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$\mathscr{D}_{\langle M_k\rangle}$ OPERATORS AND SPECTRAL OPERATORS

WANG SHENGWANG
(*Nanking University*)

1. $\mathscr{D}_{\langle M_k\rangle}$ operators with their spectrum on the complex plane. Throughout this paper, all notations are the same as [1, 2] and the sequence $\{M_k\}$ satisfies $(M.\ 1)$, $(M.\ 2)$ and $(M.\ 3)$[1], i.e., logarithmic convexity, non-quasianalyticity and differentiability. By means of $\{M_k\}$, we can define the associated function $M(t_1, t_2)$ (cf. [7])

$$M(t_1, t_2)=\sup_{\substack{k_i\geq 0\\(i=1,2)}}\left(\sum_{i=1}^{2}k_i\ln|t_i|-\ln M_{k_1+k_2}\right)\quad (t_i\neq 0,\ i=1,\ 2),$$

and the space $\mathscr{D}_{\langle M_k\rangle}$ of two variables

$$\mathscr{D}_{\langle M_k\rangle}=\left\{\varphi\ \middle|\ \varphi\in\mathscr{D};\ \|\varphi\|_\nu=\sup_{\substack{s\in R^2\\k_i\geq 0\\(i=1,2)}}\left|\frac{\partial^{k_1+k_2}}{\partial s_1^{k_1}\partial s_2^{k_2}}\varphi(s)\Big/\nu^k M_k\right|<+\infty\right.$$

$$\left.\text{for some integer } \nu>0\right\},$$

where $s=(s_1,\ s_2)$, $k=k_1+k_2$. It is evident that for any $\varphi\in\mathscr{D}_{\langle M_k\rangle}$

$$\sup_{\substack{s\in R^2\\k\geq 0}}\left|\frac{\partial^k\varphi(s)}{\nu^k M_k}\right|<+\infty$$

for some integer $\nu>0$, where $\partial=\frac{1}{2}\left(\frac{\partial}{\partial s_1}-i\frac{\partial}{\partial s_2}\right)$, and $|\partial^k\varphi(s)|\leqslant\|\varphi\|_\nu\nu^k M_k$. $\|\cdot\|_\nu$ will be called ν–norm. For the definition and properties of bounded $\mathscr{D}_{\langle M_k\rangle}$ operators with their spectrum on the complex plane, we refer the reader to see [3, 4]. Let X be a Banach space, $B(X)$ be the ring of all linear bounded operators defined on X. If $T\in B(X)$ is a $\mathscr{D}_{\langle M_k\rangle}$ operator, we have $T=T_1+iT_2$, $T_1=U_{\mathrm{Re}t}$, $T_2=U_{\mathrm{Im}t}$, where U is a spectral ultradistribution of T. Since $\mathrm{supp}(U)$ is compact, U may be easily extended to the whole space $\mathscr{E}_{\langle M_k\rangle}$.

By few computations, as a function of $(s_1,\ s_2)$, $e^{i(t_1s_1+t_2s_2)}$ satisfies: for given $\mu_l>0$ $(l=1,\ 2)$, there exist $A>0$ and an integer $\nu>0$ such that

$$\|e^{i(t_1s_1+t_2s_2)}\|_\nu\leqslant Ae^{M(\mu_1t_1,\ \mu_2t_2)},$$

where $\|e^{i(t_1s_1+t_2s_2)}\|_\nu$ denotes the ν–norm of $e^{i(t_1s_1+t_2s_2)}$. For every $\varphi\in\mathscr{D}_{\langle M_k\rangle}$, there exist $h_l>0\ (l=1,\ 2)$ and $A'>0$ such that

$$|\hat\varphi(t_1,\ t_2)|\leqslant A'e^{-M(h_1t_1,\ h_2t_2)},$$

where
$$\hat\varphi(t_1,\ t_2)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(t_1s_1+t_2s_2)}\varphi(s_1,\ s_2)\,ds_1ds_2$$

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is the Fourier transform of $\varphi(s_1, s_2)$. Using the same argument as in [1] theorem 3, we can easily prove that one of the spectral ultradistributions of $\mathscr{D}_{\langle M_k\rangle}$ operator T can be expressed as

$$U_\varphi=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(t_1T_1+t_2T_2)}\hat{\varphi}(t_1, t_2)dt_1dt_2.$$

Let T be a spectral operator, S, N, $E(\cdot)$ be the scalar part, radical part and the spectral measure of T respectively. The following theorem gives a sufficient condition of a spectral operator to be a $\mathscr{D}_{\langle M_k\rangle}$ operator.

