - i_0)k$$

satisfy the requirements of this corollary.

Lemma 5.3. *Let F and $\theta_t(-\infty<t<\infty)$ be the same as in Lemma* 5.1. *Suppose that $\theta_t(-\infty<t<\infty)$ has no fixed points. Then for any given numbers $\bar{T}>0$ and $\tau>0$, there corresponds a number $\varepsilon>0$ possessing the following property, namely: If $\bar{T} \leqq T < \infty$ and $g(t)$ is a continuous strictly increasing function on $\langle 0, T\rangle$, $g(0)=0$, and if a and b are such that $\phi_t(a)$ and $\phi_{g(t)}(b)$ are in a distance $<\varepsilon$ for all $t \in \langle 0, T\rangle$, then*

$$(1-\tau)T < g(T) < (1+\tau)T.$$

Proof Clearly, we may assume that $\bar{T}$ is small so that $\theta_t(-\infty<t<\infty)$ has no periodic orbits of period $<3\bar{T}$. Since $\langle 0, T\rangle$ can be divided into subintervals of length between $\bar{T}$ and $2\bar{T}$, the conclusion of the lemma holds if it holds in case $\bar{T} \leqq T \leqq 2\bar{T}$. In this case $\{\phi_t(a) \mid t \in \langle -\delta, T+\delta\rangle\}$ is always a simple arc for small constant $\delta>0$. It is then easy to use usual arguments in dynamical systems to complete the proof.

Proof of Theorem 4.1. Under the hypotheses of Theorem 4.1, S has only a finite number of singularities, each being hyperbolic, and has only a finite number of contratible periodic orbits by Theorem 3.1. So it remains to show that S has hyperbolic structure over Ω_1. We shall show that $\check{E}=\Omega_1\cap\check{E}=0$. Then by Lemma 4.3, there are numbers $\eta_1>0$ and $d_1>0$ such that

$$\log\|\psi_t(v)\|=\int_0^t\bar{\omega}(\psi_s^{\#}(v))ds=\int_0^t\zeta(\phi_s(x))ds$$
$$\leqq -t\eta_1 \quad \text{for} \quad v\in\overline{\mathscr{D}}\cap D_-(x),\ x\in\Omega_1 \ \text{ and } \ t\geqq d_1.$$

Applying this result to $-S$ we shall get a similar inequality for $u\in\overline{\mathscr{D}}\cap D_+(x)$, $x\in\Omega_1$, i.e., there are numbers $\eta_2>0$ and $d_2>0$ such that

$$\log\|\psi_t(u)\|\geqq t\eta_2 \quad \text{for} \quad u\in\overline{\mathscr{D}}\cap D_+(x) \quad \text{and} \quad t\geqq d_2.$$

This gives the hyperbolic structure of S over Ω_1 (see [4]), and the Theorem is proved.

Therefore it remains to show $\check{E}=0$. To prove this, let us fix topological metrics ρ_0 and ρ respectively on M^3 and on its tangent bundle $\mathscr{C}$, and for any $K\subset\mathscr{C}$ and any number $\lambda>0$ denote by $U(K,\lambda)$ the λ-nieghbourhood of K in $\mathscr{C}$. Under the hypotheses of Theorem 4.1, by the continuity of the function $\zeta(x)$ and the transformation group $\phi_t(-\infty<t<\infty)$ and by the compactness of Ω_1, there are $\epsilon>0$ and $\tau>0$ such that if $\rho_0(x,y)<\epsilon$ for x, $y\in\Omega_1$ and $|t-t'|<\tau t$ for $\tilde{T}\leqq t\leqq 2\tilde{T}$, then

$$\left|\int_0^t\zeta(\phi_s(x))ds-\int_0^{t'}\zeta(\phi_s(y))ds\right|<\frac{t\tilde{\eta}}{6},\quad \tau\sup_{w\in\Omega_1}|\zeta(w)|<\frac{\tilde{\eta}}{6}$$

