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SOME INTRINSIC INVARIANTS OF A PARAMETRIC CURVE IN AFFINE HYPERSPACE

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§ 1 Introduction

In a previous paper[1] the author has determined certain invariants of a parametric curve in affine plane. The purpose of the present paper is to generalize the method to affine hyperspace $\mathscr{A}^m$ of m (>2) dimensions.

Let the parametric equations of a curve of degree n in $\mathscr{A}^m$ be

(E) $$\boldsymbol{x}=\sum_{i=0}^{n}\frac{1}{i!}\boldsymbol{a}_i t^i,$$

where
$$\boldsymbol{x}=(x_p),\qquad p=1,2,\cdots,m,$$
$$\boldsymbol{a}_i=(a_{pi}),\qquad i=0,1,\cdots,n$$

and
$$2<m<n.$$

In what follows, we denote the determinant of order n by

$$\det|\boldsymbol{a}_{i_1},\ \boldsymbol{a}_{i_2},\ \cdots,\ \boldsymbol{a}_{i_m}|=p_{i_1,i_2,\cdots,i_m}$$

and assume that

$$p_{n-m+1,\,n-m+2,\,\cdots,\,n-1,\,n}\neq 0,$$

and, furthermore, that in the last determinant all the diagonal cofactors from below of the orders 1, 2, $\cdots$, $n-1$ are different from zero. Then there exists a unique regular affine transformation, called the *canonical affinity*:

A^*: $$\boldsymbol{x}\to\boldsymbol{x}^*$$

with $J^*\equiv\det|A^*|\neq 0$, so also a canonical parameter transformation:

T^*: $$t=t^*-R,$$

where we have put

(1.1) $$R=p_{n-m,\,n-m+2,\,\cdots,\,n}/p_{n-m+1,\,n-m+2,\,\cdots,\,n},$$

such that the equations in (E) are reduced to the *canonical* form:

(E^*) $$x^*_p=\sum_{i=1}^{n-m+p}\frac{1}{i!}a^*_{p,i}t^{*i},$$

where

(1.2) $$a^*_{p,\,n-m+p}=(n-m+p)!,$$
$$a^*_{1,\,n-m}=0\quad(p=1,2,\cdots,m).$$

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This new system (E^*) will be called the *associate* system of (E), and is characterized by two facts:

(1) The highest degrees in t^* of the 1st, 2nd, $\cdots$, mth coordinates of $\boldsymbol{x}^*$ are equal to $n-m+1$, $n-m+2$, $\cdots$, n respectively. (2) The coefficient of the term in x_1^* of second to the highest degree is equal to zero.

We shall demonstrate the following

Theorem *A parametric curve of degree n in m-dimensional affine space $\mathscr{A}^m$ with $n>m>2$ possesses, in general, $m(n-m)-2$ intrinsic affine invariants.*

§ 2 The Pan-Infexion Equation

As an extension of the inflexion equation of a plane curve, we now consider in $\mathscr{A}^m$ the equation

(2.1) $$D(t)\equiv\det\left|\frac{d\boldsymbol{x}}{dt},\ \frac{d^2\boldsymbol{x}}{dt^2},\ \cdots,\ \frac{d^m\boldsymbol{x}}{dt^m}\right|=0$$

and call it the *pan-inflexion equation* of the curve defined by the system (E).

Obviously, the equation (2.1) is covariantly connected with respect to any non-singular affinity A:

(2.2) $$\bar{x}_r=\sum a_{rs}x_s+a_r,\quad r=1,\ 2,\ \cdots,\ m,$$

where

(2.3) $$J\equiv\det|a_{rs}|\neq 0,$$

and any linear transformation T of parameter t:

(2.4) $$t=c\bar{t}+f\quad (c\neq 0).$$

Accordingly, the equation (2.1) can be rewritten in the form:

(F^*) $$D^*(t^*)\equiv\det\left|\frac{d\boldsymbol{x}^*}{dt^*},\ \frac{d^2x^*}{dt^{*2}},\ \cdots,\ \frac{d^mx^*}{dt^{*m}}\right|=0.$$