Theorem 1. *Let $T\in B(X)$ be a spectral operator satisfying*

$$\sup_{k>0}\ \sup_{\substack{|\mu_j|\leq 1\\ \delta_j\in\mathfrak{B}\\ j=1,2,\cdots,k}}\left(\left\|\frac{N^n}{n!}\sum_{j=1}^{k}\mu_jE(\delta_j)\right\|M_n\right)^{\frac{1}{n}}\to 0\quad(n\to\infty),\tag{1}$$

where $\mathfrak{B}$ denotes the class of Borel subsets in the complex plane, then T is a $\mathscr{D}_{\langle M_k\rangle}$ operator and one of its spectral ultradistributions can be expressed as

$$U_\varphi=\sum_{n=0}^{\infty}\frac{N^n}{n!}\int\partial^n\varphi(s)dE(s).$$

Proof Let $\varphi\in\mathscr{D}_{\langle M_k\rangle}$ satisfy

$$|\partial^n\varphi|\leq\|\varphi\|_\nu\nu^nM_n\quad(n=0, 1, 2, \cdots),$$

by (1), we have

$$\left\|\frac{N^n}{n!}\sum_{j=1}^{k}\mu_jE(\delta_j)\right\|\leq\frac{A}{(2\nu)^nM_n},$$

where $A>0$ only depends on ν, $|\mu_j|\leq 1$. By a simple computation, we get

$$\sum_{n=0}^{\infty}\left\|\frac{N^n}{n!}\int\partial^n\varphi dE\right\|\leq 2A\|\varphi\|_\nu.\tag{2}$$

Put

$$U_\varphi=\sum_{n=0}^{\infty}\frac{N^n}{n!}\int\partial^n\varphi dE,$$

then $U_1=I$, $U_s=T$ and

$$U_{\varphi\psi}=\sum_{n=0}^{\infty}\frac{N^n}{n!}\int\partial^n(\varphi\psi)dE=\sum_{n=0}^{\infty}N^n\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\int\partial^k\varphi\partial^{n-k}\psi dE$$
$$=\sum_{k=0}^{\infty}\frac{1}{k!}N^k\int\partial^k\varphi dE\cdot\sum_{j=0}^{\infty}\frac{1}{j!}N^j\int\partial^j\psi dE=U_\varphi U_\psi,$$

i.e., U: $\mathscr{D}_{\langle M_k\rangle}\to B(X)$ is a continuous homomorphism. Therefore T is a $\mathscr{D}_{\langle M_k\rangle}$ operator. Theorem is proved.

Corollary 1. *If N, $E(\cdot)$ satisfy*

$$\left(\frac{M_n}{n!}\vee(N^nE)\right)^{\frac{1}{n}}\to 0\quad(n\to\infty),$$

then T is a $\mathscr{D}_{\langle M_k\rangle}$ operator.

Corollary 2. *Let N be a quasinilpotent, then if and only if*

$$\left(\frac{\|N^n\|}{n!}M_n\right)^{\frac{1}{n}}\to 0\quad(n\to\infty),\tag{3}$$

$S+N$ is a $\mathscr{D}_{\langle M_k \rangle}$ operator for every scalar operator S commuting with N.

We shall call N a $\{M_k\}$–quasinilpotent if it satisfies (3) (cf. [4]). The following proposition gives some properties of a $\{M_k\}$–quasinilpotent.

Proposition. *Let N be a quasinilpotent, the following assertions are equivalent:*

(i) *N is a $\{M_k\}$–quasinilpotent.*

(ii) *For every $\lambda>0$, there exists $B_\lambda>0$ such that*[1)]

$$\|R(\zeta, N)\| \leqslant B_\lambda e^{M^*\left(\frac{\lambda}{|\zeta|}\right)} \quad (|\zeta| \text{ is sufficiently small}).$$

(iii) *For every $\mu>0$, there exists $A_\mu>0$ such that*

$$\|e^{izN}\| \leqslant A_\mu e^{M(\mu|z|)}.$$

Proof By [9] proposition 4.5, the equivalence of (i), (iii) is evident (cf. [4]). It remains to prove the equivalence of (ii), (iii).

(ii)$\Rightarrow$(iii). By putting $\lambda=\frac{\mu}{2}$ in (ii), $r=\delta_\mu(|z|)$ (cf. [1]) and using lemma 4 in [1], we have

$$\|e^{izN}\| \leqslant \frac{1}{2\pi}\int_{|\zeta|=r} |e^{iz\zeta}|\,\|R(\zeta, N)\|\,|d\zeta| \leqslant B_\lambda r e^{|z|r} e^{M^*\left(\frac{\lambda}{r}\right)}$$

$$\leqslant B_\lambda e^{|z|\delta_\mu(|z|)} e^{M^*\left(\frac{\mu}{2\delta_\mu(|z|)}\right)} \leqslant 2B_\lambda e^{M(\mu|z|)},$$

where $0<r<1$.

