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关于 Varma 的缺插值样条函数

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最近，A. K. Varma 在[1,2]中讨论了五次、六次缺插值样条函数.

设 $n=2m+1$, $x_i=\frac{i}{2m}$, $i=0, 2, \cdots, 2m$. 用 $S_{n5}^{(2)}(x)$ 表示在 $[0, 1]$ 上满足下列条件的五次样条函数

(1) $S_n(x)\in C^2[0, 1]$,

(2) $S_n(x)$ 在 $[x_{2i}, x_{2i+2}]$ 上, $i=0, 1, \cdots, m-1$, 是五次多项式.

Varma 在[1]中证明了

定理 1 设 $n=2m+1$, $x_i=\frac{i}{2m}$, $i=0, 2, \cdots, 2m$, $t_{2i}=x_{2i}+\frac{2}{3}h$, $i=0, 1, \cdots m-1$, $2h=\frac{1}{m}$. 给定 $f(x_0), f(x_2), \cdots; f(x_{2m}); f(t_0), f(t_2), \cdots, f(t_{2(m-1)}); f''(t_2), \cdots; f''(t_{2(m-1)})$; $f'(x_0)$, $f'(x_{2m})$. 则存在着唯一的五次样条函数 $S_n(x)\in C^2[0, 1]$, 满足

$$\begin{cases} S_n(x_{2i})=f(x_{2i}), \; i=0, 1, \cdots, m, \\ S_n(t_{2i})=f(t_{2i}), \; S_n''(t_{2i})=f''(t_{2i}), \; i=0, 1, \cdots, m-1, \\ S_n'(x_0)=f'(x_0), \; S_n'(x_{2m})=f'(x_{2m}) \end{cases}$$

定理 2 设 $f(x)\in C^2[0, 1]$, $S_n(x)$ 是满足定理 1 中条件的五次样条函数, 则

$$\|S_n^{(p)}(x)-f^{(p)}(x)\|\leqslant 89\, n^{p-2}\omega_2\left(\frac{1}{m}\right), \; p=0, 1, 2,$$

这里 $\omega_2(\delta)$ 是 $f''(x)$ 的连续模.

定理 3 设 $f(x)\in C^4[0, 1]$, $S_n(x)$ 是满足定理 1 中条件的五次样条函数, 则

$$\|S_n^{(p)}(x)-f^{(p)}(x)\|\leqslant 103\cdot m^{p-4}\omega_4\left(\frac{1}{m}\right)+8\, m^{p-4}\|f^{(4)}\|_\infty, \; p=0, 1, 2,$$

这里 $\omega_4(\cdot)$ 表示 $f^{(4)}(x)$ 的连续模.

我们首先指出, Varma 的证明过程是错误的. [1]中式(5.2)是不成立的.

本文给出正确的证明, 并改进定理的结果.

1. 预备知识[1]如果 $P(x)$ 是 $[0, 1]$ 上的五次多项式, 则

$$P(x)=P(0)A_0(x)+P\left(\frac{1}{3}\right)A_1(x)+P(1)A_2(x)+P'(0)B_0(x)$$
$$+P'(1)B_1(x)+P''\left(\frac{1}{3}\right)C_1(x), \tag{1}$$

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其中

$$\left.\begin{aligned}&C_1(x)=\frac{9}{8}x^2(x-1)^2(3x-1), \quad B_2(x)=\frac{1}{8}x^2(x-1)(3x-1)(9x-5),\\&B_0(x)=x(x-1)^2(1-3x), \quad A_2(x)=x^2(3x-1)\left[\frac{1}{2}-\frac{7}{4}(x-1)-\frac{99}{16}(x-1)^2\right]\\&A_1(x)=\frac{81}{16}x^2(x-1)^2(1+9x), \quad A_0(x)=(x-1)^2(1-3x)(1+5x+9x^2)\end{aligned}\right\} \tag{2}$$

在[1]中还给出 $A_0(x)$, $A_1(x)$, $A_2(x)$, $B_0(x)$, $B_2(x)$, $C_1(x)$各阶导数在 $x=0$, $\frac{1}{3}$, 1 的数值.