As the latter is of $N\equiv m(n-m)$ degrees, we obtain

(2.5) $$\sum_{k=0}^{N}\frac{1}{k!}g_k^*t^{*k}=0,$$

where we have in particular

(2.6) $$g_0^*=\begin{vmatrix} a_{1,1}^* & a_{2,1}^* & \cdots & a_{m,1}^* \\ a_{1,2}^* & a_{2,2}^* & \cdots & a_{m,2}^* \\ \cdots & \cdots & & \cdots \\ a_{1,m}^* & a_{2,m}^* & \cdots & a_{m,m}^* \end{vmatrix}.$$

Suppose that $g_0^*\neq 0$, then the equation (2.5) takes the canonical form as follows:

(2.7) $$1+\sum_{k=1}^{N}\frac{1}{k!}G_kt^{*k}=0$$

with the coefficients

(2.8) $$G_k=g_k^*/g_0^*,\qquad k=1,\ 2,\ \cdots,\ N.$$

In the 4th section we shall demonstrate that

$$(2.9)\qquad \begin{aligned} g^*_{m(n-m)} &= \frac{n!\,[m(n-m)]!}{(n-m)!}\prod_{p=1}^{m}(m-p)!, \\ g^*_{m(n-m)-1} &\equiv 0, \end{aligned}$$

so that $G_{N-1}\equiv 0$, and therefore we have $N-1$ G_k's only. It is evident from (2.1) that these G_k's are affine invariants with respect to A. In the next section we are going to show that they are relative invariants with respect to T.

§ 3 Proof of the Theorem

In order to show this, we have to calculate certain coefficients in the transformed equations

$$(3.1)\qquad \bar{x}_r \equiv \sum_{j=0}^{n} \frac{1}{j!}\,\bar{a}_{r,j}\bar{t}^j,$$

when the original ones (E) are subjected to A and T simultaneously.

A simple calculation gives

$$(3.2)\qquad \begin{aligned} \bar{a}_{r,j} &= c^j \sum_{s=1}^{m} \alpha_{rs} \sum_{i=j}^{n} \frac{1}{(i-j)!}\,a_{s,i} f^{i-j} + \delta_{0j}\alpha_r, \\ j &= 0, 1, \cdots, n, \\ r &= 1, 2, \cdots, m, \end{aligned}$$

where $\delta_{0j}=1$ or 0 according as $j=0$ or $\neq 0$.

In the following we need merely to evaluate $\bar{a}_{r,j}$ for $j=1, 2, \cdots, n$, and therefore assume that

$$i, j, l = 1, 2, \cdots, n,$$
$$p, q, r = 1, 2, \cdots, m.$$

Putting

$$(3.3)\qquad A_{s,j} = \sum_{i=j}^{n} \frac{1}{(i-j)!}\,a_{s,i} f^{i-j},$$

we are led to the relations

$$(3.4)\qquad \bar{a}_{r,j} = c^j \sum_{s=1}^{m} \alpha_{rs} A_{s,j},$$

or in matrix form

$$(3.5)\qquad (\bar{a}_{r,j}) = c^j (\alpha_{rs})(A_{s,j}),$$

the left side being an $m\times n$ matrix.

From (3.4) and the definition of $\bar{p}_{j_1, j_2, \cdots, j_m}$ we easily conclude that

$$(3.6)\qquad \bar{p}_{j_1, j_2, \cdots, j_m} = c^j \cdot J \cdot P_{j_1, j_2, \cdots, j_m},$$

where $j=\sum_{r=1}^{m} j_r$, and

$$(3.7)\qquad P_{j_1, j_2, \cdots, j_m} \equiv \det\,|A_{s,j_1} A_{s,j_2} \cdots A_{s,j_m}|.$$

We have now to compute two of them, namely,

$$\bar{p}_{n-m+1,\, n-m+2,\, \cdots,\, n} \quad \text{and} \quad \bar{p}_{n-m,\, n-m+2,\, \cdots,\, n}.$$

To this end, (3.3) is utilized to derive the following A's:

$$A_{s,n}=a_{s,n},$$
$$A_{s,n-1}=a_{s,n-1}+a_{s,n}f,$$
$$A_{s,n-2}=a_{s,n-2}+a_{s,n-1}f+\frac{1}{2!}a_{s,n}f^2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$A_{s,n-m+1}=a_{s,n-m+1}+a_{s,n-m+2}f+\cdots+\frac{1}{(m-1)!}a_{s,n}f^{m-1},$$
$$A_{s,n-m}=a_{s,n-m}+a_{s,n-m+1}f+\cdots+\frac{1}{m!}a_{s,n}f^{m}.$$