(iii)$\Rightarrow$(ii). Since $\|R(\zeta, N)\| = \|R(|\zeta|e^{\frac{\pi}{2}i}, e^{\left(\frac{\pi}{2}-\arg\zeta\right)i}N)\|$, we can easily obtain (ii) by applying the sufficient part of theorem 5 in [1] to the operator $e^{\left(\frac{\pi}{2}-\arg\zeta\right)i}N$.

2. $\mathscr{D}_{\langle M_k \rangle}$ operators with their spectrum on the real line. In this section, all functions in $\mathscr{D}_{\langle M_k \rangle}$ are of one variable, hence if $T\in B(\boldsymbol{X})$ is a $\mathscr{D}_{\langle M_k \rangle}$ operator, then $\sigma(T)\subset \boldsymbol{R}$, the real line. Now we consider the conditions to guarantee a bounded $\mathscr{D}_{\langle M_k \rangle}$ operator T to be a spectral. In the sequel, the conjugate space of $\boldsymbol{X}$ will be denoted by $\boldsymbol{X}^*$.

If $f\in\mathscr{D}'_{\langle M_k \rangle}$, from [8, 9], there exist countable many regular measures μ_n $(n\geqslant 0)$ satisfying

$$\langle f, \varphi\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\int\varphi^{(n)}(t)\,d\mu_n(t) \tag{4}$$

and for every $h>0$, there exists $A>0$ such that

$$\sum_{n=0}^{\infty}\frac{h^n}{n!}M_n\int|d\mu_n| \leqslant A. \tag{5}$$

By a similar method used in [10], it can be easily proved that:

1° for $f\in\mathscr{D}'_{\langle M_k \rangle}$, if $f'=0$, then $f=\text{const}$;

2° for $f\in\mathscr{D}'_{\langle M_k \rangle}$, if f' is a measure, then f is a function of bounded variation in every finite interval.

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1) In (ii), we have to suppose that $\left\{\frac{M_k}{k!}\right\}$ is logarithmically convex. As for $M^*\left(\frac{\lambda}{|\zeta|}\right)$, $M(\mu|z|)$, we refer the reader to see [1].

In general, the sequence $\{\mu_n\}$ $(n\geqslant 0)$ in (4), (5) is not unique. Now we introduce the following:

Definition. *Let n_0 be a positive integer, $f\in\mathscr{D}'_{\langle M_k\rangle}$ with compact support (i. e., $f\in\mathscr{E}'_{\langle M_k\rangle}$) is called n_0-singular, if for all $n\geqslant n_0$, there exist μ_n in (4), (5) with* Supp(μ_n) *contained in a closed subset F satisfying* mes $F=0$. *If $n_0=1$, f is called singular. Suppose that $T\in B(\boldsymbol{X})$ is a $\mathscr{D}_{\langle M_k\rangle}$ operator, $\boldsymbol{U}$ is its spectral ultradistribution, we say that T is n_0-singular (singular), if for every $x\in\boldsymbol{X}$, $x^*\in\boldsymbol{X}^*$, $x^*\boldsymbol{U}.x$ is n_0-singular (singular).*

In the sequel, we shall often suppose that $f\in\mathscr{D}'_{\langle M_k\rangle}$ has compact support and when f is n_0-singular, μ_n $(n\geqslant n_0)$ will denote what satisfy the conditions discribed in the above definition. Therefore all of these μ_n $(n\geqslant n_0)$ are singular with respect to Lebesgue measure, but the inverse is false.

Lemma 1. *If $f\in\mathscr{D}_{\langle M_k\rangle}$ is n_0-singular, then for every $n\geqslant n_0$, μ_n is unique, especially, μ_0 is also unique when $n_0=1$.*

Proof It suffices to prove that when $f=0$, then $\mu_n=0$ $(n\geqslant n_0)$. In fact, $f=0$ is equivalent to

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\mu_n^{(n)}=0 \quad\text{or}\quad \Big(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\mu_n^{(n-1)}\Big)'=-\mu_0. \tag{6a}$$

Since $\mathscr{D}_{\langle M_k\rangle}$ is differentiable, it follows that $g_1=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\mu_n^{(n-1)}\in\mathscr{D}'_{\langle M_k\rangle}$. Since all of supp$(\mu_n)$ $(n\geqslant 0)$ are contained in a neighbourhood of supp(f), we may suppose that supp(g_1) is compact. By (6a), g_1 is a function of bounded variation. Similarly, $\sum_{n=k}^{\infty}\frac{(-1)^n}{n!}\mu_n^{(n-k)}=g_k$ are also functions of bounded variation for all $k>1$. By the hypothesis, we can easily see that the subset where $g_k\neq 0$ is of Lebesgue measure zero, hence as ultradistributions in $\mathscr{D}'_{\langle M_k\rangle}$, $g_k=0$ $(k\geqslant n_0)$, i. e.,

$$\left.\begin{aligned}(-1)^{n_0}\frac{\mu_{n_0}}{n_0!}+(-1)^{n_0+1}\frac{\mu'_{n_0+1}}{(n_0+1)!}+\cdots&=0,\\ (-1)^{n_0+1}\frac{\mu_{n_0+1}}{(n_0+1)!}+\cdots&=0,\\ \cdots\cdots\cdots\cdots\cdots&=0.\end{aligned}\right\} \tag{6b}$$