with $\tilde{\eta}$, $\tilde{T}$ given as in Lemma 4.2. We may assume $\tilde{T}\geqq 1$ and choose ϵ so small that the conclusion of Lemma 5.3 holds with respect to $F=\Omega_1$, $\theta_t=\phi_t$, $\overline{T}=\tilde{T}$, τ. Also, using Theorem 3.1 and Lemma 4.2 we can choose ϵ so small that the ϵ-neighbourhood of Ω_1 in M^3 contains no points of $\mathscr{P}_1$ and singularities of S. We shall consider the so-called $(\tilde{\eta}/4,\ 2\tilde{T};\ 1)$ quasi-hyperbolic orbit arc in the sense of the paper [6]. Then, by the main theorem in that paper, there are numbers $\delta_*>0$ and $T_*\geqq 2\tilde{T}$ such that: If S has an orbit arc

$$Q=\{\phi_t(a)\,|\,t\in\langle 0,\ T\rangle\}\subset\Omega_1,\quad T_*\leqq T<\infty, \tag{5.16}$$

which is $(\tilde{\eta}/4,\ 2\tilde{T};\ 1)$ quasi-hyperbolic with respect to the decomposition

$$\mathscr{D}_a=D_-(a)\oplus D_+(a)$$

such that

$$\psi_T(D_-(a))\cap\overline{\mathscr{D}}\subset U(D_-(a)\cap\overline{\mathscr{D}},\ \delta_*), \tag{5.17}$$

then there is a periodic orbit P of S through a point b together with a continuous strictly increasing function $h(t)$ on $\langle 0,\ T\rangle$, $h(0)=0$, such that

$$\phi_{h(T)}(b)=b,\ \rho_0(\phi_t(a),\ \phi_{h(t)}(b))<\epsilon \quad \text{for} \quad t\in\langle 0,\ T\rangle.$$

Also, by Lemma 4.2 and the compactness of Ω_1, there is $\delta>0$ such that if $\rho_0(x,\ y)<\delta$ for x and $y\in\Omega_1$, then $D_-(y)\cap\overline{\mathscr{D}}\subset U(D_-(x)\cap\overline{\mathscr{D}},\ \delta_*)$.

Now, suppose on the contrary that $\check{E}\neq 0$. We shall show that a quasi-hyperbolic arc Q as above exists actually. In fact, let $a_*\in\check{E}$. Then

$$\int_0^t (\phi_s(a_*))ds \geqq 0 \quad \text{for all} \quad t \geqq 0.$$

This together with Lemma 4.4 gives that Ω_1, $\phi_t(-\infty<t<\infty)$ and $\zeta(x)$ are the sort of spaces, transformation groups and the functions considered in Lemma 5.1 with $\eta=\tilde{\eta}$. Thus, by Corollary 5.2, we have a point $a\in\Omega_1$ and a number $T\geqq T_*$ such that $\rho_0(a, \phi_T(a))<\delta$ and hence (5.17) holds, and that

$$\log\|\psi_t(v)\|=\int_0^t \zeta(\phi_s(a))ds \leqq \frac{-t\tilde{\eta}}{4} \quad \text{for} \quad v\in D_-(a)\cap\overline{\mathscr{D}},\ 0\leqq t\leqq T, \tag{5.18}$$

$$\log\|\psi_{T-t}(v)\| \leqq \frac{t\tilde{\eta}}{3} \quad \text{for} \quad v\in D_-(\phi_T(a))\cap\overline{\mathscr{D}},\ 0\leqq t\leqq T, \tag{5.19}$$

$$\log\|\psi_T(v)\|=\int_0^T \zeta(\phi_s(a))ds \geqq \frac{-T\tilde{\eta}}{3} \quad \text{for } v\in D_-(a)\cap\overline{\mathscr{D}}. \tag{5.20}$$

