Chin. Ann. of Math.
1 (3, 4) 1980

QUALITATIVE INVESTIGATION OF A TORAL DIFFERENTIAL EQUATION HAVING CRITICAL POINTS

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Since the fundamental paper of H. Poincaré [1], there were many classical works concerning toral dynamical systems without critical points (see, e. g., [2—6]). More recently, Chin Yuanchun [7] investigated certain concrete toral dynamical systems given by differential equations, but still without critical points. On the other hand, in the past few years there appeared many papers studying global structure or topological classification of continuous flows on 2-manifolds (see, e. g., [8—15]), in which one may have critical points, but there is no differential equation. It seems to us that an investigation of dynamical systems on 2-manifolds having both critical points and differential equations is coming up to date.

In this paper, parallel to the simplest but also most fundamental linear autonomous plane system

$$\frac{dx}{dt}=ax+by, \quad \frac{dy}{dt}=cx+dy,$$

we study the global structure of the toral dynamical system

$$\frac{dx}{dt}=A\sin x+B\sin y, \ \frac{dy}{dt}=C\sin x+D\sin y \tag{1}$$

in detail.[1)]

In order that (1) can define a toral dynamical system exactly, we must first take a certain fundamental square in (x, y) plane, such that on opposite sides of this square (1) will define the same vector fields. And then, if we identify each pair of opposite sides of the square, we will get a dynamical system on the torus.

In § 1 we take

$$S_1: \ 0\leqslant x\leqslant 2\pi, \ 0\leqslant y\leqslant 2\pi$$

as the fundamental square. Since $\sin x$ and $\sin y$ both have period 2π, we get an analytic system on the torus. This system has 4 elementary critical points, but the global structure of its phase-portrait is comparatively simple.

Manuscript received Jan. 9, 1980.

1) In subsequent papers, global structure of similar systems on torus, Klein Bottle and projective plane will be given.

In § 2 we take

$$S_2:\ 0\leqslant x\leqslant\pi,\ \ 0\leqslant y\leqslant\pi$$

as the fundamental square, and the toral dynamical system so obtained is only a C^1 system. It has a unique critical point with index zero, but global structure of the phase-portrait is of various kinds.

§ 1. Suppose that system (1) has S_1 as its fundamental square. We assume $AD-BC\neq 0$, since when $AD-BC=0$, (1) is integrable, and the phase-portrait is very simple. System (1) has 9 elementary critical points in S_1, which are divided into two groups. $(0, 0)$, $(0, 2\pi)$, $(2\pi, 2\pi)$, (π, π), $(2\pi, 0)$ are in the first group; when opposite sides of S_1 were identified, we get from them 2 elementary critical points of the same kind on the torus. $(0, \pi)$, $(2\pi, \pi)$, $(\pi, 0)$, $(\pi, 2\pi)$ are in the second group, and they give rise to 2 elementary critical points of the other kind. If we assume $AD-BC>0$ (<0), the critical points of the first (second) kind will both be nodes, foci or centers, while that of the second (first) kind will be saddles.

After a detailed analysis of all possible cases corresponding to different values of A, B, C, D, together with Theorem 1 at the end of this section, which asserts that system (1) can not have periodic orbits of the second kind, we are able to conclude that possible topological structures of the phase-portrait of system (1) are of the following three types:[1)]

I. Center-Saddle type. This can appear only in 2 cases:

(1) $A=-D,\ B=-C,\ |B|>|A|$. We have at this time Fig. 1·1, and system (1) has the general integral

$$\left(\cos\frac{x+y}{2}\right)^{D+C}=K\left(\cos\frac{x-y}{2}\right)^{D-C}. \tag{2}$$

Notice that the 2 saddles are on the same singular cycle.

(2) $A=D=0,\ B\neq -C$. We have Fig. 1·2, and the general integral of (1) is

$$\cos y=\frac{C}{B}\cos x+K. \tag{3}$$

Here the 2 saddles are on different singular cycles.

II. Node-Saddle type. We have Fig. 2, which is drawn under the assumption $AD-BC<0$ (so that (π, π) is a saddle point) in order to show the trend of the 4 separatrixes through (π, π). The case $A=-D,\ B=-C,\ |B|<|A|$ also belongs to this case, of course, its general integral is still (2).

III. Focus-Saddle type. We have Figs. 3·1 and 3·2, the difference of which his: in Fig. 3·1, the 4 separatrixes go from saddles into (π, π) within the dotted lined square; while in Fig. 3·2, there are only 2 such separatrixes, the other 2 will wind around the

1) Since system (1) is unchanged under the transformaton $x=x'+\pi,\ y=y'+\pi,\ t=-t'$, we have only to draw the portrait around the critical point (π, π).

torus once before going into (π, π). Around (π, π) there may or may not exist limit cycle of the first kind, for which see the analysis given later.

Now let us examine system (1) in detail, when it is not of the Center-Saddle type.

Case I. $A+D=-(B+C)\neq 0$.

Let $\bar{A}=\frac{A-D}{2}$, $\bar{B}=\frac{B-C}{2}$, $\delta=\frac{A+D}{2}$, then

$$A=\bar{A}+\delta,\quad B=\bar{B}-\delta,\quad C=-\bar{B}-\delta,\quad D=-\bar{A}+\delta, \tag{4}$$

and (1) becomes

$$\frac{dx}{dt}=(\bar{A}+\delta)\sin x+(\bar{B}-\delta)\sin y,\quad \frac{dy}{dt}=(-\bar{B}-\delta)\sin x+(-\bar{A}+\delta)\sin y. \tag{5}$$

It is easily seen that the system (5) determines a family of generalized rotated vector fields as δ varies (See [16], § 3). When $\delta=0$, (5) has 2 centers as shown in Fig. 1·1, hence (5) will not have closed orbit of the first kind when $\delta\neq 0$.

