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ON A FREE BOUNDARY PROBLEM

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1. Introduction

D. G. Schaeffer[1, 2] discussed a free boundary problem rising in the supersonic plane flow past a curved wedge. By means of the Nash-Moser implicit function theorem, he proved that the flow possesses the same qualitative features as the corresponding flow past a straight wedge. In this note we shall prove that, using the general results on the free boundary problems for quasi-linear hyperbolic systems in [4, 5, 6], the same result can be easily obtained without the additional hypothesis on the Hölder continuity.

For the isentropic irrotational steady plane flow (the number of equations is equal to 2), using the hodograph method we have transformed this problem in to a linear one and obtained the local solution[7]; for the general steady plane flow (the number of equations is equal to 3 with entropy as the third unknown), by means of the stream function, this problem has been transformed in to a problem with the fixed boundary curves and the similar result has been obtained[8].

In this note we restrict our discussion to the general steady plane flow for polytropic gases. For the isentropic irrotational steady plane flow, a similar discussion can be given (see also [11]).

2. A theorem for typical free boundary problems with a characteristic boundary curve

We consider the following free boundary problem for the quasi-linear hyperbolic system

$$\sum_{j=1}^{n}\zeta_{lj}(t,\ x,\ u)\left(\frac{\partial u_j}{\partial t}+\lambda_l(t,\ x,\ u)\frac{\partial u_j}{\partial x}\right)=\mu_l(t,\ x,\ u)\quad (l=1,\ \cdots,\ n). \tag{2.1}$$

On the known boundary curve $x=g(t)$ $(g(0)=0)$:

$$\sum_{j=1}^{n}\zeta^0_{rj}u_j=G_r(t,\ x,\ u)\quad (r=1,\ \cdots,\ m). \tag{2.2}$$

On the free boundary curve $x=x(t)$ $(x(0)=0)$:

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$$\sum_{j=1}^{n}\zeta_{sj}^{0}u_j=G_s(t,\ x,\ u)\quad (s=m+1,\ \cdots,\ n)\tag{2.3}$$

and

$$\frac{dx}{dt}=F(t,\ x,\ u),\tag{2.4}$$

where ζ_{lj}, λ_l, μ_l, G_l $(l,\ j=1,\ \cdots,\ n)$ and F are continuously differentiable functions with respect to all arguments on the considered region,

$$\det|\zeta_{lj}|\neq 0\tag{2.5}$$

and $g(t)$ has a continuous second derivative.

We assume that at the origin the boundary conditions (2.2), (2.3) possess a unique solution $u=u^0=\{u_1^0,\ \cdots,\ u_n^0\}$ and

$$\zeta_{lj}^0=\zeta_{lj}(0,\ 0,\ u^0).\tag{2.6}$$

We assume

$$\lambda_r^0<F^0<g'(0)<\lambda_{s'}^0\quad (r=1,\ \cdots,\ m;\ s'=m+2,\ \cdots,\ n),\tag{2.7}$$

where

$$\lambda_l^0=\lambda_l(0,\ 0,\ u^0),\ F^0=F(0,\ 0,\ u^0)\quad (l=1,\ \cdots,\ n).\tag{2.8}$$

We assume again that the condition (2.2) implies the following condition: on $x=g(t)$:

$$\frac{dg(t)}{dt}=\lambda_{m+1}(t,\ x,\ u).\tag{2.9}$$

Thus, $x=g(t)$ is a $(m+1)$-th characteristic curve.

We call this problem a typical free boundary problem with a characteristic boundary curve.

Let

$$v_j=\sum_{k=1}^{n}\zeta_{jk}^0u_k.\tag{2.10}$$

Introducing the characterizing matrix of this problem

$$\Theta\equiv(\theta_{lk})=\left(\frac{\partial G_l}{\partial v_k}(0,\ 0,\ u^0)\right).\tag{2.11}$$

We have proved in [5] the following

Theorem 1. *Suppose*

$$|\Theta|_{\min}\equiv\inf_{\substack{\gamma_j\neq 0\\ j=1,\cdots,n}}\ \max_{l=1,\cdots,n}\sum_{k=1}^{n}\left|\frac{\gamma_l}{\gamma_k}\theta_{lk}\right|<1,\tag{2.12}$$

where γ_j $(j=1,\ 2,\ \cdots,\ n)$ *are arbitrary real numbers, then the problem* (2.1)—(2.4) *possesses a unique solution on a suitably small region*

$$R(\delta)=\{(t,\ x)\,|\,0\leqslant t\leqslant\delta,\ x(t)\leqslant x\leqslant g(t)\}.\tag{2.13}$$

i. e., there exist uniquely a twice continuously differentiable function $x=x(t)$ $(x(0)=0;$ $x(t)>g(t),\ t>0)$ *on the interval* $0\leqslant t\leqslant\delta$ *and a continuously differentiable vector-function* $u=u(t,\ x)$ *defined on the region* $R(\delta)$ *such that they satisfy* (2.1)—(2.4) *on* $R(\delta)$, *where* $\delta>0$ *is suitably small.*