满足定理1中条件的样条函数可表成

$$\begin{aligned}S_n(x)=&f(x_{2i})A_0\left(\frac{x-2ih}{2h}\right)+f(x_{2i+2})A_2\left(\frac{x-2ih}{2h}\right)+f(t_{2i})A_1\left(\frac{x-2ih}{2h}\right)\\&+2hS_n'(x_{2i})B_0\left(\frac{x-2ih}{2h}\right)+2hS_n'(x_{2i+2})B_2\left(\frac{x-2ih}{2h}\right)\\&+4h^2f''(t_{2i})C_1\left(\frac{x-2ih}{2h}\right) \quad (2ih<x<12i+2)h\end{aligned} \tag{3}$$

其中 $S_n'(x_{2i})$ 满足

$$\begin{aligned}&\frac{5}{2}hS_n'(x_{2i+2})+43hS_n'(x_{2i})-8hS_n'(x_{2i-2})=\frac{93}{4}f(x_{2i})+\frac{63}{8}f(x_{2i+2})\\&+60f(x_{2i-2})+\frac{81}{8}f(t_{2i})-\frac{405}{4}f(t_{2i-2})-9h^2f''(t_{2i})-18h^2f''(t_{2i-2}),\\&\qquad (i=1,2,\cdots,m-1)\end{aligned} \tag{4}$$

$$\begin{aligned}4h^2S_n''(x_{2i})=&-18f(x_{2i})+\frac{63}{8}f(t_{2i+2})+\frac{81}{8}f(t_{2i})-20hS_n'(x_{2i})\\&-\frac{5}{2}hS_n'(x_{2i+2})-9h^2f''(t_{2i}).\end{aligned} \tag{5}$$

当 $f(x)\in C^r[0,1]$ 时,

$$\left.\begin{aligned}&f^{(p)}(x_{2i+2})=f^{(p)}(x_{2i})+\sum_{k=p+1}^{r-1}\frac{f^{(k)}(x_{2i})}{(k-p)!}(2h)^k+\frac{f^{(r)}(\theta_{p,r,2i})}{(r-p)!}(2h)^{r-p},\\&f^{(p)}(x_{2i-2})=f^{(p)}(x_{2i})+\sum_{k=p+1}^{r-1}\frac{f^{(k)}(x_{2i})}{(k-p)!}(-2h)^k+\frac{f^{(r)}(\xi_{p,r,2i})}{(r-p)!}(-2h)^{r-p},\\&f^{(p)}(t_{2i})=f^{(p)}(x_{2i})+\sum_{k=p+1}^{r-1}\frac{f^{(k)}(x_{2i})}{(k-p)!}\left(\frac{2}{3}h\right)^k+\frac{f^{(r)}(\eta_{p,r,2i})}{(r-p)!}\left(\frac{2}{3}h\right)^{r-p}, \quad (0\leqslant p\leqslant r)\\&f^{(p)}(t_{2i-2})=f^{(p)}(x_{2i})+\sum_{k=p+1}^{r-1}\frac{f^{(k)}(x_{2i})}{(k-p)!}\left(-\frac{4}{3}h\right)^k+\frac{f^{(r)}(\zeta_{p,r,2i})}{(r-p)!}\left(-\frac{4}{3}h\right)^{r-p}\end{aligned}\right\} \tag{6}$$

2. 定理2的证明 首先指出, [1]中等式(原文中式(5.2))

$$\begin{aligned}S_n''(x)=&f''(x_{2k})A_0\left(\frac{x-2kh}{2h}\right)+f''(x_{2k+2})A_2\left(\frac{x-2kh}{2h}\right)+f''(t_{2k})A_1\left(\frac{x-2kh}{2h}\right)\\&+2hS_n'''(x_{2k}+)B_0\left(\frac{x-2kh}{2h}\right)+2hS_n'''(x_{2k+2}-)B_2\left(\frac{x-2kh}{2h}\right)\\&+4h^2S_n^{(4)}(t_{2k})C_1\left(\frac{x-2kh}{2h}\right)\end{aligned} \tag{*}$$

是不正确的.