From (5.19) and (4.2), we have $\log\|\psi_{T-t}(u)\| \leqq -\frac{5t\tilde{\eta}}{3}$ for $u\in D_+(\phi_T(a))\cap\overline{\mathscr{D}}$ and $\tilde{T}\leqq t\leqq T$, or equivalently,

$$\log\|\psi_t(u)\| \geqq \frac{5t\tilde{\eta}}{3} \quad \text{for} \quad u\in D_+(\phi_{T-t}(u))\cap\overline{\mathscr{D}},\ \tilde{T}\leqq t\leqq T. \tag{5.21}$$

Also from (4.2) and $\tilde{T}\geqq 1$,

$$\log\|\psi_t(u)\|-\log\|\psi_t(v)\| \geqq 2\tilde{\eta} \quad \text{for} \quad u\in D_+(\phi_s(a))\cap\overline{\mathscr{D}},$$
$$v\in D_-(\phi_s(a))\cap\overline{\mathscr{D}} \quad \text{and} \quad 0\leqq s<s+\tilde{T}\leqq s+t\leqq s+2\tilde{T}\leqq T. \tag{5.22}$$

Combining (5.18), (5.21) and (5.22) we get thus an arc Q as in (5.16) which is $(\tilde{\eta}/4, 2\tilde{T}; 1)$ quasi-hyperbolic with respect to $\mathscr{D}_a=D_-(a)\oplus D_+(a)$ such that (5.17) holds, and it is easy to see that $\langle 0, T\rangle$ has a division

$$0=t_0<t_1<\cdots<t_m=T$$

with $\tilde{T}\leqq t_k-t_{k-1}<2\tilde{T}$, fulfilling the requirements in the definition of quasi-hyperbolicity[6]. Finally, there is a periodic orbit P of S together with a function $h(t)$ on $\langle 0, T\rangle$ as described in the above paragraph.

Due to the construction of the ϵ-neighbourhood of Ω_1 in M^3 and the property of the function $h(t)$, the periodic orbit P cannot be a contractible one, and thus $\subset\Omega_1$. Let T_0 be the period of P. By the property of $h(t)$, $h(T)=pT_0$ for a certain integer $p\geqq 1$. Also, by the choice of ϵ and the property of $h(t)$ we have

$$\left|\frac{1}{h(T)}\int_0^{h(T)} \zeta(\phi_s(b))ds-\frac{1}{T}\int_0^T \zeta(\phi_s(a))ds\right|$$
$$\leqq\left|\left(\frac{1}{h(T)}-\frac{1}{T}\right)\int_0^{h(T)} \zeta(\phi_s(b))ds\right|+\frac{1}{T}\left|\int_0^T \zeta(\phi_s(a))ds-\int_0^{h(T)} \zeta(\phi_s(b))ds\right|$$
$$\leqq\left|\frac{\tau}{h(T)}\int_0^{h(T)} \zeta(\phi_s(b))ds\right|+\frac{1}{T}\sum_{k=0}^{m-1}\left|\int_{t_{k-1}}^{t_k} \zeta(\phi_s(a))ds-\int_{h(t_{k-1})}^{h(t_k)} \zeta(\phi_s(b))ds\right|$$
$$<\tau\sup_{w\in\Omega_1}|\zeta(w)|+\frac{1}{T}\sum_{k=0}^{m-1}(t_k-t_{k-1})\tilde{\eta}/6<\tilde{\eta}/3.$$

This together with (5.20) gives

$$\frac{1}{h(T)}\int_0^{h(T)} \zeta(\phi_s(b))ds>-\frac{2\tilde{\eta}}{3}>-\tilde{\eta},$$

or equivalently $\frac{1}{pT_0}\log\|\psi_{pT_0}(v)\|>-\tilde{\eta}$, and hence $\frac{1}{T_0}\log\|\psi_{T_0}(v)\|>-\tilde{\eta}$ for

$$v\in D_-(a)\cap\overline{\mathscr{D}}.$$

But this contradics to Corollary 2.2. The proof of Theorem 4.1 is now complete.