From the demonstration of theorem 1 ([4, 5]; see also [10]), we can obtain

Remark 1. *The height δ of the existence region $R(\delta)$ of the solution depends continuously only on:*

1° *the C^1 norm of the functions ζ_{lj}, λ_l, μ_l, G_l, F $(l, j=1, \cdots, n)$ and the modulus of continuity of $\frac{\partial G_l}{\partial u_j}$ $(l, j=1, \cdots, n)$;*

2° *the C^2 norm of $g(t)$;*

3° *the C^0 norm of $1/\det|\zeta_{lj}|$.*

Remark 2. *If all the known functions have the additional Hölder continuity, then the solution $u=u(t, x)$ and $x=x(t)$ have also the corresponding additional Hölder continuity.*

We point out that if Θ is a 3×3 matrix as follows

$$\Theta=\begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix}, \tag{2.14}$$

then

$$|\Theta|_{\min}=\sqrt{|ac|}. \tag{2.15}$$

3. Statement for the supersonic plane flow past a curved wedge

For a uniform supersonic oncoming flow (velocity $\boldsymbol{q}=(u, v)=(q_0, 0)$, pressure $p=p_0$, sound speed $c=c_0$, density $\rho=\rho_0$, entropy $S=S_0$), if the equation of the surface of the curved wedge is $y=f(x)$ $(f(0)=0)$ (cf. Fig. 1), we are require to find an unknown shock wave curve $y=y(x)$ $(y(0)=0)$ and the functions $u=u(x, y)$, $v=v(x, y)$, $p=p(x, y)\cdots$ defined on a suitable angular region between the wedge and the shock such that they satisfy the following system

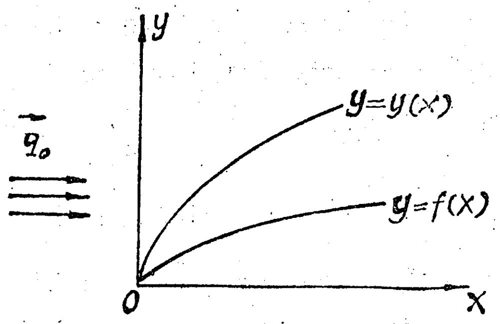

Fig. 1

$$\begin{cases} \dfrac{\partial(\rho u)}{\partial x}+\dfrac{\partial(\rho v)}{\partial y}=0, \\ u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}=0, \\ u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}=0, \\ u\dfrac{\partial S}{\partial x}+v\dfrac{\partial S}{\partial y}=0, \\ p=A(S)\rho^\gamma \quad (A(S)=(\gamma-1)\exp C_v^{-1}(S-S_*)) \end{cases} \tag{3.1}$$

with the Bernoulli's law

$$\mu^2(u^2+v^2)+(1-\mu^2)c^2=c_*^2 \quad \left(\mu^2=\frac{\gamma-1}{\gamma+1}\right), \tag{3.2}$$

and the following boundary conditions (cf. [3]):

on $y=f(x)$:

$$v=f'(x)u, \tag{3.3}$$

on $y=y(x)$:

$$v=(q_0-u)\sqrt{\frac{u-\tilde{u}}{U_0-u}}, \tag{3.4}$$

$$p-p_0=\rho_0 q_0(q_0-u) \tag{3.5}$$

and

$$\frac{dy}{dx}=\frac{q_0-u}{v}, \tag{3.6}$$

where γ is the adiabatic exponent, C_v is the specific heat at constant volume, S_* is an appropriate constant, c_* is the critical speed and

$$\tilde{u}=\frac{c_*^2}{q_0}, \quad U_0=(1-\mu^2)q_0+\tilde{u}. \tag{3.7}$$

If $\theta_0=\mathrm{tg}^{-1}f'(0)>0$ is not too great, according to the property of the shock polars, (3.4) and

$$v=f'(0)u \tag{3.8}$$

possess a unique supersonic solution (u_1, v_1): $u_1^2+v_1^2>c_*^2$. Thus we can determine the direction of the shock curve at the origin

$$y'(0)\equiv \mathrm{tg}\,\beta_0=\frac{q_0-u_1}{v_1} \tag{3.9}$$

and the corresponding values $p=p_1$, $S=S_1$ etc.