事实上, 由于在 $x_{2k}\leqslant x\leqslant x_{2k+2}$ 中, $S_n''(x)$ 是一个次数低于 5 的多项式, 由 (1) 得

$$S_n''(x)=S_n''(x_{2k})A_0\left(\frac{x-2kh}{2h}\right)+S_n''((x_{2k+2})A_2\left(\frac{x-2kh}{2h}\right)+S_n''(t_{2k})A_1\left(\frac{x-2kh}{2h}\right)$$

$$+2hS_n'''(x_{2k}+)B_0\left(\frac{x-2kh}{2h}\right)+2hS_n'''(x_{2k+2}-)B_2\left(\frac{x-2kh}{2h}\right)$$

$$+4h^2S_n^{(4)}(t_{2k})C_1\left(\frac{x-2kh}{2h}\right),$$

而 $A_0(x)$, $A_1(x)$, $A_2(x)$, $B_0(x)$, $B_2(x)$, $C_1(x)$ 是线性无关的函数. 如果 (*) 式成立, 则必须有

$$S_n''(x_{2k})=f''(x_{2k}),\quad S_n''(x_{2k+2})=f''(x_{2k+2}),$$

而这是不可能的.

为了证明定理 2, 我们需要几个引理.

引理 1[1]　设 $f(x)\in C^2[0,1]$ 令

$$M_{2m}=\max_{1\leqslant i\leqslant m-1}|S_n'(x_{2i})-f'(x_{2i})|,$$

则

$$M_{2m}\leqslant\frac{138}{65m}\omega_2\left(\frac{1}{m}\right),$$

其中 $\omega_2(\delta)$ 是 $f''(x)$ 的连续模.

引理 2[1]　设 $f(x)\in C^2[0,1]$, 则

$$|S_n'''(x_{2i}+)|\leqslant\frac{102}{h}\omega_2\left(\frac{1}{m}\right),$$

$$|S_n'''(x_{2i}-)|\leqslant\frac{244}{h}\omega_2\left(\frac{1}{m}\right),$$

$$|S_n^{(4)}(t_{2i})|\leqslant\frac{70}{h^2}\omega_2\left(\frac{1}{m}\right).$$

今后, 我们用 C_t 表示仅依赖于 t 的常数.

引理 3　设 $f(x)\in C^2[0,1]$, 则

$$\max_{1\leqslant i\leqslant m-1}|S_n''(x_{2i})-f''(x_{2i})|\leqslant C_2\omega_2\left(\frac{1}{m}\right).$$

证　把 (6) 式代入 (5) 式得

$$4h^2S_n''(x_{2i})=20h[f'(x_{2i})-S_n'(x_{2i})]+\frac{5}{2}h[f'(x_{2i})-f'(x_{2i+2})]$$

$$+\frac{5}{2}h[f'(x_{2i+2})-S_n'(x_{2i+2})]+\frac{63}{4}h^2f''(\theta_{0,2,2i})$$

$$+\frac{9}{4}h^2f''(\eta_{0,2,2i})-9h^2f''(t_{2i})$$

$$=20h[f'(x_{2i})-S_n'(x_{2i})]+\frac{5}{2}h[f'(x_{2i+2})-S_n'(x_{2i+2})]$$

$$-5h^2f''(\theta_{1,2,2i})+\frac{63}{4}h^2f''(\theta_{0,2,2i})+\frac{9}{4}h^2f''(\eta_{0,2,2i})-9h^2f''(t_{2i}),$$

因此

$$4h^2[S_n''(x_{2i})-f''(x_{2i})]=20h[f'(x_{2i})-S_n'(x_{2i})]+\frac{5}{2}h[f'(x_{2i+2})-S_n'(x_{2i+2})]$$
$$+5h^2[f''(\theta_{0,2,2i})-f''(\theta_{1,2,2i})]+9h^2[f''(\theta_{0,2,2i})-f''(t_{2i})]$$
$$+\frac{7}{4}h^2[f''(\theta_{0,2,2i})-f''(x_{2i})]+2h^2[f''(\eta_{0,2,2i})-f''(x_{2i})]$$
$$+\frac{1}{4}h^2[f''(\eta_{0,2,2i})-f''(x_{2i})].$$

利用引理1及连续模的性质即可证得引理.