Hence we arrive at the relations

$$\bar{p}_{n-m+1,\, n-m+2,\, \cdots,\, n}=c^{\rho}\cdot J\cdot P_{n-m+1,\, n-m+2,\, \cdots,\, n},$$
$$\bar{p}_{n-m,\, n-m+2,\, \cdots,\, n}=c^{\rho-1}\cdot J\cdot\{P_{n-m,\, n-m+2,\, \cdots,\, n}+fP_{n-m+1,\, n-m+2,\, \cdots,\, n}\}$$

with

$$\rho=\frac{1}{2}m(2n-m+1).$$

Let $\bar{R}$ be the corresponding expression of R as defined by (1.1). The above two equations give immediately the important equality

$$\bar{R}=\frac{1}{c}(R+f). \tag{3.8}$$

Just as we have used t^* to denote the canonical parameter of t, $\bar{t}^*$ is utilized to denote the corresponding one to $\bar{t}$, that is,

$$t=t^*-R,\quad \bar{t}=\bar{t}^*-\bar{R}. \tag{3.9}$$

Substituting these expressions into (2.4) and reducing by means of (3.8), we obtain finally

$$\bar{t}^*=\frac{1}{c}t^*. \tag{3.10}$$

There is no difficulty in proving the relative invariant property of each G in (2.7). In fact, let

$$1+\sum_{r=1}^{N}\bar{G}_r\bar{t}^{*r}=0 \tag{3.11}$$

be the canonical equation corresponding to (2.7). If (3.10) is substituted into (3.11), then we get

$$\bar{G}_r=c^rG_r. \tag{3.12}$$

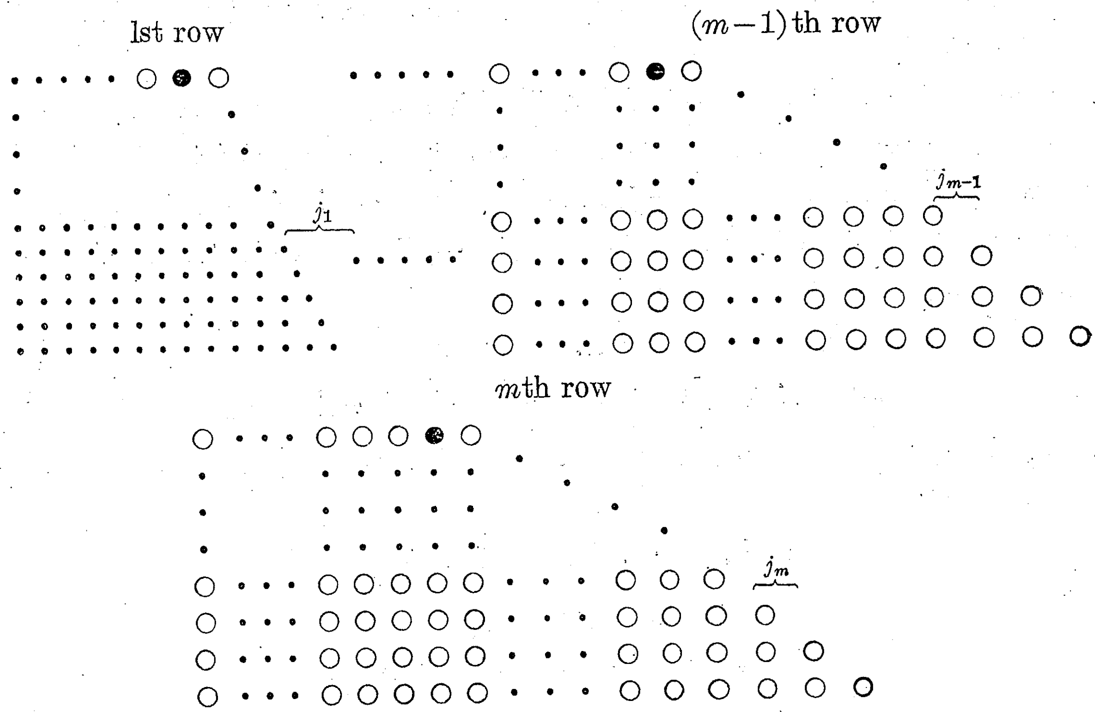
In general case $G_1\neq 0$. We have then $N-2$ *intrinsic* affine invariants as given by

$$I_r=G_r/(G_1)^r, \tag{3.13}$$
$$r=2,\cdots,N-2,N.$$

Thus we have completed the proof of our theorem stated in the introduction.