§ 6. Proofs of Theorems I and II.

After establishing Theorem 4.1, it is now easy to prove the following theorems.

Theorem 6.1. *Suppose that $S\in\mathscr{X}(M^3)$ has no singularities. Then, a necessary and sufficient condition for S to be Ω-stable is that S satisfies Axiom A and the no cycle condition.*

Proof Here, the sufficient part is known before by Pugh-Shub [13]. It remains to prove the necessity. Suppose that S is Ω-stable. Then, $S\in\mathscr{X}^*(M^3)$ [5], and using the closing lemma ([7], [12]), we see easily that $\mathscr{P}$ is dense in Ω. But now, S has no singularities. So we can apply Theorem 4.1 to conclude that S has hyperbolic structure over Ω. It follows that S satisfies Axiom A. So the proof of the theorem is complete, if the following lemma is proved.

Lemma 6.2. *Suppose that $S\in\mathscr{X}^*(M^3)$ has no singularities and satisfies Axiom A. Then S satisfies the no cycle condition.*

Proof Suppose on the contrary that there is a cycle $B_0, B_1, \cdots, B_m, B_{m+1} (m\geqq 1)$ of basic sets S with $B_i\neq B_j$ for $0\leqq i<j\leqq m$ and $B_0=B_{m+1}$ such that, for each $0\leqq i\leqq m$ there is a point b_i in $M^3-\Omega$, belonging to the intersection of the unstables manifold W_i^u of an orbit in B_i and the stable manifold W_i^s of an orbit in B_{i+1} (see [13] for reference). We can choose vectors u_i and $v_i\in\mathscr{D}_{b_i}$ tangent to W_i^u and W_i^s respectively, $\|u_i\|=1=\|v_i\|$. Then

$$\lim_{t\to-\infty}\|\psi_t(u_i)\|=0,\quad \lim_{t\to\infty}\|\psi_t(v_i)\|=0. \tag{6.1}$$

We see easily that $b_i\neq b_j$ for $0\leqq i<j\leqq m$. For convenience write also $b_{m+1}=b_0$. For each $0\leqq i\leqq m+1$, let Q_i and R_i be the positive half orbit $\{\phi_t(b_i)\,|\,t\in\langle 0, \infty)\}$ and the negative half orbit $\{\phi_t(b_i)\,|\,t\in(-\infty, 0\rangle\}$ respectively, and let

$$F_i=Q_i\cup B_{i+1}\cup R_{i+1}\quad\text{for}\quad 0\leqq i\leqq m,\ F=\bigcup_{0\leqq i\leqq m}F_i.$$

Then, since S satisfies Axiom A and a basic set of S is topologically transitive, therefore F is a connected closed subset of M^3, invariant under $\phi_t(-\infty<t<\infty)$, and contains none of the contractible periodic orbits of S (which are finite in number by Theorem 3.1), and by arguments in stable manifold theory we can find always in an arbitrarily given neighbourhood of F_i, orbit arcs A_i of S with two end points of A_i arbitrarily near to b_i and b_{i+1} for each $i=0, 1, \cdots, m$; and in particular, in case that S has hyperbolic structure over F, orbit arcs $A=\{\phi_t(a)\,|\,t\in\langle 0, T\rangle\}$, $T\geqq 1$, can be found with two end points a and $\phi_T(a)$ arbitrarily near to b_0 (for the technique used to reach these conclusions, see e.g. [10], [13]). But this particular case cannot occur, because $b_0 \in M^3 - \Omega$. In the former general case, we can perturb S into a system $X_k \in \mathscr{X}^*(M^3)$ for each integer $k>0$ such that $\|S - X_k\|_1 < \frac{1}{k}$ and that X_k has a periodic orbit P_k in the $\frac{1}{k}$-neighbourhood of F and all the points $b_0, b_1, \cdots, b_m$ are in the $\frac{1}{k}$-neighbourhood of P_k (with respect to a preassigned topological metric on M^3).