Before discussing the other cases, we mention that the 4 dotted lines in Fig. 3 are all segments without contact with respect to system (1). Hence if (1) has any closed orbit of the first kind, it will remain in the interior of the dotted lined square. The vertical and horizontal isoclines of system (1) will in general be located also therein, as one of the 4 curves shown in Fig. 4.[1] Notice that the divergence of system (1) vanishes on the curve

$$A\cos x+D\cos y=0, \tag{6}$$

which, according to different signs and absolute values of A and D, will take any form of one of the 4 curves shown in Figs. 5·1 and 5·2. In particular, when $A=D\neq 0$, (6) coincides with the 4 sides of the square, and therefore (1) can not have any closed orbit within this square. Hence we may assume hereafter that $A\neq D$. Furthermore, without loss of generality, we may assume also that in system (1), $A\geqslant 0$ and $B\geqslant 0$.[2]

Case II. $CD<0$. Assume $AD-BC>0$, then we must have $C<0$, $D>0$. Take a Liapounov function

$$V=-C\sin^2\frac{x}{2}+B\sin^2\frac{y}{2},$$

then $V=V_0$, $\max(B, -C)\leqslant V_0\leqslant B-C$ define a family of closed curves around the critical point (π, π). From (1), we get

$$\left.\frac{dV}{dt}\right|_{(1)}=-\frac{AC}{2}\sin^2 x+\frac{BD}{2}\sin^2 y\geqslant 0,$$

hence system (1) has no closed orbit of the first kind.

Case III. $C\geqslant 0$, $D>0$. In this time we take

$$V=C\sin^2\frac{x}{2}+B\sin^2\frac{y}{2},$$

1) When $|A|=|B|$ (or $|C|=|D|$), the vertical (or horizontal) isocline degenerates into a pair of opposite sides of the dotted lined square and their perpendicular bisector.

2) Otherwise, we may use the transformation $t=-t'$, or $x=x'$, $y=2\pi-y'$.

and the remaining of the proof is similar to that of Case II.

Case IV. $C<0$, $D<0$ (assume $AD-BC>0$).

a) $A\geqslant B>0$, $A\geqslant|D|$. Suppose that there exists a closed orbit Γ, then the relative position of Γ, the two isoclines and the curve (6) are shown in Fig. 6, where the shaded regions Q_1 and Q_2 denote those parts inside Γ in which

$$A\cos x+D\cos y>0.$$

It is easily seen that symmetric parts of Q_1 and Q_2 with respect to the line $x+y=2\pi$ lie also in the interior of Γ. Moreover, for any point P in Q_i ($i=1, 2$) and its symmetric point P', we always have

$$(A\cos x+D\cos y)_P<-(A\cos x+D\cos y)_{P'}.$$

Therefore, we have the inequality

$$\iint\limits_{\text{int.}\,\Gamma}(A\cos x+D\cos y)\,dx\,dy<0,$$

which contradicts a known theorem of Bendixson, hence Γ can not exist. It is easily seen that in this case (π, π) is a stable focus (when $A+D$ is sm all) or stable node (when$A+D$ is large).

b) $A\geqslant B>0$, $A<|D|$. From a), (π, π) is stable when $A+D\geqslant 0$, it is a stable fine focus when $A+D=0$. Since when $A+D$ becomes negative (π, π) changes it is stability, we see from Theorem 3·7 of [16] that limit cycle will appear around (π, π) if $0<-A-D\ll 1$. It seems to us that the limit cycle(stable)is unique, but it is difficult to give a rigorous proof. Notice that system (1) is quite different from those which have been investigated heretofore.

c) $0<A<B$. In this case, if $|C|\leqslant|D|$, then the transformation $x'=2\pi-y$, $y'=2\pi-x$ will transform it into case a) or b). If $|C|>|D|$, then similar to cases a) and b), we can prove that system (1) has no closed orbit of the first kind when $A\geqslant|D|$, and will have limit cycle when $A<|D|$ and $|A+D|$ is small.

Let us close this section by proving the following:

Theorem 1. *System* (1) *can not have periodic orbits of the second kind.*

Proof It suffices to examine the case in which closed orbit of the second kind, if exists, goes from the left boundary of S_1 to the right boundary.

1) $|A|\geqslant|B|$. In this case, the vertical isocline of (1) has the form of curve (3) or (4) in Fig. 4. Any orbit starts from the left boundary of S_1 will turn back to the left when it meets the vertical isocline, and thus can not arrive at the right boundary.

2) $|A|<|B|$, $|C|\geqslant|D|$. Assume $AB>0$, $CD>0$, then the two isoclines are shown in Fig. 7. For any orbit $\widehat{PQR}$ going from left to right, it is easy to show that symmetric curve of $\widehat{PQ}$ with respect to $x=\pi$ will lie entirely below $\widehat{QR}$, and hence $\widehat{PQR}$ can not be a closed orbit of the second kind. Other cases may be treated in a similar way.

3) $|A|<|B|$, $|C|<|D|$. In this case, as the vertical and horizontal isoclines are both from left to right, the proof of the nonexistence of closed orbits of the second kind is similar, and hence is omitted.

§ 2. Now let us take S_2 as the fundamental square. Then the 4 critical points O $(0, 0)$, $O'(\pi, 0)$, $O''(0, \pi)$ and $O^*(\pi, \pi)$ give rise to a single critical point on the torus with index zero. Notice that A, B, C and D change their signs at the same time under the transformation

$$x'=\pi-x, \quad y'=\pi-y,$$

the vector field and hence the phase-portrait of system (1) will therefore be symmetric with respect to $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$. Aside from this, we can use also the following transformations to simplify our discussion:

1) $x'=y$, $y'=x$ (take symmetry with respect to the line $x=y$), which interchanges A with D, and B with C.

2) $x'=x$, $y'=\pi-y$, or $x'=\pi-x$, $y'=y$ $\left(\text{take symmetry with respect to } y=\frac{\pi}{2} \text{ or to } x=\frac{\pi}{2}\right)$, which changes the signs of C and D, or signs of A and B.