(6b) shows that $\mu_{n_0}=\mu_{n_0+1}=\cdots=0$. If $n_0=1$, by $\mu_n=0$ $(n\geqslant 1)$ and (6a), we have $\mu_0=0$. Thus the lemma is proved.

For $\varphi\in\mathscr{D}_{\langle M_k\rangle}$, $\hat{\varphi}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-its}\varphi(t)\,dt$ and $\check{\varphi}(s)=\int_{-\infty}^{\infty}e^{its}\varphi(t)\,dt$ will express the Fourier and inverse Fourier transform of φ respectively. When f is an ultradistribution, $\hat{f}$, $\check{f}$ also have the same meaning.

Remark *If ultradistributions $f=\sum_{n=0}^{\infty}\frac{(-1)^n\mu_n^{(n)}}{n!}$ and $g=\sum_{n=0}^{\infty}\frac{(-1)^n\nu_n^{(n)}}{n!}$ are singular, where μ_n, ν_n $(n\geqslant 1)$ satisfy those conditions discribed in the preceding definition,* *then from* $\check{f}=\check{g}$ $\left(i.e., \sum_{n=0}^{\infty}\frac{(it)^n}{n!}\check{\mu}_n(t)=\sum_{n=0}^{\infty}\frac{(it)^n}{n!}\check{\nu}_n(t)\right)$, *we can deduce* $f=g$, *by the above lemma, we have* $\mu_n=\nu_n$, *hence* $\check{\mu}_n=\check{\nu}_n$ $(n\geqslant 0)$, *i.e., from* $\check{f}=\check{g}$, *we get* $\check{\mu}_n=\check{\nu}_n$.

Lemma 2. *For* $f\in\mathscr{D}'_{\langle M_k\rangle}$ *has compact support, we have* $\check{f}(s)=\langle f_t, e^{its}\rangle$.

Proof For every $\varphi\in\mathscr{D}_{\langle M_k\rangle}$, $\varphi(t)=\int_{-\infty}^{\infty}e^{its}\hat{\varphi}(s)ds$ is the limit of the integral sum $\sum_{j=0}^{n}e^{its_j}\hat{\varphi}(s_j)\Delta s_j$ with respect to the topology of $\mathscr{E}_{\langle M_k\rangle}$, hence

$$\int_{-\infty}^{\infty}\langle f_t, e^{its}\rangle\hat{\varphi}(s)ds=\lim\sum_{j=1}^{n}\langle f_t, e^{its_j}\rangle\hat{\varphi}(s_j)\Delta s_j$$

$$=\left\langle f_t, \lim\sum_{j=1}^{n}e^{its_j}\hat{\varphi}(s_j)\Delta s_j\right\rangle=\langle f, \varphi\rangle.$$

Lemma 3. *Suppose that* X *is reflexive,* T *is a bounded singular* $\mathscr{D}_{\langle M_k\rangle}$ *operator;* $\boldsymbol{U}$ *is its spectral ultradistribution, then there exist operator-valued measures* $u_n(\cdot)$ $(n\geqslant 0)$ *such that*

(i) *each of* $u_n(\cdot)$ *is bounded and strongly countably additive and for every* $h>0$ *there exists* $A>0$ *such that*

$$\|u_n(\delta)\|\leqslant An!h^n/M_n \quad (n\geqslant 0) \tag{7}$$

for every Borel subset δ *on the real line;*

(ii) *for every* $\varphi\in\mathscr{D}_{\langle M_k\rangle}$

$$\boldsymbol{U}_\varphi=\sum_{n=0}^{\infty}\frac{1}{n!}\int\varphi^{(n)}(t)du_n(t), \tag{8}$$

in which every integral converges in the sense of strong operator topology and the series converges in the sense of uniform operator topology.