But, if there are infinitely many terms P_k in the sequence $\{P_k\}$, which is a contractible periodic orbit of X_k, then using Theorem 2.1, by the same method (namely, taking certain limit processes) as in the proof of Lemma 4.4 we shall see that F contains a contractible periodic orbit of S. But this is impossible, as shown in the above paragraph. Therefore, there must be infinitely many terms P_k which has exactly one characteristic exponent of absolute value <1, and hence again by the same method as in the proof of Lemma 4.4 we see that there are $\bar{u}_i$ and $\bar{v}_i \in \mathscr{D}_{b_i}$, $\|\bar{u}_i\| = 1 = \|\bar{v}_i\|$, such that

$$\frac{1}{t}(\log\|\psi_t(\bar{u}_i)\| - \log\|\psi_t(\bar{v}_i)\|) \geqq 2\tilde{\eta} \quad \text{and}$$

$$\frac{1}{t}(\log\|\psi_{-t}(\bar{v}_i)\| - \log\|\psi_{-t}(\bar{u}_i)\|) \geqq 2\tilde{\eta} \quad \text{for} \quad t \geqq \tilde{T}, \tag{6.2}$$

where $\tilde{\eta}>0$ and $\tilde{T}>0$ are the same as in (4.2).

Combining (6.1) and (6.2) will lead to

$$\lim_{t\to\infty} \|\psi_t(\bar{v}_i)\| = 0, \quad \lim_{t\to-\infty} \|\psi_t(\bar{u}_i)\| = 0, \ i=0, 1, \cdots, m, \tag{6.3}$$

In fact, since dim $M^3=3$ implies dim $\mathscr{D}_{b_i}=2$. let us write $v_i = r_i\bar{u}_i + s_i\bar{v}_i$. If $r_i=0$, then $v_i = \pm\bar{v}_i$ and the first equality in (6.3) holds clearly. If $r_i \neq 0$, then from (6.1) and (6.2) the first equality still hold, because

$$\lim_{t\to\infty} \frac{\|\psi_t(\bar{v}_i)\|}{\|\psi_t(\bar{u}_i)\|} = 0 \quad \text{and}$$

$$\lim_{t\to\infty} \log \frac{\|\psi_t(v_i)\|}{\|\psi_t(\bar{u}_i)\|} = \log\|r_i\| + \lim_{t\to\infty} \log \left\| \frac{\psi_t(u_i)}{\|\psi_t(\bar{u}_i)\|} + \frac{s_i}{|r_i|} \frac{\psi_t(\bar{v}_i)}{\|\psi_t(\bar{u}_i)\|} \right\| = \log\|r_i\|.$$

Similarly, we can verify the second equality in (6.3). Then from (6.3) and the hyperbolic structure of S over Ω, we see easily that S has hyperbolic structure over F by [4]. But this cannot occur, as we have shown above already. This proves Lemma 6.2.

Theorem 6.3. *Suppose that $S \in \mathscr{X}(M^3)$ has no singularities. Then, a necessary and sufficient condition for S to be structurally stable is that S satisfies Axiom A and the strong transversality condition.*

Proof The sufficiency part is known before by Robinson [16]. It remains to prove the necessity. Suppose that S is structurally stable. Then S is also Ω-stable, and hence by Theorem 6.1, S satisfies Axiom A.