3) $x'=\pi-y$, $y'=\pi-x$ (take symmetry with respect to $x+y=\pi$), gives which

$$A'=-D, \quad D'=-A, \quad B'=-C, \quad C'=-B.$$

It is easily seen that, by means of these transformations, we may confine our discussion to the case $AD-BC>0$ under one of the following conditions:

I $\qquad B>0, \quad C<0;$

II $\qquad A>0, \quad D>0.$

Of course, several subcases of I and those of II may overlap. This we shall point out in the sequel.

Notice that, for analysing the topological structure of the phase-portrait of system (1), whether O is a focus or a node is a matter of indifference, but it is important to know in every case the slope of the 2 exceptional directions at O' and O'', and also at O and O^*, when they are nodes. This determines whether there were orbits running out from these critical points and going into S_2, or orbits going into these points from S_2. Let $k_1 \geqslant k_2$ $(k_1' \geqslant k_2')$ be the slopes of the exceptional directions at O and O^* (O' and O''), then k_i $(i=1, 2)$ satisfies the equation

$$Bk_i^2+(A-D)k_i-C=0 \tag{7}$$

and k_i' $(i=1, 2)$ satisfies

$$Bk_i'^2-(A+D)k_i'+C=0. \tag{8}$$

Analysis of the vertical and horizontal isoclines is the same as in § 1, but here if $AB>0$ (or $CD>0$), then the vertical (or horizontal) isocline has no locus in S_2.

Let us examine cases I and II in detail.

I $B>0$, $C<0$. There are 6 subcases.

(i) $A=D=0$. System (1) has at this time the general integral

$$B\cos y - C\cos x = K. \tag{9}$$

When $B=-C$, the line $x+y=\pi$ is an orbit, and we have Fig. 8, in which the torus is filled with closed orbits.[1] This is a special case of the later Theorem 2.

When $B\neq -C$, but B and C are commensurable, we can prove that all orbits are closed. Let us take the case $B=-3C$ as an example. From (9) we see that the orbit passing through $O''(0, \pi)$ is $-3\cos y - \cos x = 2$. (Fig. 9) It intersects $x=\pi$ at $P\left(\pi, \cos^{-1}\left(-\frac{1}{3}\right)\right)$. The symmetric point of P (by symmetric points we mean two points lying on opposite sides of S_2 and having equal abscissa or ordinate, which are actually the same point on the torus) is $P'\left(0, \cos^{-1}\left(-\frac{1}{3}\right)\right)$. The orbit passing through P' is $3\cos y + \cos x = 0$, which intersects $x=\pi$ at $Q\left(\pi, \cos^{-1}\frac{1}{3}\right)$; the orbit passing through $Q'\left(0, \cos^{-1}\frac{1}{3}\right)$ is $3\cos y + \cos x = 2$, which passes through $O'(\pi, 0)$. Therefore, these three orbits make together a closed orbit on the torus. Since now k_1 and k_2 are imaginary, and only k'_2 is negative, we see that the integral curve passing through the critical point O on the torus is unique.[2]

Similarly, for any point R_1 on the segment $\overline{O''O^*}$, the 4 orbits $\widehat{R_1R_2}$, $\widehat{R'_2R_3}$, $\widehat{R'_3R_4}$ and $\widehat{R'_4R'_1}$ make together a closed orbit on the torus. It is easily seen that the rotation number ρ of every orbit in Fig. 9 is $\frac{1}{3}$.[3]

When B and C are noncommensurable, from (9) and a well known theorem in number theory, we see that the torus is filled with ergodic orbits, all having the rotation number $\rho=\frac{|C|}{B}$.

Now suppose $|A|+|D|\neq 0$, without loss of generality, we may assume $A\neq 0$.

(ii) $A>0$, $D\leqslant 0$. As in (i), the integral curve passing through O is unique. If we have also

$$C+D=-(A+B) \quad \text{or} \quad A+D=-(B+C), \tag{10}$$

then $x+y=\pi$ is an orbit of system (1) (see Fig. 10·1). Otherwise, we will have Figs. 10·2 and 10·3.

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1) Of course, in § 2, no closed orbits of the first kind will exist.

2) In this case, as well as in cases (ii) and (iii), there is no difference between O and an ordinary point of the system (1), except when orbit enters (or leaves) O we have $t\to+\infty$ $(-\infty)$. O is called in these cases "a removable critical point".

3) For the sake of definiteness, by the rotation number ρ of an closed orbit on torus we mean the ratio of the number of its longitudinal crossing in S_2 to the number of its transversal crossing. When the orbit is not closed we take the limit of this ratio as the definition of ρ, which coincides with the classical definition of rotation number on torus.

Theorem 2. *Suppose* (10) *holds, then if* $A=-D$ (*so that* $B=-C$), *the torus is filled with closed orbits; while if* $A\neq -D$, *system* (1) *has one and only one closed orbit* $x+y=\pi$, which *is the* α *and* ω *limit set of all other orbits, in other words,* $x+y=\pi$ *is the unique limit cycle (semi-stable) of system* (1).

Proof 1) When $A=-D$, $B=-C$, the vector field in S_2 is symmetric with respect to $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, the line $x+y=\pi$, and the line $x=y$. From this it is easily seen that for any point P_1 on the left side of S_2, the orbits $\widehat{P_1P_2}$ and $\widehat{P'_2P'_1}$ will make a closed orbit on the torus. In fact, system (1) has now the general integral (2).

2) When $A+D=-(B+C)\neq 0$, the same as in § 1 case I, we construct a family of generalized rotated vector fields (5). Since (5) for $\delta=0$ has a family of closed orbits filling up the torus, it will certainly have no closed orbit for $\delta\neq 0$, except a part of the stationary set, i. e., $x+y=\pi$, which is still a closed orbit of (5). The α and ω limit property of $x+y=\pi$ with respect to other orbits is trivial.

