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ON THE RIEMANNIAN SPACES ADMITTING PARALLEL YANG-MILLS FIELDS

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1. Introduction

A Yang–Mills field is called parallel if all gauge derivatives of its field strength are zero. Parallel gauge fields are generalizations of the symmetric spaces in differential geometry and are special solutions of the Yang–Mills equations. In the 2–dimensional case all solutions of the Yang–Mills equations are parallel. They can be determined completely. In the present paper we consider the 4–dimensional case. The main results are the following two theorems.

Theorem 1 *A 4–dimensional Riemannian space admitting nontrivial parallel Yang–Mills fields must be locally Kählerian or half–symmetric.*

Here a half–symmetric manifold is defined as a Riemannian manifold satisfying

$$R^-_{ijkl;m}=0\,(\text{or } R^+_{ijkl;m}=0), \tag{1}$$

where $R^{\mp}_{ijkl}$ are the anti-self-dual and self-dual parts of the curvature tensor respectively[1], and the semicolon is the symbol for covariant derivatives.

Theorem 2 *A half–symmetric space which is not symmetric must be a Kähler Einstein space or a conformally half–flat Einstein space.*

Here the conformal half-flatness means that the anti-self-dual or self-dual part of the Weyl tensor vanishes.[2]

2. The abelian case

Let M be a 4–dimensional Riemannian manifold and F a Yang–Mills field over M. We use moving orthonormal frames to express all geometrical and physical quantities. Let G be the gauge group and g its Lie algebra. For a parallel Yang–Mills field we have

$$f_{ij\,|\,k}=0, \tag{2}$$

where f_{ij} is the field strength and "$|$" the symbol for gauge derivatives.

If G is Abelian, then (2) is reduced to

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$$f_{ij;k}=0. \tag{3}$$

Consequently, M admits a parallel skew symmetric tensor field. Let $\pm\lambda_1 i$, $\pm\lambda_2 i$ be the eigenvalues of f_{ij}. It is obvious that if $|\lambda_1|\neq|\lambda_2|$, then the local holonomy[3] group of M keeps two 2-dimensional tangent subspaces unchanged and that if $|\lambda_1|=|\lambda_2|\neq 0$, then there exists a parallel tensor field

$$(J_{ij})=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

provided the frames are suitably chosen. In both cases M admits Kählerian structure[4], at least locally.

Remark It is easily seen that in the 2-dimensional case the Yang-Mills equations are equivalant to (2) and can be solved completely. The field strength takes value in an Abelian subalgebra and equals to the area form of the Riemannian metric up to a constant factor such that. the quantization condition be satisfied, or equivalently, the first Chern number should be an integer, if the manifold is compact and oriented.

3. The proof of theorem 1

For the general cases we recall the generalized Ricci identity[5] for a g-valued skew symmetric tensor ϕ_{ij}

$$\phi_{ij|kl}-\phi_{ij|lk}=R_{hikl}\phi_{hj}+R_{hjkl}\phi_{ih}+[f_{kl},\ \phi_{ij}]. \tag{4}$$

Replacing ϕ_{ij} by f_{ij} and using (2) we obtain

$$R_{hikl}f_{hj}+R_{hjkl}f_{ih}+[f_{kl},\ f_{ij}]=0. \tag{5}$$

The differentiation of (5) gives

$$f_{ih}R_{hjkl;m}-R_{ihkl;m}f_{hj}=0. \tag{6}$$

Let P be an arbitrary point and Σ the subalgebra of so_4 generated by the set of 4×4 skew symmetric matrices: $\{(\langle f_{ij},\ a\rangle)\,|\,a\in g\}$. Here $\langle f_{ij},\ a\rangle$ is the invariant inner product of $f_{ij}(P)$ and a.

Suppose that P is a generic point in the sense that Σ is of maximal dimension. If Σ is so_3 and keeps a 3-dimensional plane unchanged, from (6) we see that

$$R_{hjkl;m}=0. \tag{7}$$

Consequently, the space is a symmetric space. If Σ has two invariant 2-dimensional planes, then (f_{ij}) takes value in an Abelian subalgebra of g. From (5) we have

$$f_{ih}R_{hjkl}-R_{ihkl}f_{hj}=0. \tag{8}$$

By differentiation it is seen that all the covariant derivatives of the curvature tensor satisfy the equations which are analogous to (8). So the holonomy group has two invariant 2-planes and the case has been considered in 2. If Σ is so_4 or a 4-dimensional subalgebra, then M_4 is symmetric, If Σ is an irreducible representation of su_2 in R^4, then f_{ij} must be self-dual or anti-self-dual. Without loss of generality we suppose that f_{ij} is anti-self-dual. From (6) it is easy to obtain (1). So the space is half-symmetric. Noting that a symmetric space is also a half-symmetric space, we have proved theorem 1.