Proof For $x\in\boldsymbol{X}$, $x^*\in\boldsymbol{X}^*$, $\|x\|\leqslant 1$, $\|x^*\|\leqslant 1$, the class of ultradistributions $x^*\boldsymbol{U}.x$ is bounded in $\mathscr{E}'_{\langle M_k\rangle}$ and their supports are contained in $\sigma(T)$. By [8, 9], there exist regular measures $\mu_n(\cdot; x, x^*)$ $(n\geqslant 0)$ satisfying: for every $h>0$, there exists $A>0$ such that

$$\|\mu_n(\cdot; x, x^*)\|\leqslant An!h^n/M_n \quad (n\geqslant 0) \tag{9}$$

uniformly for all $\|x\|\leqslant 1$, $\|x^*\|\leqslant 1$, where $\|\mu\|$ denotes the total variation of μ. In addition, we have

$$x^*\boldsymbol{U}_\varphi x=\sum_{n=0}^{\infty}\frac{1}{n!}\int\varphi^{(n)}(t)d\mu_n(t; x, x^*),$$

in which the series converges absolutely and uniformly with respect to all $\|x\|\leqslant 1$, $\|x^*\|\leqslant 1$. By the singularity of T, we may suppose that for every $n\geqslant 1$, supp$(\mu_n(\cdot; x, x^*))$ is contained in a fixed closed subset F with Lebesgue measure zero. From lemma 1, $\mu_n(\cdot; x, x^*)$ $(n\geqslant 0)$ is unique, hence for every Borel subset δ, $\mu_n(\delta; x, x^*)$ is a bounded bilinear functional of x, x^*. By the reflexivity of $\boldsymbol{X}$, there exists for every $n\geqslant 0$ a bounded linear operator $u_n(\delta)$ defined on $\boldsymbol{X}$ such that

$$x^*u_n(\delta)x=\mu_n(\delta; x, x^*).$$

From (9), $u_n(\cdot)$ satisfies (i). Evidently, we may suppose that supp(u_n) $(n\geqslant 0)$ is contained in a fixed neighbourhood G of $\sigma(T)$. As for (ii), using the following inequality

$$\left|x^*\int\varphi^{(n)}(t)\,du_n(t)x\right|=\left|\int\varphi^{(n)}(t)\,d\mu_n(t;\,x,\,x^*)\right|$$
$$\leqslant A\sup_{t\in G}|\varphi^{(n)}(t)|\,n!h^n\|x\|\,\|x^*\|/M_W,$$

we get the result that the series in (8) converges with respect to the uniform operator topology. Finally, by [12] Theorem IV. 10.8 and Definition IV. 10.7, every integral in (8) converges in the strong operator topology.

Lemma 4. *Under the hypotheses of the preceding lemma, we have*

$$\check{u}_n(\tau)\check{u}_m(\sigma)=\check{u}_{n+m}(\tau+\sigma)\quad(m,\,n\geqslant 0),\tag{10}$$

in which τ, σ *are real numbers.*

Proof For every $x\in X$, $x^*\in X^*$, we have

$$x^*U_{e^{i(\tau+\sigma)t}}x=x^*U_{e^{i\tau t}}U_{e^{i\sigma t}}x=\sum_{n=0}^{\infty}\frac{(i\tau)^n}{n!}x^*\check{u}_n(\tau)U_{e^{i\sigma t}}x;\tag{11}$$

$$x^*U_{e^{i(\tau+\sigma)t}}x=\sum_{k=0}^{\infty}\frac{[i(\tau+\sigma)]^k}{k!}x^*\check{u}_k(\tau+\sigma)x=\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{(i\tau)^n}{n!}\frac{(i\sigma)^{k-n}}{(k-n)!}x^*\check{u}_k(\tau+\sigma)x$$
$$=\sum_{n=0}^{\infty}\frac{(i\tau)^n}{n!}\sum_{m=0}^{\infty}\frac{(i\sigma)^m}{m!}x^*\check{u}_{n+m}(\tau+\sigma)x=\sum_{n=0}^{\infty}\frac{(i\tau)^n}{n!}\int e^{i\tau t}dx^*v_n(t)x,\tag{12}$$

in which

$$v_n(\delta)=\sum_{m=0}^{\infty}\frac{(i\sigma)^m}{m!}\int_\delta e^{i\sigma t}du_{n+m}(t)\tag{13}$$

depends on σ and δ is a Borel subset. Putting σ fixed, from (9), for every $h>0$ there exists $A>0$ such that for all $\|x\|\leqslant 1$, $\|x^*\|\leqslant 1$,

$$\|x^*v_n(t)x\|\leqslant\sum_{m=0}^{\infty}\frac{|\sigma|^m}{m!}\|x^*u_{n+m}(t)\,x\|\leqslant A\sum_{m=0}^{\infty}\frac{|\sigma|^m}{m!}(n+m)!\left(\frac{h}{2}\right)^{m+n}\Big/M_{n+m}$$
$$\leqslant(An!h^n/M_n)\left(\sum_{m=0}^{\infty}\frac{|\sigma|^m(n+m)!}{m!n!2^{n+m}}h^m\Big/M_m\right)$$
$$\leqslant(An!h^n/M_n)\sum_{m=0}^{\infty}(|\sigma|h)^m/M_m.$$