Remark. *The first part of the conclusion of Theorem 2 holds also for the case* $A<0$, $D\geqslant 0$, *and the second part holds also for* $A\geqslant D>0$.

Let us examine now the case when (10) is not satisfied. Suppose $C_0+D=-(A+B)$, and $C\neq C_0$. Under the condition $AB>0$, system (1) represents a family of rotated vector fields when C varies. Since (1) has a family of closed orbits with $\rho=1$ when $A=-D$, $C=C_0$, it will have no closed orbit with $\rho=1$ when $A=-D$, $C\neq C_0$; and since (1) has a semi-stable limit cycle with $\rho=1$ when $A\neq -D$, $C=C_0$, it will have no closed orbit with $\rho=1$ when $C>C_0$, $A\neq -D$. In these two cases, the orbits of system (1) will either be all ergodic, or (1) will have closed orbit with $\rho\neq 1$.

On the other hand, if $A\neq -D$, $C<C_0$, then (1) will have at least two limit cycles (symmetric with respect to $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$) of opposite stability, which are generated from the semi-stable cycle of (1) when $A\neq -D$, $C=C_0$.

(iii) $A\geqslant D>0$. Suppose first $C=-D<0$, then the horizontal isoclines are the lines $x=y$ and $x+y=\pi$ (see Fig. 11). It is easily seen from the contraction mapping principle that the mapping of $\overline{O''P}$ onto $\overline{MQ}$ induced by flows of system (1) has at least a fixed point. It means that there exists at least a closed orbit $l_1=\widehat{R_1S_1}=\widehat{R_1R'_1}$ lying above the orbit $\widehat{PQ}$ passing through $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$. By symmetry, there exists at least a closed orbit $l_2=\widehat{R_2S_2}=\widehat{R_2R'_2}$ below $\widehat{PQ}$. Let us prove that these are the only closed orbits of system (1). Integrating (1) along l_1, we get

$$y_{S_1}=y_{R_1}+\int_0^{\pi}\frac{C\sin x+D\sin y}{A\sin x+B\sin y}\,dx.\ ^{1)}$$

Since $y_{S_1}=y_{R_1}$, this implies

1) Integrals which follow are all taken along periodic orbits that we considered.

$$\int_0^\pi \frac{C\sin x+D\sin y}{A\sin x+B\sin y}\,dx=0. \tag{11}$$

Notice that

$$\frac{\partial}{\partial y}\left(\frac{C\sin x+D\sin y}{A\sin x+B\sin y}\right)=\frac{(AD-BC)\sin x\cos y}{(A\sin x+B\sin y)^2}, \tag{12}$$

which is non-positive in the region D_1: $0\leqslant x\leqslant\pi$, $\frac{\pi}{2}\leqslant y\leqslant\pi$, in which l_1 lies. Therefore, as (11) holds along l_1, it can not hold along any other closed orbit above $\widehat{PQ}$, thus the uniqueness of l_1 is proved. The uniqueness of l_2 in D_2: $0\leqslant x\leqslant\pi$, $0\leqslant y\leqslant\frac{\pi}{2}$ follows from symmetry.

When C increases from $-D$, the phase-portrait and uniqueness of l_1 and l_2 are similar to Fig. 11. As $C\to 0$, l_1 and l_2 approach the semi-stable cycle $y=0$ (i. e., $y=\pi$) from different sides. When C decreases from $-D$, l_1 and l_2 approach each other in S_2, until both of them coincide with a closed orbit l^* passing through $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\left(0,\frac{\pi}{2}\right)=\left(\pi,\frac{\pi}{2}\right)$ when $C=C^*$ $(<-D)$. But as long as l_i intersects the line $y=\frac{\pi}{2}$ (when $C>C^*$), the proof of uniqueness given above will no longer be valid. Let us give a new proof by means of a more delicate estimation of integrals.

Assume that apart from the closed orbit l_2 in the lower part of S_2 there is another closed orbit l_3 (see Fig. 12). Let $y=y_i(x)$ $(i=2, 3)$ be their equations, and set $\delta(x)=y_2(x)-y_3(x)$. Then $\delta(x)$ satisfies the equation

$$\frac{d\delta(x)}{dx}=\frac{\partial}{\partial y}\left(\frac{C\sin x+D\sin y}{A\sin x+B\sin y}\right)_{y=\xi(x)}\cdot\delta(x)=\frac{(AD-BC)\sin x\cos\xi(x)}{(A\sin x+B\sin\xi(x))^2}\,\delta(x), \tag{13}$$

in which $y_3(x)<\xi(x)<y_2(x)$. Since $\delta(0)=y_2(0)-y_3(0)=y_2(\pi)-y_3(\pi)=\delta(\pi)$, from (13) we get

$$\int_0^\pi \frac{\sin x\cos\xi(x)\cdot\delta(x)}{(A\sin x+B\sin\xi(x))^2}\,dx=0. \tag{14}$$

We shall prove that the equality (14) can not exist. Let the portion of l_2 lying above $y=\frac{\pi}{2}$ correspond to $x\in(x_1, x_2)$, for which $\xi(x)$ may be greater than $\frac{\pi}{2}$, and $\cos\xi(x)$ may be negative. Let the equation of $\widehat{PQ}$ be $y=\bar{y}(x)$, then $\bar{y}(\pi-x)=\pi-\bar{y}(x)$. Since

$$\xi(x)<y_2(x)<\bar{y}(x),\quad \xi(\pi-x)<y_2(\pi-x)<\bar{y}(\pi-x)=\pi-\bar{y}(x),$$

hence if $\xi(x)>\frac{\pi}{2}$, then

$$0>\cos\xi(x)>\cos\bar{y}(x),\quad \cos\xi(\pi-x)>\cos\bar{y}(\pi-x)=-\cos\bar{y}(x)>0. \tag{15}$$