现在可以证明定理2. 在$[x_{2k}, x_{2k+2}]$中, $S_n''(x)$是一个三次多项式, 因此成立

$$S_n''(x)=S_n''''(x_{2k}+)\frac{(x_{2k+2}-x)^2(x-x_{2k})}{(2h)^2}-S_n''''(x_{2k+2}-)\frac{(x-x_k)^2(x_{2k+2}-x)}{(2h)^2}$$
$$+S_n''(x_{2k})\frac{(x-x_{2k+2})^2[2(x-x_{2k})+2h]}{(2h)^3}$$
$$+S_n''(x_{2k+2})\frac{(x-x_{2k})^2[2(x_{2k+2}-x)+2h]}{(2h)^3},$$

且
$$\frac{(x-x_{2k+2})^2[2(x-x_{2k})+2h]}{(2h)^3}+\frac{(x-x_{2k})^2[2(x_{2k+2}-x)+2h]}{(2h)^3}\equiv 1,$$

于是
$$S_n''(x)-f''(x)=S_n''''(x_{2k}+)\frac{(x_{2k+2}-x)^2(x-x_{2k})}{(2h)^2}-S_n''''(x_{2k+2}-)\frac{(x-x_{2k})^2(x_{2k+2}-x)}{(2h)^2}$$
$$+[S_n''(x_{2k})-f''(x_{2k})]\frac{(x_{2k+2}-x)^2[2(x-x_{2k})^2+2h]}{(2h)^3}$$
$$+[S_n''(x_{2k+2})-f''(x_{2k+2})]\frac{(x-x_{2k})^2[2(x_{2k+2}-x)+2h]}{(2h)^3}$$
$$+[f''(x_{2k})-f''(x)]\frac{(x_{2k+2}-x)^2[2(x-x_{2k})+2h]}{(2h)^3}$$
$$+[f''(x_{2k+2})-f''(x)]\frac{(x-x_{2k})^2[2(x_{2k+2}-x)+2h]}{(2h)^3}.$$

利用引理1、2、3, 以及连续模的性质可知

$$|S_n''(x)-f''(x)|\leqslant C_2\omega_2\left(\frac{1}{m}\right).$$

对于$p=0, 1$, 由通常方法可证.

我们把定理3改进成

定理4 设$f(x)\in C^4[0, 1]$, 则

$$\|S_n^{(p)}(x)-f^{(p)}(x)\|\leqslant C_p m^{p-4}\omega_4\left(\frac{1}{m}\right), \ (p=0, 1, 2, 3, 4)$$

这里$\omega_4(\cdot)$是$f^{(4)}(x)$的连续模.

证明还需要[1]中的一个引理.

引理4 设$f(x)\in C^4[0, 1]$, 则

$$|f'(x_{2i})-S_n'(x_{2i})|\leqslant\frac{272}{195}h^3\omega_4\left(\frac{1}{m}\right),$$

$$|S_n'''(x_{2i}+)-f'''(x_{2i})|\leqslant 35h\omega_4\left(\frac{1}{m}\right),\quad |S_n'''(x_{2i}-)-f'''(x_{2i})|\leqslant 82h\omega_4\left(\frac{1}{m}\right),$$

$$|S_n^{(4)}(x_{2i}+)-f'''(t_{2i})|\leqslant 78h\omega_4\left(\frac{1}{m}\right),\quad |S_n^{(4)}(x_{2i}-)-f^{(4)}(x_{2i})|\leqslant 165\omega_4\left(\frac{1}{m}\right),$$

$$|S_n''''(t_{2i})-f'''(t_{2i})|\leqslant 13h\omega_4\left(\frac{1}{m}\right),\qquad |S_n^{(5)}(t_{2i})|\leqslant\frac{47}{h}\omega_4\left(\frac{1}{m}\right).$$