Since $B=\sum_{m=0}^{\infty}\frac{(|\sigma|h)^m}{M_m}<+\infty$, it follows that

$$\|x^*v_n(t)x\|_V\leqslant ABn!h^n/M_W$$

uniformly for all $\|x\|\leqslant 1$, $\|x^*\|\leqslant 1$. Therefore $v_n(\cdot)$ $(n=0,\,1,\,2,\,\cdots)$ are bounded strongly countably additive operator-valued measures. Evidently, the utradistribution $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(x^*v_n(t)x)^{(n)}$ is singular. By (11), (12) and the remark after lemma 1, we have

$$x^*\check{u}_n(\tau)U_{e^{i\sigma\cdot}}\,x=x^*\check{v}_n(\tau)x,$$

i. e.,

$$\sum_{m=0}^{\infty}\frac{(i\sigma)^m}{m!}x^*\check{u}_n(\tau)\check{u}_m(\sigma)x=\sum_{m=0}^{\infty}\frac{(i\sigma)^m}{m!}x^*\check{u}_{n+m}(\tau+\sigma)x.\tag{14}$$

For fixed τ, the two sides of (14) are the inverse Fourier transform of $\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}$ $\cdot(x^*\check{u}_n(\tau)u_m(\cdot)x)^{(m)}$, $\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}(x^*w_{n+m}(\cdot)x)^{(m)}$ respectively, where $w_{n+m}(\delta)=\int_\delta e^{i\tau t}$ $\cdot du_{n+m}(t)$, δ is a Borel subset. Still by the remark after lemma 1, we get (10). Lemma 4 is thus proved.

Lemma 5. *Putting* $E(\cdot)=u_0(\cdot)$, $N=u_1(R)$, *we have*

(i) $E(\cdot)$ *is a spectral measure;*

(ii) *for every Borel subset* δ *and every* $n\geqslant 0$,

$$u_n(\delta)=N^nE(\delta)=E(\delta)N^n;$$

(iii) N *is a quasinilpotent satisfying*

$$\lim_{n\to\infty}\left(\frac{\|N^n\|}{n!}M_n\right)^{\frac{1}{n}}=0. \tag{15}$$

Proof (i) From lemma 3 (i), it remains to prove that

$$E(\delta)E(\varepsilon)=E(\delta\cap\varepsilon),\quad E(\delta)+E(\varepsilon)-E(\delta)E(\varepsilon)=E(\delta\cup\varepsilon), \tag{16}$$

for all Borel subsets δ, ε. Putting $n=m=0$ in (10), we obtain $\check{E}(\tau)\check{E}(\sigma)=\check{E}(\tau+\sigma)$, i. e.,

$$\int e^{i\tau t}dE(t)\check{E}(\sigma)=\int e^{i(\tau+\sigma)t}dE(t)=\int e^{i\tau t}d_t\left(\int_{-\infty}^{t}e^{i\sigma t}dE(\tau)\right).$$

Since the inverse Fourier transform is 1–1, it follows that

$$E(\delta)\check{E}(\sigma)=\int_\delta e^{i\sigma t}dE(t). \tag{17}$$

(17) may be written as

$$\int e^{i\sigma t}dE(\delta)E(t)=\int_\delta e^{i\sigma t}dE(t),$$

still by the property (1–1) of the inverse Fourier transform, we have $E(\delta)E(\varepsilon)$ $=E(\delta\cap\varepsilon)$. Finally, by the additivity of $E(\cdot)$,

$$E(\delta\cup\varepsilon)=E(\delta)+E(\varepsilon\setminus(\delta\cap\varepsilon))=E(\delta)+E(\varepsilon)-E(\delta\cap\varepsilon).$$

(16) holds.

(ii) and (iii) Putting $\sigma=0$, $m=1$ and substituting n by $n-1$ in (10), we have

$$\check{u}_n(\tau)=\check{u}_{n-1}(\tau)N. \tag{18}$$

Similarly, $\check{u}_n(\tau)=N\check{u}_{n-1}(\tau)$. Let $n=0, 1, 2, \cdots$, it follows that

$$\check{u}_n(\tau)=\check{E}(\tau)N^n=N^n\check{E}(\tau).$$

By the property (1–1) of the inverse Fourier transform again, we get

$$u_n(\cdot)=E(\cdot)N^n=N^nE(\cdot),$$

especially, $N^n=u_n(R)$. Therefore by (9), for every $h>0$,

$$\|N^n\|=\|u_n(R)\|\leqslant An!h^n/M_n,$$

i. e., (15) holds.