4. The proof of theorem 2

Now let M be a half-symmetric space with $R^{-}_{ijkl;m}=0$. Differentiating these equations and using the Ricci identity we obtain

$$R^{-}_{hjkl}R_{hipq}+R^{-}_{ihkl}R_{hjpq}+R^{-}_{ijhl}R_{hkpq}+R^{-}_{ijkh}R_{hlpq}=0. \tag{9}$$

Differentiating again and using (1) we see that

$$R^{-}_{hjkl}R^{+}_{hipq;r}+R^{-}_{ihkl}R^{+}_{hjpq;r}+R^{-}_{ijhl}R^{+}_{hkpq;r}+R^{-}_{ijkh}R^{+}_{hlpq;r}=0. \tag{10}$$

Since a 4×4 self-dual skew symmetric matrix commutes with any 4×4 anti-self-dual skew symmetric matrix, from (10) it follows that

$$R^{-+}_{ijhl}R^{+}_{hkpq;r}+R^{-+}_{ijkh}R^{+}_{hlpq;r}=0, \tag{11}$$

where R^{-+}_{ijkl} is the self-dual part of R^{-}_{ijkl} with respect to indices k, l. If $R^{-+}_{ijpq}\neq0$ and $R^{+}_{klpq;r}\neq0$, then the two sets of matrices (R^{-+}_{ijkl}) and $(R^{+}_{klqq;r})$ with k, l as row and column indices commute with each other. Consequently

$$R^{-+}_{ijkl}=A_{ij}B_{kl},$$
$$R^{+}_{ijpq;r}=B_{ij}B_{pq}\sigma_r, \tag{12}$$

where (B_{kl}) is self-dual. Now the Bianchi identity is reduced to

$$B_{pq}\sigma_r+B_{qr}\sigma_p+B_{rp}\sigma_q=0, \tag{13}$$

or equivalently

$$B_{pq}\sigma_q=0, \tag{13}$$

since (B_{pr}) is self-dual. It is easily seen that $\det(B_{pq})\neq0$ if $(B_{pq})\neq0$. From (13) it follows that $\sigma_q=0$. Hence $R^{+}_{ijkl;r}=0$, which is contradictory to the hypothesis, so we obtain the conclusion that if M is not a symmetric space, then we must have $R^{+-}_{ijkl}=0$, and hence M must be an Einstein space.

Let M be non-symmetric. The matrix (R^{--}_{ijkl}), with (i, j) ane (k, l) as row and column indices respectively, represents a symmetric linear transformation L on the space of anti-self-dual skew symmetric 2-tensors. If the eigenvalues are not equal, then there should be a parallel anti-self-dual tensor field. Consequently, the space is Kählerian. Conversely, for a Kähler Einstein space there is an orthonormal frame at each point, such that the anti-self-dual part of the connection is proportional to the coefficients of the fundamental 2-form. Hence

$$R^{-}_{ijkl;m}=R^{--}_{ijkl;m}=A_{ij}A_{kl}\sigma_m, \tag{14}$$

where (A_{ij}) is anti-self-dual. From Bianchi identity we obtain that the space is half-symmetric.

If the eigenvalues of L are equal, then $(R_{\bar{i}\bar{j}\bar{k}\bar{l}})$ is diagonal, i. e., the anti-self-dual part of the Weyl tensor $W_{\bar{i}\bar{j}\bar{k}l}=0$. Then the space is a conformally self-dual Einstein space. Conversely, it is known that the curvature of a conformally half-flat Einstein space satisfies

$$\left\{R_{ijkl}+\frac{R}{12}(g_{ik}g_{jl}-g_{il}g_{jk})\right\}^{-}=0, \tag{15}$$

where the minus sign is the symbol for the anti-self-dual part with respect to the indices i, j. From (15) it is easily seen that the space is half-symmetric. Theorem 2 is proved.

Remark The Kähler Einstein spaces with $R=0$ are very interesting in physics[6]. In the compact case the existence of such a metrics is a consequence of the famous theorem of Yau[7]. Besides the Kähler Einstein space with $R=0$, the known conformally half-flat Einstein spaces are Kähler spaces with constant holomorphic section curvature, but they are symmetric spaces. It is desirable to find other conformally half-flat Einstein spaces. It was noticed that they should not be Kählerian.

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Appendix

In [8] some results on 2-dimensional Yang-Mills equations, from the point of view of Morse theory, were announced. We note that the global solutions of the 2-dimensional Yang-Mills equations can be obtained directly.