Let $\xi(x)=\frac{\pi}{2}+\eta(x)$, $\bar{y}(x)=\frac{\pi}{2}+\bar{\eta}(x)$, then

$$\bar{\eta}(x)>\eta(x),\quad \xi(\pi-x)<\bar{y}(\pi-x)=\frac{\pi}{2}-\bar{\eta}(x), \tag{16}$$

and we have

$$\sin\xi(x)>\sin\bar{y}(x)=\cos\bar{\eta}(x)=\sin\left(\frac{\pi}{2}-\bar{\eta}(x)\right)>\sin\xi(\pi-x). \tag{17}$$

Now divide the integral in (15) into five parts

$$\int_0^\pi=\int_0^{x_1}+\int_{x_1}^{x_2}+\int_{x_2}^{\pi-x_2}+\int_{\pi-x_2}^{\pi-x_1}+\int_{\pi-x_1}^{\pi}=I_1+I_2+I_3+I_4+I_5, \tag{18}$$

in which only I_2 may be negative in value. In the notations of Fig. 12, it is easily seen that $\delta_1>\delta(0)$, $\delta_2<\bar{\delta}_2<\bar{\delta}_1<\delta(\pi)$. If $\delta_2\geqslant\delta_1$, then $I_2\geqslant0$, which proves the nonexistence of (14). If $\delta_2<\delta_1$, we can prove that $I_2+I_4\geqslant0$. In fact,

$$I_2+I_4=\int_{x_1}^{x_2}[\quad]dx+\int_{\pi-x_2}^{\pi-x_1}[\quad]dx,$$

and from (16), (17) and $\bar{\delta}_2>\delta_2$, we see that the value of the integrand in I_4 at $\pi-x$ is greater than the absolute value of the integrand in I_2 at x, hence $I_2+I_4\geqslant0$, and (14) can not exist, too.

Finally, let us say a little more about the rotation number ρ of the orbits of system (1) when C varies from 0 to $-\infty$. We have already seen, when $C\in[C^*,0]$, $\rho\equiv0$. If C decreases from C^*, both of the separatrixes $\widehat{O''M}$ and $\widehat{O'N}$ in Fig. 11 turn clockwise, and they coincide with $x+y=\pi$ when $C=-D-A-B$. At this time, the semi-stable limit cycle $x+y=\pi$, as well as all orbits approaching it, have rotation number $\rho=1$. If C decreases furthermore and ultimately approachs $-\infty$, then the limit of system (1) is $\frac{dx}{dy}=0$, i. e., all orbits have their limiting position $x=$const., consequently, $\rho\to\infty$. Since the right hand side of (1) varies monotonically and continuously with C, and ρ varies monotonically and continuously with the right hand side of (1) (see [7]), we see that ρ must take all values in $(0,+\infty)$ as C varies from 0 to $-\infty$. When ρ takes an irrational value, obviously every orbit will be ergodic; but if ρ takes a rational value other than 0 and 1, two possibilities of the phase-portrait of system (1) may happen: a) there is a family of closed orbits, b) only one or two orbits are closed, while all other orbits are open. Evidently,in case b) ρ will correspond to a closed interval of values of C, say $[C_1, C_2]$, and system (1) will have a semi-stable limit cycle when $C=C_1$ or C_2, but two limit cycles when $C\in(C_1,C_2)$. We hope that the actual state will be case b), but we are still unable to give a rigorous proof.

(iv) $D>A>0$. Now if $(D-A)^2+4BC<0$, then (7) has imaginary roots, and the phase-portrait is similar to case (iii). If $(D-A)^2+4BC\geqslant0$, then $k_i>0$ $(i=1,2)$, and there will be at least two orbits passing through the critical point O. In this time, apart from limit cycles like l_1 and l_2, there will appear a region filled with singular cycles (with $\rho=0$) all passing through O (see Fig. 13). Moreover, according as the x coordinate of S is equal to $\frac{\pi}{2}$ or not, this latter region will reduce to a singular cycle, or on the contrary, may fill up the whole torus, as is easily seen for $A=B=-C=1$, $D=3$ (see Fig. 14).

(v) $A<0$, $D<0$. This case can be transformed into cases (iii) and (iv) by the transformation $x'=\pi-y$, $y'=\pi-x$.

(vi) $A<0$, $D\geqslant 0$. In this case k_1 and k_2 are both imaginary or both positive, $k'_2<0<k'_1$, and all kinds of topological structures of the phase-portrait described in (i)—(v) may happen.

II $A>0$, $D>0$. In view of case I, it suffices to consider the following seven subcases:

(i) $B=C=0$. System (1) has a first integral

$$\left(\operatorname{tg}\frac{y}{2}\right)^{A}=K\left(\operatorname{tg}\frac{x}{2}\right)^{D}.$$

The four sides of S_2 are all orbits, and the torus is filled with singular cycles all passing through O (see Fig. 15).

(ii) $C=0$, $B>0$. Now $y=0$ and $y=\pi$ are orbits, and there are 4 different kinds of phase-portrait as shown in Figs. 16·1—16·4. In Fig. 16·1, O is a removable critical point[1)], and $y=0$ $(y=\pi)$ is a semi-stable limit cycle. In Figs. 16·2—16·4, O is a true critical point, through which there are infinitely many orbits. There are two different possible structures in Fig. 16·2: a) The two separatrixes $\widehat{OM}$ and $\widehat{NO^*}$ become ultimately the same orbit on torus when continued in different directions. Suppose it has the rotation number $\frac{1}{p}$ $(p\geqslant 2)$, then all other orbits, except $y=0$, have rotation number $\frac{1}{p+1}$. b) $\widehat{OM}$ and $\widehat{NO^*}$ are different orbits on torus, then together with $y=0$, they divide the torus into two regions, one is filled with singular cycles having rotation number $\frac{1}{p}$, and the other is filled with singular cycles having rotation number $\frac{1}{p+1}$. Figs. 16·3 and 16·4 correspond to cases a) and b) of Fig. 16·2 respectively with $p=1$. It is easily seen that if in Fig. 16·2 $B\gg D>A$, then p may be very large.