§ 4 Evaluation of g_N^* and g_{N-1}^*

It remains for us to demonstrate the validity of (2.9). For this purpose we express the m rows of the determinant D^* in the left side of (F^*) symbolically as follows (see figure):



In each row a series of small circles on one and the same column is used to denote a polynomial in t^* such that the degree of each term reaches the highest at the right end, and becomes 0 at the left. In every top series we have described a black circle to denote a lacunary term, since $a_{1,n-m}^*=0$. Take, for example, the mth row. The mth column of this row is composed of all the terms of $d^m x_m^*/(dt^*)^m$, and the degrees in t^* from the right to the left are $n-m$, $n-m-1$, $\cdots$, 0, respectively. Moreover, each term on the same verticle line of this row is of the same degree in t^*. We take one verticle, the j_mth line from the right, say; then each term on it is of degree $n-m-j_m+1$. Here j_m runs over the range 1, 2, $\cdots$, m. In this verticle only the j_m terms counted from the bottom are different from zero, the others ($m-j_m$ in total) being lacunary. The top term is found to be

$$(4.1)\qquad T_{m,j_m}=\frac{(n-j_m+1)\,!}{(n-m-j_m+1)\,!}\,t^{*n-m-j_m+1}.$$

When j_m runs over the range 1, 2, $\cdots$, m, corresponding non-zero terms form an equilateral triangle of the hypotenuse composed of m small circles, as shown in the above figure.

This illustration is applicable to any row of D^*, so that we obtain the non-zero top term on the j_μth verticle, counted from the right, as

$$T_{\mu, j_\mu}=\frac{(n-j_\mu+1)!}{(n-\mu-j_\mu+1)!}t^{*n-\mu-j_\mu+1} \quad (\mu=1, 2, \cdots, m). \tag{4.2}$$

It should be noted that under the assumption $j_\mu=j_\nu$, but $\mu\neq\nu$, the corresponding terms of the j_μth and the j_ν verticles on one and the same column of D^* must be proportional, and therefore any determinant of order m containing these two verticles vanishes identically.

Now we notice that the terms in D^* of the highest degree appear when and only when we select those verticles from each corresponding equilateral triangle of the rows above stated, and have to render $j_\mu\neq j_\nu$ for $\mu\neq\nu$. Hence $(j_1, j_2, \cdots, j_m)$ must be one permutation of $(1, 2, \cdots, m)$. If we use $\sigma(j_1, j_2, \cdots, j_m)$ to denote $+1$ or -1 according as the substitution

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_{m-1} & j_m \\ m+1-j_1 & m+1-j_2 & \cdots & m+1-j_{m-1} & m+1-j_m \end{pmatrix}$$

is even or odd, then the determinant of order m thus formed by selecting the above m verticles is equal to

$$\sigma(j_1, j_2, \cdots, j_m)T_{1,j_1}T_{2,j_2}\cdots T_{m,j_m}. \tag{4.3}$$

Since

$$\sum_{\mu=1}^{m}(n-\mu-j_\mu+1)=m(n-m)=N,$$

the term of the highest degree in t^*, denoted by At^{*N}, is found to be

$$\sum\sigma(j_1, j_2, \cdots, j_m)T_{1,j_1}T_{2,j_2}\cdots T_{m,j_m}, \quad (j_1, j_2, \cdots, j_m)=(1, 2, \cdots, m).$$

Hence we arrive at

$$A=\sum\sigma(j_1, j_2, \cdots, j_m)\frac{(n-j_1+1)!(n-j_2+1)!\cdots(n-j_m+1)!}{(n-j_1)!(n-j_2-1)!\cdots(n-j_m-m+1)!}, \quad (j_1, j_2, \cdots, j_m)=(1, 2, \cdots, m).$$

Whence follows immediately the determinant form of A, namely,

$$D_m^N=\begin{vmatrix} \frac{(n-m+1)!}{(n-m)!} & \frac{(n-m+1)!}{(n-m-1)!} & \cdots & \frac{(n-m+1)!}{(n-2m+1)!} \\ \frac{(n-m+2)!}{(n-m+1)!} & \frac{(n-m+2)!}{(n-m)!} & \cdots & \frac{(n-m+2)!}{(n-2m+2)!} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \frac{n!}{(n-1)!} & \frac{n!}{(n-2)!} & & \frac{n!}{(n-m)!} \end{vmatrix}.$$

By means of direct calculation or mathematical induction we obtain finally

$$D_m^N=\frac{n!}{(n-m)!}\prod_{\nu=1}^{m}(m-\nu)!,$$

or

$$g_N^* = \frac{N!n!}{(n-m)!} \prod_{\nu=1}^{m} (m-\nu)! . \tag{4.4}$$

Thus the first of (2.9) is proved.