3．定理 4 的证明　令

$$P(x)=3(x-1)\left(x-\frac{1}{3}\right),\quad Q(x)=\frac{3}{2}x\left(x-\frac{1}{3}\right),\quad R(x)=-\frac{9}{2}x(x-1).$$

容易验证，当 $x_{2k}\leqslant x\leqslant x_{2k+2}$ 时，对于二次多项式 $S_n''''(x)$ 成立着

$$S_n''''(x)=S_n''''(x_{2k}+)P\left(\frac{x-x_{2k}}{2h}\right)+S_n''''(x_{2k+2}-)Q\left(\frac{x-x_{2k}}{2h}\right)+S_n''''(t_{2k})R\left(\frac{x-x_{2k}}{2h}\right),$$

且
$$P\left(\frac{x-x_{2k}}{2h}\right)+Q\left(\frac{x-x_{2k}}{2h}\right)+R\left(\frac{x-x_{2k}}{2h}\right)\equiv 1,$$

$$(x-x_{2k})P\left(\frac{x-x_{2k}}{2h}\right)+(x-x_{2k+2})Q\left(\frac{x-x_{2k}}{2h}\right)+(x-t_{2k})R\left(\frac{x-x_{2k}}{2h}\right)\equiv 0.$$

因此
$$\begin{aligned}S_n''''(x)-f'''(x)&=[S_n''''(x_{2k}+)-f'''(x)]P\left(\frac{x-x_{2k}}{2h}\right)\\&\quad+[S_n''''(x_{2k+2}-)-f'''(x)]Q\left(\frac{x-x_{2k}}{2h}\right)\\&\quad+[S_n''''(t_{2k})-f'''(x)]R\left(\frac{x-x_{2k}}{2h}\right)\\&=[S_n''''(x_{2k}+)-f'''(x_{2k})]P\left(\frac{x-x_{2k}}{2h}\right)\\&\quad+[S_n''''(x_{2k+2}-)-f'''(x_{2k+2})]Q\left(\frac{x-x_{2k}}{2h}\right)\\&\quad+[S_n''''(t_{2k})-f'''(t_{2k})]R\left(\frac{x-x_{2k}}{2h}\right)\\&\quad+[f'''(x_{2k})-f'''(x)]P\left(\frac{x-x_{2k}}{2h}\right)\\&\quad+[f'''(x_{2k+2})-f'''(x)]Q\left(\frac{x-x_{2k}}{2h}\right)\\&\quad+[f'''(t_{2k})-f'''(x)]R\left(\frac{x-x_{2k}}{2h}\right)\\&=\Sigma_1+\Sigma_2,\end{aligned}$$

其中
$$\begin{aligned}\Sigma_1=&[S_n''''(x_{2k}+)-f'''(x_{2k})]P\left(\frac{x-x_{2k}}{2h}\right)+[S_n''''(x_{2k+2}-)\\&-f'''(x_{2k+2})]Q\left(\frac{x-x_{2k}}{2h}\right)+[S_n''''(t_{2k})-f'''(t_{2k})]R\left(\frac{x-x_{2k}}{2h}\right),\end{aligned}$$

由引理 4，得
$$|\Sigma_1|\leqslant C_3h\omega_4\left(\frac{1}{m}\right)\leqslant C_3m^{-1}\omega_4\left(\frac{1}{m}\right)$$

利用
$$\begin{aligned}&f'''(x)=f'''(x_{2k})+f^{(4)}(\zeta_{1,2k})(x-x_{2k}),\\&f'''(x)=f'''(x_{2k+2})+f^{(4)}(\zeta_{2,2k})(x-x_{2k+2}),\\&f'''(x)=f'''(t_{2k})+f^{(4)}(\zeta_{3,2k})(x-t_{2k}),\quad(x_{2k}\leqslant x\leqslant x_{2k+2})\end{aligned}$$