(iii) $C=0$, $B<0$. Here are three different kinds of phase-portrait as shown in Figs. 17·1—17·3. The situation is similar to that of Figs. 16·2—16·4, i. e., there are two different possible structures in Fig. 17·1, while 17·2 and 17·3 correspond to these two cases respectively with $p=0$.

(iv) $B=0$, $C\neq 0$. This case may be transformed into (ii) and (iii) by the transformation $x=y'$, $y=x'$.

(v) $C<0$, $B<0$. There are 3 different kinds of phase-portrait as shown in Figs. 18·1—18·3. In Fig. 18·1 three separatrixes $\widehat{OO'}$, $\widehat{O'O''}$ and $\widehat{O''O^*}$ divide the torus into two regions, orbits in one region all have rotation number $\rho=0$, and in the other region $\rho=\infty$. In Fig. 18·2, four separatrixes $\widehat{OO'}$, $\widehat{OO''}$, $\widehat{O'O^*}$ and $\widehat{O''O^*}$ divide the torus into three regions filled with orbits having rotation numbers 0, 1 and ∞ respectively.

1) See footnote 1) of case I (i).

In Fig. 18·3, there are two possibilities, one similar to Fig. 18·1, the other similar to Fig. 18·2, just as we have discussed in cases (ii) and (iii).

(vi) $C>0$, $B<0$.

(vii) $C>0$, $B>0$.

These two subcases can be transformed into subcases of case I by one of the four transformations used before.

Summary. The different kinds of topological structures of phase-portrait of system (1) in § 2 are the following:

1) Ergodic type: The torus is filled with ergodic orbits. This occurs when $A=D=0$, B and C are noncommensurable; and also in case I (iii), when ρ is irrational.

2) Periodic motion type: The torus is filled with a family of periodic orbits. This occurs when $A=-D$, $B=-C$; or $A=D=0$, and B, C are commensurable.

3) Limit cycle type: (i) There are two simple limit cycles, one is stable, the other is unstable; they are the ω and α limit sets of all other orbits respectively. (ii) The two simple limit cycles coincide, becoming a semi-stable limit cycle.

In all the above types, the point O is a removable critical point.

4) Singular cycle type: The torus is filled with singular cycles all passing through the critical point O. According to different values of A, B, C and D, these singular cycles (except the separatrixes) may form a single simply connected region (Figs. 15, 16·3 and 17·2), two simply connected regions (Figs. 16·4, 17·3 and 18·1), or three simply connected regions (Fig. 18·2). Cycles in each region all have the same rotation number.

5) Mixed type or Intermediate type: There are both limit cycles and actual critical point, through the latter we have singular cycles and also orbits approaching limit cycles.

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环面上一个含奇点的微分方程的定性研究

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摘　　要

关于环面上无奇点的动力体系的研究．自从 H. Poincaré 的开创性工作以后，较早的研究工作已见于 Coddington 与 Levinson 的书中．近期则有秦元勋的工作，他已研究了具体的微分方程，但仍保持无奇点的假设．近年来国内外又出现了不少研究一般二维流形上动力体系的拓扑结构或分类的文章，其中考虑了奇点，但却没有具体的微分方程．本文类比于平面线性定常系统，研究了环面上的微分方程

$$\frac{dx}{dt}=A\sin x+B\sin y,\ \frac{dy}{dt}=C\sin x+D\sin y,\quad (AD-BC\neq 0) \tag{1}$$

的轨线的全局结构．

在 §1 中假设(1)定义在(x, y)平面上的正方形 S_1: $0\leqslant x\leqslant 2\pi$, $0\leqslant y\leqslant 2\pi$ 内，然后把 S_1 的两对对边等同起来，从而得到环面上的解析系统．它有两个初等的非鞍点和两个初等鞍点．经过分析，得到中心-鞍点，结点-鞍点和焦点-鞍点等三种可能拓扑结构，在最后一种情况有时能出现极限环，但唯一性未能证明．此外，环面上不存在第二类周期轨线．

在 §2 中假设(1)定义在正方形 S_2: $0\leqslant x\leqslant \pi$, $0\leqslant y\leqslant \pi$ 内，再把 S_2 的两对对边等同起来，从而得到环面上的 C^1 系统．此系统只有一个指标为零的奇点，但它的轨线拓扑结构的可能情况要比 §1 多一些．环面可以被具有相同旋转数的一族闭轨线所充满，也可以被一族各态历经的轨线所充满．它可能具有唯一的半稳定极限环，或是一个稳定环和一个不稳定环，一切其它轨线都从正负向趋向它们．环面还可能被分成一个，两个或三个单连通域，每一域中充满着具有相同旋转数的奇闭轨线．最后，环面上也可能既存在极限环，又存在为奇闭轨线所充满的区域．此外，我们还固定(1)式右边的三个系数 A, B, D，而让 C 从零变到 $-\infty$，以观察方程的全局结构和轨线的旋转数的变化．