Summarizing the above discussions, we obtain

Theorem 2. *Suppose* X *is reflexive.* $T\in B(X)$ *is a singular* $\mathscr{D}_{(M_k)}$ *operator if and* *only if T is a spectral operator satisfying*

(i) *For every $x\in \boldsymbol{X}$ and $x^*\in \boldsymbol{X}^*$,* supp $(x^*N^nE(\cdot)x)$ *is contained in a fixed closed subset F of Lebesgue measure zero for all $n\geqslant 1$ (F may depend on x, x^*), where $E(\cdot)$, N are the spectral measure and radical part of T respectively;*

(ii) *N satisfies* (15).

Corollary. *Suppose $\boldsymbol{X}$ is reflexive. $T\in B(\boldsymbol{X})$ is a singular $\mathscr{D}_{\langle M_k\rangle}$ operator and* mes $\sigma(T)=0$ *if and only if T is a spectral operator satisfying*

(i) mes supp$(E(\cdot))=0$;

(ii) *N satisfies* (15).

Theorem 2 is an extention of some results of [5][1)].

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$\mathscr{D}_{\langle M_k\rangle}$ 型算子与谱算子

王 声 望
(南京大学)

摘 要

1. 谱位于平面上的有界 $\mathscr{D}_{\langle M_k\rangle}$ 型算子 记号与 [1, 2] 相同，不再一一赘述. 设序列 $\{M_k\}$ 满足 $(M.1)$, $(M.2)$, $(M.3)$ 即对数凸性、非拟解析性、可微性[1]. 由 $\{M_k\}$ 我们可以定义二元相关函数 $M(t_1, t_2)$（详见 [7]）以及二元 $\mathscr{D}_{\langle M_k\rangle}$ 空间

$$\mathscr{D}_{\langle M_k\rangle}=\left\{\varphi\,|\,\varphi\in\mathscr{D};\text{存在正整数 }\nu\text{ 使 }\|\varphi\|_\nu=\sup_{\substack{s\in R^2\\k_i\geq 0\\(i=1,2)}}\left|\frac{\partial^{k_1+k_2}}{\partial s_1^{k_1}\partial s_2^{k_2}}\varphi(s)\right|\Big/\nu^k M_k<+\infty\right\},$$

其中 $s=(s_1, s_2)$、$k=k_1+k_2$. 关于谱位于复平面上的有界 $\mathscr{D}_{\langle M_k\rangle}$ 型算子的定义及性质可参看 [3, 4]. 设 $\boldsymbol{X}$ 为 Banach 空间, $B(\boldsymbol{X})$ 为 $\boldsymbol{X}$ 上有界线性算子的全体组成的环. 当 $T\in B(\boldsymbol{X})$ 为 $\mathscr{D}_{\langle M_k\rangle}$ 型算子时，有 $T=T_1+iT_2$; $T_1=\boldsymbol{U}_{\mathrm{Re}\,t}$、$T_2=\boldsymbol{U}_{\mathrm{Im}\,t}$, 此处 $\boldsymbol{U}$ 为 T 的谱超广义函数, t 为复变量. 由于 $\mathrm{supp}(\boldsymbol{U})$ 为紧集，故可将 $\boldsymbol{U}$ 延拓到 $\mathscr{E}_{\langle M_k\rangle}$ 上且保持连续性.

经过简单的计算，若 $T\in B(\boldsymbol{X})$ 为谱位于平面上的一个 $\mathscr{D}_{\langle M_k\rangle}$ 型算子，则 T 的一个谱超广义函数(1) $\boldsymbol{U}$ 可表成

$$\boldsymbol{U}_\varphi=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(t_1T_1+t_2T_2)}\hat{\varphi}(t_1, t_2)\,dt_1\,dt_2.$$

设 $T\in B(\boldsymbol{X})$ 为谱算子, S、N、$E(\cdot)$ 分别为 T 的标量部分、根部、谱测度. 下面的定理给出了谱算子成为 $\mathscr{D}_{\langle M_k\rangle}$ 型算子的一个充分条件

定理 1 设 T 为谱算子适合下面的条件

$$\sup_{k>0}\sup_{\substack{|\mu_j|\leq 1\\ \delta_j\in\mathfrak{B}\\ j=1,2,\cdots k}}\left(\left\|\frac{N^n}{n!}\sum_{j=1}^{k}\mu_jE(\delta_j)\right\|M_n\right)^{\frac{1}{n}}\to 0\quad(n\to\infty),$$

其中 $\mathfrak{B}$ 为平面上的 Borel 集类. 则 T 为 $\mathscr{D}_{\langle M_k\rangle}$ 型算子且它的一个谱广义函数可表为

$$\boldsymbol{U}_\varphi=\sum_{n=0}^{\infty}\frac{N^n}{n!}\int\partial^n\varphi(s)\,dE(s).$$

推论 1 设 $E(\cdot)$, N 满足

$$\left(\frac{M_n}{n!}\vee(N^nE)\right)^{\frac{1}{n}}\to 0\quad(n\to\infty),$$

则 T 为 $\mathscr{D}_{\langle M_k\rangle}$ 型算子.