我们得到 $\Sigma_2=[f'''(x_{2k})-f'''(x)]P\left(\frac{x-x_{2k}}{2h}\right)+[f'''(x_{2k+2})-f'''(x)]Q\left(\frac{x-x_{2k}}{2h}\right)$

$$+[f'''(t_{2k})-f'''(x)]R\left(\frac{x-x_{2k}}{2h}\right)=-f^{(4)}(\zeta_{1,2k})(x-x_{2k})P\left(\frac{x-x_{2k}}{2h}\right)$$

$$-f^{(4)}(\zeta_{2,2k})(x-x_{2k+2})Q\left(\frac{x-x_{2k}}{2h}\right)-f^{(4)}(\zeta_{3,2k})(x-t_{2k})R\left(\frac{x-x_{2k}}{2h}\right)$$

$$=[f^{(4)}(x_{2k})-f^{(4)}(\zeta_{1,2k})](x-x_{2k})P\left(\frac{x-x_{2k}}{2h}\right)$$

$$+[f^{(4)}(x_{2k})-f^{(4)}(\zeta_{2,2k})](x-x_{2k+2})Q\left(\frac{x-x_{2k}}{2h}\right)$$

$$+[f^{(4)}(x_{2k})-f^{(4)}(\zeta_{3,2k})](x-t_{2k})R\left(\frac{x-x_{2k}}{2h}\right).$$

利用连续模的性质，得

$$|\Sigma_2|\leqslant C_3h\omega_4\left(\frac{1}{m}\right).$$

由是 $\quad |S_n'''(x)-f'''(x)|\leqslant|\Sigma_1|+|\Sigma_2|\leqslant C_3h\omega_4\left(\frac{1}{m}\right)\leqslant C_3m^{-1}\omega_4\left(\frac{1}{m}\right).$

利用罗尔定理可以证得当 $p=0, 1, 2$ 时，有

$$\|S_n^{(p)}(x)-f^{(p)}(x)\|\leqslant C_pm^{p-4}\omega_4\left(\frac{1}{m}\right).$$

对于 $p=4$，由引理 4，得

$$|S_n^{(4)}(x_{2i}+)-f^{(4)}(x_{2i})|\leqslant 78\omega_4\left(\frac{1}{m}\right),\quad |S_n^{(4)}(x_{2i}-)-f^{(4)}(x_{2i}-)|\leqslant 165\omega_4\left(\frac{1}{m}\right).$$

并注意到当 $x_{2k}\leqslant x\leqslant x_{2k+2}$ 时，$S_n^{(4)}(x)$ 是一线性函数，因此

$$|S_n^{(4)}(x)-S_n^{(4)}(x_{2k}+)|\leqslant|S_n^{(4)}(x_{2k+2}-)-S_n^{(4)}(x_{2k}+)|$$

$$\leqslant|S_n^{(4)}(x_{2k+2}-)-f^{(4)}(x_{2k+2})|+|f^{(4)}(x_{2k+2})-f^{(4)}(x_{2k})|+|f^{(4)}(x_{2k})-S_n^{(4)}(x_{2k}+)|$$

$$\leqslant C_4\omega_4\left(\frac{1}{m}\right),$$

$$|S_n^{(4)}(x)-f^{(4)}(x)|\leqslant|S_n^{(4)}(x)-S_n^{(4)}(x_{2k}+)|+|S_n^{(4)}(x_{2k}+)-f^{(4)}(x_{2k})|$$

$$+|f^{(4)}(x_{2k})-f^{(4)}(x)|\leqslant C_4\omega_4\left(\frac{1}{m}\right).$$

证毕.

4. 进一步的推广 用定理 2，定理 4 类似方法可以证得

定理 5 设 $f(x)\in C^3[0, 1]$，则

$$\|f^{(p)}(x)-S_n^{(p)}(x)\|\leqslant C_pn^{p-3}\omega\left(f''';\ \frac{1}{n}\right),\quad p=0, 1, 2, 3.$$

定理 6 设 $f(x)\in C^5[0, 1]$，则

$$\|f^{(p)}(x)-S_n^{(p)}(x)\|\leqslant C_pn^{p-5}\omega\left(f^{(5)};\ \frac{1}{n}\right),\quad p=0, 1, 2, 3, 4, 5.$$

我们还可用二阶连续模来估计收敛速度.