To show that S satisfies the strong transversality condition, suppose on the contrary that there is a point $b \in M^3 - \Omega$ which lies on a stable manifold W^s of an orbit P in a basic set B_0 of S and lies also on an unstable manifold W^u of an orbit Q in a basic set B_1 of S, but W^s and W^u do not intersect transversally at b. We must have $B_0 \neq B_1$ since $b \in M^3 - \Omega$ [13, p. 154]. Then, by arbitrarily small C^1 perturbation of S, we may assume that P and Q are periodic orbits of S. Of course, P and Q are not contractible and hence dim $W^s = \dim W^u = 2$. Again by arbitrarily small C^1 perturbation of S around the point b, we may assume further dim $(W^s \cap W^u) = 2$. But then, using the theorem of Kupka-Small [19, p. 804], this will lead to a contradiction to the structural stability of S. Theorem 6.3 is thus proved.

We now prove the main results (see § 1) of our paper.

Proof of Theorem I. Apply the necessity conditions in Theorems 6.1 and 6.3 to the suspension of $f \in \mathrm{Diff}^1(M^2)$ (see Appendix below). The conclusions of Theorem I follow immediately.

Proof of Theorem II. The condition is necessary, because a Ω-stable $f \in \mathrm{Diff}^1(M^2)$ and hence any conjugacy of f has at most a countable number of periodic points. To show that the condition is sufficient, we note first that the periodic points of $f \in \mathrm{Diff}^1(M^2)$ are dense in $\Omega(f)$ [8]. Consider the suspended vector field S_f for f over the suspended manifold M^3_f. Then the periodic orbits of S_f are dense in the nonwandering set of S_f, and $S_f \in \mathscr{X}^*(M^3_f)$ has no singularities. If follows then from Theorem 4.1 and Lemma 6.2 that S_f satisfies Axiom A and the no cycle condition. Hence applying results in [13] to the vector field S_f, and using properties of suspension, we conclude that f is Ω-stable. This proves Theorem II.

Appendix

Let us say few words about the differentiable structure of the suspended manifold (denoted by) M^{n+1}_f of a given $f \in \mathrm{Diff}^1(M^n)$, because there is something which need be taken into consideration, and which seems not explicit in the literature at present at least. As usual, M^{n+1}_f is the quotient manifold $((-\infty, \infty) \times M^n)/\sim$ where $\sim$ is the eqivalence relation $(t, x) \sim (t+k, f^{-k}(x))$ for integers k, and for coordinate neighbourhoods U on M^n, $\left(-\frac{1}{2}, \frac{1}{2}\right) \times U$ and $(0, 1) \times U$ after taking quotient give an atlas on M^{n+1}_f in a natural fashion. The suspension S_f of f is the vector field on M^{n+1}_f induced from $\left(\frac{d}{dt}, 0\right)$ on $(-\infty, \infty) \times M^n$ by the quotient mapping. We see thus that the differentiable structure so introduced on M^{n+1}_f is only of class C^1 and hence S_f is C^0, because f is C^1.

However, this C^1 structure on M^{n+1}_f contains a C^∞ structure, with respect to which

S_f is a C^1 vector field, and M^n is a C^∞ submanifold of M_f^{n+1} (through identification of x and $(0, x)$). This can be shown as follows. Since M^n is compact, there is a $\delta>0$ such that at any $x\in M^n$, the exponential mapping $\exp_x$ is a diffeomorphism from

$$\Gamma_x=\{\beta\in\pi^{-1}(x) \mid \|\beta\|<\delta\}$$

into M^n, where π denotes the projection of the tangent bundle of M^n. For any $y\in\exp_x\Gamma_x$, write $\beta_y\in\Gamma_x$ such that $y=\exp_x\beta_y$. Let $\eta(t)$ be a C^∞ real function on $(-\infty, \infty)$ with values in $\langle 0, 1\rangle$ and with $\eta(t)=0$ for $t<0$, $\eta(t)=1$ for $t>1$, and let $f_0\in\mathrm{Diff}^\infty(M^n)$ be such that $f_0^{-1}f(x)\in\exp_x\Gamma_x$ for all $x\in M^n$. Then, when $f_0^{-1}f$ is sufficiently near to the identity in $\mathrm{Diff}^1(M^n)$ (under C^1 topology), a C^1 diffeomorphism from $\langle 0, 1\rangle\times M^n$ onto itself can be defined by