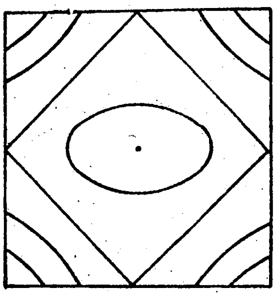

Fig.1.1

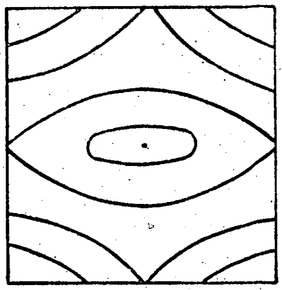

Fig.1.2

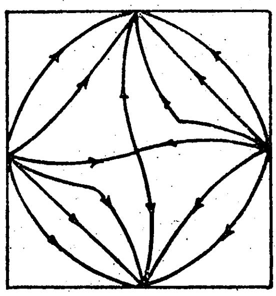

Fig.2

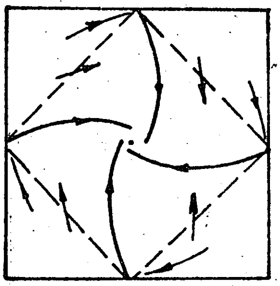

Fig.3.1

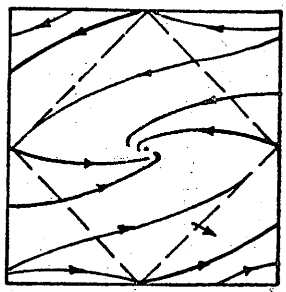

Fig.3.2

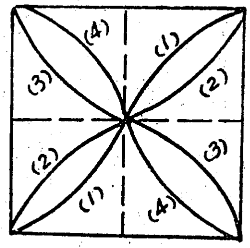

Fig.4

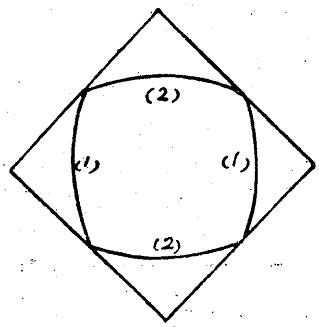

Fig.5.1 $AD>0$

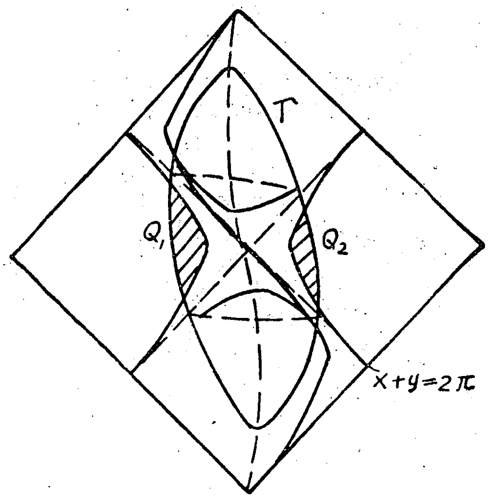

Fig.5.6

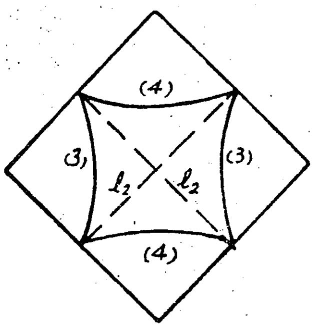

Fig.5.2 $AD<0$

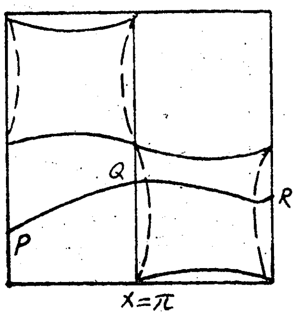

Fig.7

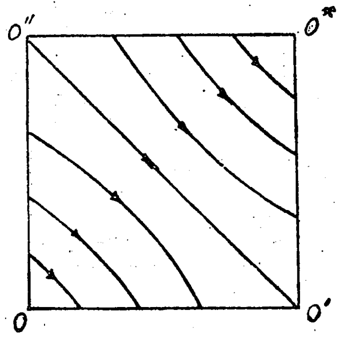

Fig.8

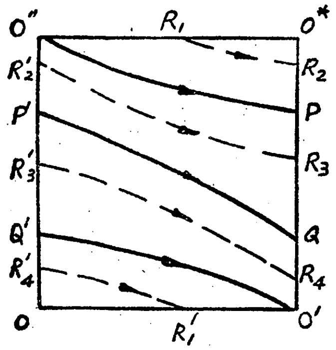

Fig.9

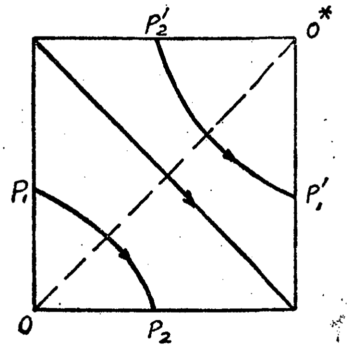

Fig.10.1

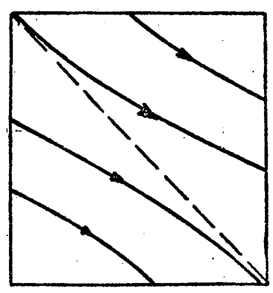

Fig.10.2 $(C+D>-A-B)$

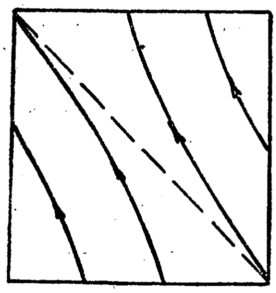

Fig.10.3$(C+D>-A-B)$

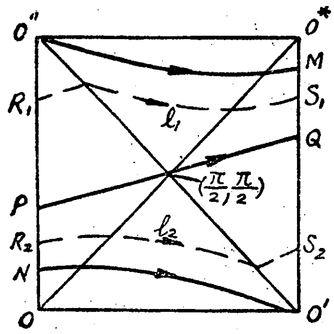

Fig.11

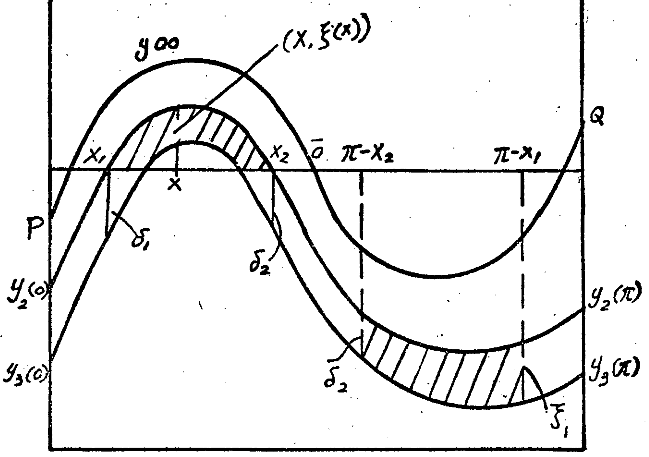

Fig.12