定理 7 设 $f(x)\in C^r[0, 1]\ (2\leqslant r\leqslant 4)$，则

$$\|S_n^{(p)}(x)-f^{(p)}(x)\|\leqslant C_{p,r}\omega_2\left(f^{(r)};\ \frac{1}{m}\right),\quad (0\leqslant p\leqslant r)$$

其中 $\omega_2(f^{(r)};\delta)=\sup\limits_{\substack{x,x\pm h\in[0,1]\\|h|\leqslant\delta}}|f^{(r)}(x+h)+f^{(r)}(x-h)-2f^{(r)}(x)|$ 是 $f^{(r)}(x)$ 的二阶连续模.

为了证明,我们需要几个引理.

引理 5　设 $f(x)$ 是以 2π 为周期的周期连续函数, $S_{n,m}(x, f; \Delta_m)$ 是以 Δ_m: $0=x_0^{(m)}<x_1^{(m)}<\cdots<x_m^{(m)}=2\pi$ 为样条节点的 n 次样条函数. 如果满足

(i) 对一切 $f\in C_{2\pi}$, 当 $\|\Delta_m\|=\max\limits_{1\leqslant i\leqslant m}(x_i^{(m)}-x_{i-1}^{(m)})\to 0$ 时, $\|f(x)-S_{n,m}(x, f; \Delta_m)\|=0(1)$,

(ii) 当 $\|f^{(n+1)}(x)\|\leqslant M$ 时, 有

$$\|f(x)-S_{n,m}(x, f; \Delta_m)\|\leqslant C\|\Delta_m\|^{n+1}\|f^{(n+1)}(x)\|,$$

则对一切 $f\in C_{2\pi}$,

$$\|f(x)-S_{n,m}(x, f; \Delta_m)\|=O(\omega_{n+1}(x, f; \|\Delta_m\|)),$$

这里 $\omega_{n+1}(f; \delta)$ 是 $f(x)$ 的 $n+1$ 阶连续模.

证明过程类似于[4]中定理 5, 不再赘述.

引理 6[5]　设 $f(x)\in C[0, 1]$, 则周期函数

$$g(x)=\begin{cases} g(-x), & x\in[-1, 0],\\ g(x-k), & x\in[k, k+1],\ k=\pm1,\ \pm2,\ \cdots. \end{cases}$$

满足 $\omega_2(g; \delta)\leqslant 5\omega_2(f; \delta)$.

利用引理 5、引理 6 以及定理 2、4、6 容易得到定理 7.

最后,我们叙述 $S_n(x)$ 逼近 $f(x)$ 的逆命题和饱和问题. 我们需要[3]中的定理.

定理 7　设 $\varepsilon_n\searrow 0$, $S_{k,n}(x)$ 是以 Δ_n: $x_\nu^{(n)}=\dfrac{\nu}{n}$, $\nu=0, 1, \cdots, n$ 为节点的逐段 k 次多项式,若

$$\|S_{k,n}(x)-f(x)\|\leqslant\varepsilon_n,$$

则

$$\omega_{k+1}\left(f; \frac{1}{n}\right)=O(\varepsilon_n).$$

应用到本文讨论的缺插值样条函数 $S_n(x)$, 我们得到

定理 8　设 $\varepsilon_n\searrow 0$, 若对某一 $p(=0, 1, 2, 3, 4, 5)$

$$\|S_n^{(p)}(x)-f^{(p)}(x)\|\leqslant\varepsilon_n,$$

则

$$\omega_{6-p}\left(f^{(p)}; \frac{1}{n}\right)=O(\varepsilon_n),$$

如对某一 p,

$$\|S_n^{(p)}(x)-f^{(p)}(x)\|=O(n^{p-6}),$$

则 $f(x)$ 是一五次多项式.

Varma 在[2]中讨论的二种缺插值样条函数也有同样问题,这里不再重复.

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ON VARMA'S LACUNARY INTERPOLATION BY SPLINES

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ABSTRACT

In articles [1], [2], A. K. Varma investigated some lacunary interpolations by spline. In this article we point out a mistake in the proof of Theorem 2 in [1] and give a new proof. Moreover, the results obtained in [1], [2] are improved.