a. First we construct local solutions. Near an arbitrary point p of M_2 there exists a local coordinate system (u_1, u_2) such that the metric is in the form

$$ds^2=du_1^2+a(u_1, u_2)du_2^2 \quad (|u_\lambda|<1,\ \lambda=1, 2). \tag{A}$$

There exists a special gauge such that

$$b_1(u_1, u_2)=0, \quad b_2(0, u_2)=0. \tag{B}$$

The Yang-Mills equations become

$$f_{21,1}-\frac{1}{2}f_{21}\frac{a_{,1}}{a}=0,$$

$$f_{12,2}-\frac{1}{2}f_{12}\frac{a_{,2}}{a}+[b_2, f_{12}]=0. \tag{C}$$

Here the comma is the notation for partial derivatives and $f_{21}=-f_{12}=b_{2,1}$. Using (B) we can solve (C) explicitly

$$b_1=0,\quad b_2=c\sigma(u),\quad f_{12}=c\sqrt{a},\tag{D}$$

where c is an arbitrary element of the Lie algebra g and

$$\sigma(u_1,\ u_2)=-\int_0^{u_1}\sqrt{a(\tau,\ u_2)}\,d\tau.\tag{E}$$

In a general coordinate system the local solution is expressed as

$$b_\lambda=ch_\lambda,\tag{F}$$

$$f=\frac{1}{2}f_{\lambda\mu}dx^\lambda\wedge dx^\mu=c\sqrt{g}\,dx^1\wedge dx^2=c\cdot\text{area element}.$$

b. From the above result we see that there exists a global solution of the Yang–Mills equations on M_2 which may be represented as follows. M_2 is covered by a system of neighborhoods $M_2=UU_\alpha$, in U_α we have

$$\underset{\alpha}{b_\lambda}=\underset{\alpha\alpha}{ch_\lambda},\quad \underset{\alpha}{f}=\underset{\alpha}{c}\sqrt{g}\,dx^1\wedge dx^2\tag{G}$$

and for non-empty $U_\alpha\cap U_\beta$ there are relations

$$\underset{\beta}{c}=(ad\zeta_{\beta\alpha})\underset{\alpha}{c},\tag{H}$$

where $\zeta_{\beta\alpha}$ are transition functions.

Let M_2 be connected. It is easily seen that

$$c_\beta=(ad\zeta_\beta)c,$$

where c is a certain c_α and for each β, ζ_β is a fixed element of G. By the gauge transformation in U_β via ζ_β we obtain

$$\underset{\beta}{b_\lambda}=\underset{\beta}{h_\lambda}c,\quad \underset{\beta}{f}=c\sqrt{g}\,dx^1\wedge dx^2.\tag{I}$$

If M_2 is compact, then c satisfies the following quantization condition

(1) c generates a compact subgroup of G,

(2) the 1st Chern number is an integer, i.e.

$$\exp(Ac)=\text{unit element } e,$$

or equivalently,

$$c=kc_1,\quad \frac{kA}{2\pi}=\text{integer}.\tag{J}$$

Here A is the area of M_2 and c_1 is an element of g, satisfying

$$\exp(2\pi c_1)=e,\quad \exp(2\pi\lambda c_1)\neq e \text{ if } (i<|\lambda|<1).\tag{K}$$

In any case there exists a solution which is reducible to an Abelian gauge field with the gauge group $\{\exp(tc)\}$[9,10]. The element c can be determined by the field strength at one point. From the element c and a homomorphism of the group $\Pi_1(M_2)$ to $H=\{\alpha\in G\,|\,(ad\alpha)c=c\}$ more solutions can be constructed.

c. *Remarks*

(1) If M_2 is the sphere S^2 with the metric $ds^2=d\theta^2+\sin^2\theta\,d\phi^2$, the solutions are standard monopole solutions[5]

$$b_1^N=0,\ b_2^N=c(\cos\theta-1)\quad(0\leqslant\theta<\pi),\tag{L}$$

$$b_1^S=0,\ b_2^S=c(\cos\theta+1)\quad(0<\theta\leqslant\pi),$$

and

$$\exp(4\pi c)=e.$$

(2) If G is U_1, for a principal U_1 bundle over a compact and oriented M_2 each solution of the Yang–Mills equation minimizes the Yang–Mills functional. If G is non-abelian there are critical values of the Yang–Mills functional.

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容有平行 Yang–Mills 场的黎曼空间

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摘 要

如果一个 Yang–Mills 场(规范群为任意李群)的场强的所有规范导数均为 0, 则称这个场为平行的 Yang–Mills 场. 平行规范场是微分几何中对称空间的推广, 它是 Yang-Mills 方程的特解.

本文的主要结果是下列两个定理:

定理 1 容有非平凡的平行 Yang–Mills 场的四维黎曼空间必须是 Kähler 流形或半对称空间. 这里半对称流形是满足

$$R^-_{ijkl;m}=0 \quad (\text{或 } R^+_{ijkl;m}=0)$$

的黎曼流形, 其中 $R^{\pm}_{ijkl}$ 分别是曲率张量的自对偶部份及反自对偶部份, 而"; "表示共变导数.

定理 2 半对称空间如果不是对称空间, 则必为 Kähler–Einstein 空间或共形半平坦 Einstein 空间. 这里共形半平坦是指 Weyl 张量的反自对偶部份或自对偶部份为 0.

在附录中作者给出了二维黎曼流形上 Yang–Mills 方程的所有的整体解.