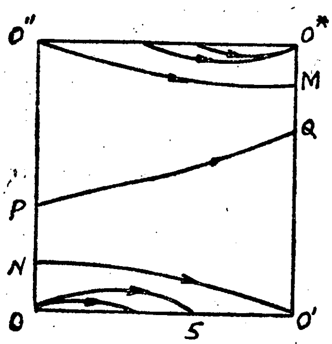

Fig.13

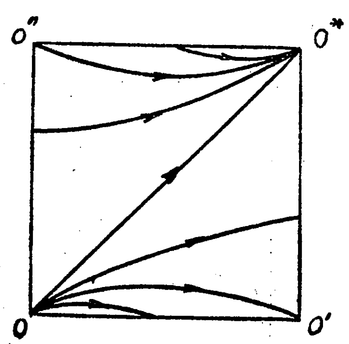

Fig.14

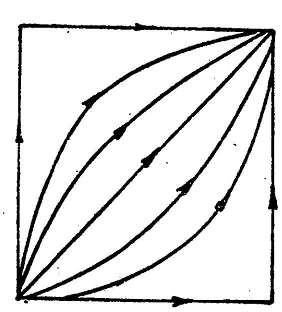

Fig.15

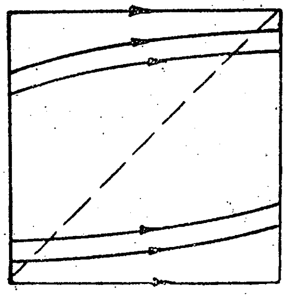

Fig.16.1($A>D>0$)

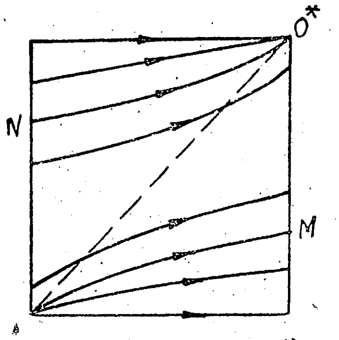

Fig.16.2($A+B>D>A$)

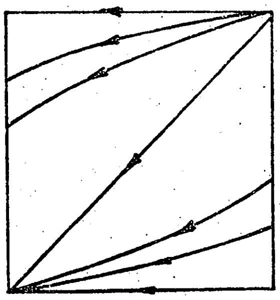

Fig.16.3($A+B=D$)

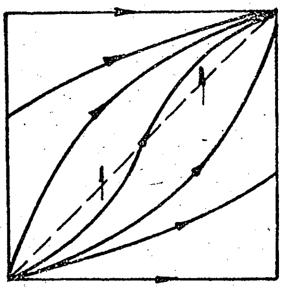

Fig.16.4($A+B<D$)

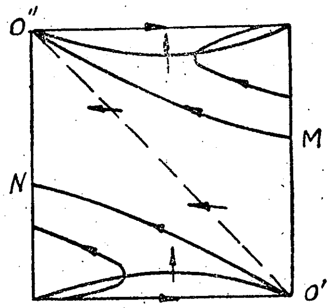

Fig.17.1

$\left(\begin{matrix} D\geqslant A, \dfrac{D}{A+B}>-1 \\ A+B<0 \end{matrix}\right)$

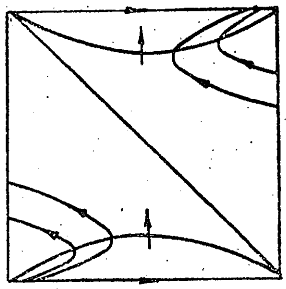

Fig.17.2

($D=-A-B, D\geqslant A$)

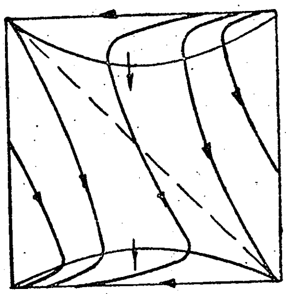

Fig.17.3(others)

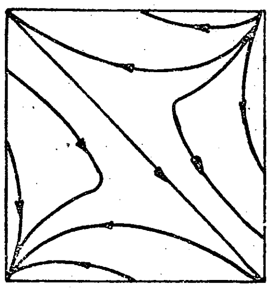

Fig.18.1

($C+D=-A-B$)

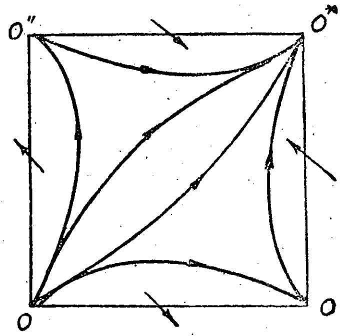

Fig.18.2

$\left(\begin{matrix} C+D>0 \\ A+B\geqslant 0 \end{matrix}\right)$

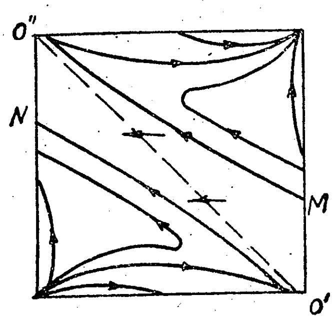

Fig.18.3

($0<C+D<-A-B$)