$$\Delta(t, x)=(t, \exp_x(\eta(t)\beta_{f_0^{-1}f(x)}))$$

and will induce a C^1 diffeomorphism from M_f^{n+1} onto $M_{f_0}^{n+1}$, where the latter is a differentiable manifold of class C^∞ because f_0 is C^∞. Then

$$d\Delta\left(\frac{d}{dt}, 0\right)_x=\left(\frac{d}{dt}, \frac{d\eta(t)}{dt}\beta_{f_0^{-1}f(x)}\right)$$

where the right is a C^1 vector field on $\langle 0, 1\rangle\times M^n$. It follows that Δ^{-1} carries the C^∞ structure on $M_{f_0}^{n+1}$ into a C^∞ structure on M_f^{n+1} satisfying all of our requirements. Our construction here is actually a generalization of that given in [17, p. 399] which deals with C^1 diffeomorphisms on circles. Clearly, a similar treatment can be done for any given diffeomorphism $f\in\mathrm{Diff}^r(M^n)$ with $r\geqq 2$.

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关于稳定性推测

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摘 要

目前微分动力体系理论中，一个主要问题是问关于离散体系的所谓稳定性推测是否成立. 设 M^n 是一 n 维紧致的 C^∞ Riemann 流形, $\mathrm{Diff}^1(M^n)$ 是 M^n 上所有 C^1 微拓变换作成的空间,赋以 C^1 拓扑. 考虑一任给的 $f\in\mathrm{Diff}^1(M^n)$. 这推测说, 在 $n\geqq2$ 情况下,若 f 是结构稳定的, 则它满足公理 A 及强匀断条件; 若 f 是 Ω-稳定的,则它满足公理 A 及无环性条件. 关于这里出现的名词, 例如可参看[18], [19], [14], [4]等. 这推测即令在 $n=2$ 情况下,直到最近, Mañé[8] 才在 $\Omega(f)=M^2$ 这一强的附加条件下证明过有正面的答案. 这里 $\Omega(f)$ 表 f 的非游荡集.

本文的一个目的是给出这推测在 $n=2$ 情况下的正面答案 (没有 $\Omega(f)=M^2$ 这附加假定). 我们的主要结果如下:

定理 1 命 $f\in\mathrm{Diff}^1(M^2)$. 则: f 结构稳定的必要条件是它满足公理 A 及强匀断条件; f 是 Ω-稳定的必要条件是它满足公理 A 及无环性条件.

这些条件的充分性也成立, 见以前的 [14], [15], [19]. 这样, 我们就得出了 $f\in\mathrm{Diff}^1(M^2)$ 结构稳定与 Ω-稳定的特征性质.

定理 2 $f\in\mathrm{Diff}^1(M^2)$ 是 Ω-稳定的,当且仅当它 $\in\mathscr{F}^*(M^2)$.

这里 $\mathscr{F}^*(M^n)$ 表所有具有下述性质的 $g\in\mathrm{Diff}^1(M^n)$ 作成的集合, 即: g 在 $\mathrm{Diff}^1(M^n)$ 中有一邻域 G 使得, 每一 $h\in G$ 的周期点都是双曲的 (或等价地, 每一 $h\in G$ 都至多只有可数个周期点)。根据一些周知的论断, 容易看出对于 $f\in\mathrm{Diff}^1(M^1)$, 定理 2 的结论仍然成立. 由此可看出, 文献[8, 383 页]中提到的一问题在 $\dim M^n\leqq2$ 情况下的解答是肯定的,文献[5, 318 页]中提到的一推测的微拓变换类比形式的答案也是正面的.

本文大部分内容(在较有限制的情况下)讨论了 M^n 上的 C^1 切向量场, 然后借助于通常的扭扩的办法完成上述定理 1 及 2 的证明.