As to the term Bt^{*N-1} in D^*, we have merely to discuss how the coefficient B is formed. During the procedure of finding out A it was revealed that in the present case two of $j_1, j_2, \cdots, j_m$ must be equal. If both of them are less than m, the determinant thus formed would vanish identically. On the other hand, if $j_\mu = m$, then we should select the left neighbour of the mth verticle in the μth column instead of the mth verticle itself. But the top term of the left neighbour verticle is lacunary. Therefore the corresponding determinant thus constructed is also equal to zero. Thus we have proved $B=0$, that is, $g_{N-1}^*=0$.

Xin Yuanlong also evaluated these coefficients g_N^* and g_{N-1}^* by an alternative method (cf. [2]).

§ 5 Applications

In the case $m=2$ we have obtained certain invariants of parameteric cubics ($n=3$) and of parametric quintics ($n=5$) with two additional conditions (cf. [3] [4]). We shall apply the above theorem to these special cases, and obtain simply the affine invariants.

For the parametric plane cubic ($m=2$, $n=3$) we consider the vectors

$$\boldsymbol{a} = (a_1, a_2, a_3), \ \boldsymbol{b} = (b_1, b_2, b_3),$$

as well as their vector product

$$\boldsymbol{a} \times \boldsymbol{b} = (p, q, r).$$

The inflexion equation (2.1) takes the form

$$D(t) \equiv pt^2 - 2qt + 2r = 0.$$

In general, $p \neq 0$.

Our theorem gives that

$$D^*(t^*) \equiv D(t^* - R). \tag{5.1}$$

In other words:

$$1 + \sum_{k=1}^{N} \frac{1}{k!} G_k t^{*k} \equiv \frac{1}{D(-R)} D(t^* - R). \tag{5.2}$$

In the present case we are led to the affine invariant

$$\mathrm{I} = \left(\frac{q}{p}\right)^2 - 2\frac{r}{p}, \tag{5.3}$$

which is a relative invariant of weight -2 with respect to T.

A second application is found in the case of parametric plane quintics with additional conditions

(5.4) $$p_{35}=0,\ p_{45}=0,\ p_{35}\neq 0.$$

In this case we can put

(5.5) $$\begin{cases} a_3=\lambda a_5, & b_3=\lambda b_5, \\ a_4=\mu a_5, & b_4=\mu b_5. \end{cases}$$

Using (5.2), we reach three affine invariants, namely,

$$a=5\left(\mu-\frac{p_{15}}{p_{25}}\right),$$

$$b=20\left\{\frac{p_{23}}{p_{25}}-\mu\frac{p_{15}}{p_{25}}+\frac{1}{2}\left(\frac{p_{15}}{p_{25}}\right)^2\right\},$$

$$g=-120\left\{\frac{p_{12}}{p_{25}}-\frac{1}{2}\lambda\left(\frac{p_{15}}{p_{25}}\right)^2+\frac{1}{6}\mu\left(\frac{p_{15}}{p_{25}}\right)^3-\frac{1}{24}\left(\frac{p_{15}}{p_{25}}\right)^2\right\},$$

which are of weights -1, -2, -4 with respect to T, respectively.

References

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[3] Su Buchin, Notes on parametric cubic splines. *Acta Math. Appl. Sinica* (1976), 49—58 (Chinese).

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高维仿射空间参数曲线的几个内在不变量

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摘 要

作者证明下列定理:

m 维仿射空间 $n(>m>2)$ 次参数曲线一般具有 $m(n-m)-2$ 个内在仿射不变量.

这定理在一些特殊情形下有着应用，它起到迅速找出仿射不变量的作用. 详情参见苏步青、忻元龙合著论文，应用数学学报，3(1980).