推论 2 设 N 为广义幂零算子，则对于任何与 N 可换的标量算子 S, $S+N$ 为 $\mathscr{D}_{\langle M_k\rangle}$ 型算子的充分必要条件是

$$\left(\frac{\|N^n\|}{n!}M_n\right)^{\frac{1}{n}}\to 0\quad(n\to\infty).$$

(1) 本文中的谱超广义函数即为[1]中的谱广义函数.

在[4]中称满足上式的算子为 $\{M_k\}$ 广义幂零算子. 显然 $\{M_k\}$ 广义幂零算子必为通常的广义幂零算子. 下面的命题给出了 $\{M_k\}$ 广义幂零算子的一些性质.

命题 设 N 为广义幂零算子,则下列事实等价:

(i) N 为 $\{M_k\}$ 广义幂零算子;

(ii) 对于任给的 $\lambda>0$, 存在 $B_\lambda>0$ 使[(1)]

$$\|R(\zeta, N)\|\leqslant B_\lambda e^{M^*\left(\frac{\lambda}{|\zeta|}\right)}\quad (|\zeta|\text{充分小});$$

(iii) 对于任给的 $\mu>0$, 存在 $A_\mu>0$ 使

$$\|e^{izN}\|\leqslant A_\mu e^{M(\mu|z|)}.$$

2. 谱位于实轴上的有界 $\mathscr{D}_{\langle M_k\rangle}$ 型算子 本节讨论有界 $\mathscr{D}_{\langle M_k\rangle}$ 型算子 T 成为谱算子的条件,这里假定 $\mathscr{D}_{\langle M_k\rangle}$ 中的函数是一元的,于是 T 的谱位于实轴上. $\boldsymbol{X}^*$ 表示 $\boldsymbol{X}$ 的共轭空间.

设 $f\in\mathscr{D}'_{\langle M_k\rangle}$, 由[8, 9],存在测度 $\mu_n(n\geqslant0)$ 使得对任何 $h>0$, 存在 $A>0$ 适合

$$\sum_{n=0}^{\infty}\frac{h^n}{n!}M_n\int|d\mu_n|\leqslant A$$

且

$$\langle f, \varphi\rangle=\sum_{n=0}^{\infty}\frac{1}{n!}\int\varphi^{(n)}(t)d\mu_n(t).$$

一般说,上述 $\mu_n(n\geqslant0)$ 不是唯一的,为此我们引入

定义 设 n_0 为正整数,如果对一切 $n\geqslant n_0$, 存在测度 μ_n, 它们的支集均包含在某一 L 零测度闭集内,则称 f 是 n_0 奇异的,若 $n_0=1$,则称 f 是奇异的. 设 $T\in B(X)$ 为 $\mathscr{D}_{\langle M_k\rangle}$ 型算子, $\boldsymbol{U}$ 为其谱超广义函数, 如果对于任何 $x\in\boldsymbol{X}$、$x^*\in\boldsymbol{X}^*$, $x^*\boldsymbol{U}.x$ 是 n_0 奇异的(奇异的),则称 T 是 n_0 奇异的(奇异的) $\mathscr{D}_{\langle M_k\rangle}$ 型算子.

经过若干准备,可以证明下面的

定理 2 设 $\boldsymbol{X}$ 为自反的 Banach 空间, 则 $T\in B(\boldsymbol{X})$ 为奇异 $\mathscr{D}_{\langle M_k\rangle}$ 型算子的充分必要条件是 T 为满足下列条件的谱算子:

(i) 对每个 $x\in\boldsymbol{X}$ 及 $x^*\in\boldsymbol{X}$, $\mathrm{supp}(x^*N^nE(\cdot)x)$ 包含在一个与 $n\geqslant1$ 无关的 L 零测度闭集 F 内(F 可以依赖于 x、x^*),此处 $E(\cdot)$、N 分别是 T 的谱测度与根部;

(ii) 算子 N 是 $\{M_k\}$ 广义幂零算子.

推论 设 $\boldsymbol{X}$ 为自反的 Banach 空间, $T\in B(\boldsymbol{X})$ 为奇异 $\mathscr{D}_{\langle M_k\rangle}$ 型算子且 $\sigma(T)$ 的测度为零的充分必要条件是 T 为满足下列条件的谱算子:

(i) $E(\cdot)$ 的支集为 L 零测度集;

(ii) 算子 N 是 $\{M_k\}$ 广义幂零算子.

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(1) 在(ii)中需假定 $\left\{\frac{M_k}{k!}\right\}$ 是对数凸的.