For the supersonic flow, the preceding system is a hyperbolic system. There are three characteristic directions

$$\frac{dy}{dx}=\mathrm{tg}(\theta\pm A)\equiv\zeta^{\pm}, \quad \frac{dy}{dx}=\mathrm{tg}\,\theta\equiv\zeta^0, \tag{3.10}$$

where θ is the angle between the flow direction and the positive x-axis: $\mathrm{tg}\,\theta=\frac{v}{u}$; A is the Mach angle defined by $\sin^2 A=c^2/q^2$. Besides, considering p, θ, S as the unknown functions, this system can be written in the following characteristic form (cf. [9])

$$\begin{cases} K(p,\theta,S)\left(\frac{\partial p}{\partial x}+\zeta^+\frac{\partial p}{\partial y}\right)+\left(\frac{\partial\theta}{\partial x}+\zeta^+\frac{\partial\theta}{\partial y}\right)=0, \\ \frac{\partial S}{\partial x}+\zeta^0\frac{\partial S}{\partial y}=0, \\ K(p,\theta,S)\left(\frac{\partial p}{\partial x}+\zeta^-\frac{\partial p}{\partial y}\right)-\left(\frac{\partial\theta}{\partial x}+\zeta^-\frac{\partial\theta}{\partial y}\right)=0, \end{cases} \tag{3.11}$$

where

$$K(p,\theta,S)=\frac{1}{\rho(u^2+v^2)\,\mathrm{tg}\,A}. \tag{3.12}$$

According to the property of the shock wave, we have $A_1\equiv A(u_1, v_1)>\beta_0-\theta_0$. From this, by means of a simple coordinate transformation

$$\begin{cases} X = x\sin\theta_0 - y\cos\theta_0, \\ Y = x\cos\theta_0 + y\sin\theta_0, \end{cases} \tag{3.13}$$

it is easy to see that the characteristic directions satisfy the conditions (2.7) and (2.9) (in which $n=3$, $m=1$) (see Fig. 2).

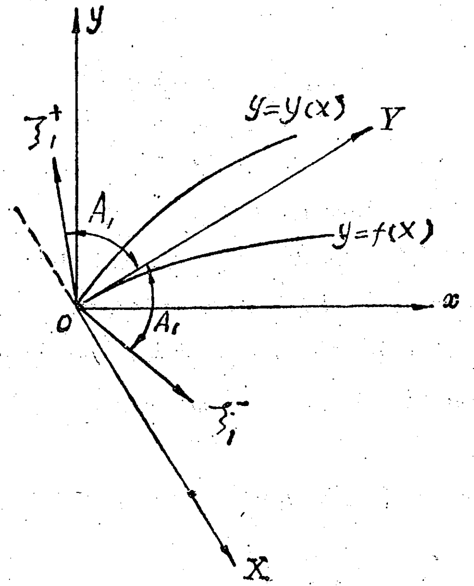

Fig. 2

Thus, in order to use theorem 1, it is sufficient to check the corresponding condition of solvability (2.12).

4. Justification of the condition of solvability and conclusions

Let

$$U = K_1 p + \theta,\ V = S,\ W = K_1 p - \theta, \tag{4.1}$$

in which

$$K_1 = K(p_1, \theta_1, S_1) > 0 \ \left(\theta_1 = \mathrm{tg}^{-1}\frac{v_1}{u_1}\right). \tag{4.2}$$

The boundary condition (3.3) can be written in the following form:

on $y=f(x)$:

$$U = W + 2\,\mathrm{tg}^{-1} f'(x). \tag{4.3}$$

Moreover, according to the property of the shock, along the shock polar we have $\frac{d\theta}{du}<0$, $\frac{dS}{du}<0$, $\frac{dp}{du}<0$, then $\frac{dU}{du}<0$. Hence the boundary condition (3.4), (3.5) can be rewritten in the following form:

on $y=y(x)$:

$$V = G(U), \tag{4.4}$$

$$W = H(U). \tag{4.5}$$

Thus, the characterizing matrix Θ of this problem can be written in the form (2.14) in which

$$a=1,\quad b=G'(U_1),\quad c=H'(U_1) \tag{4.6}$$

with $U_1=K_1p_1+\theta_1$. It is sufficient to prove

$$|H'(U_1)|<1. \tag{4.7}$$

According to the property of the shock, along the shock polar we have $\frac{dp}{d\theta}>0$. Noticing

$$H'(U_1)=\frac{K_1dp-d\theta}{K_1dp+d\theta},$$

we obtain (4.7) immediately.

Thus, from theorem 1 we obtain

Theorem 2. *If $f(x)\in C^2$ and $\theta_0=\mathrm{tg}^{-1}f'(0)>0$ is not too great, then on a suitable angular region between the wedge and the shock, the above free boundary problem (3.1)—(3.6) has a unique solution locally $y=y(x)\in C^2$ and $u=u(x, y)$, $v=(x, y)$, $p=p(x, y)$, $\cdots\in C^1$ that possesses the similar structure as the corresponding flow past a straight wedge.*

According to remark 2 of theorem 1, if $f(x)\in C^{2+\varepsilon}$, then $y(x)\in C^{2+\varepsilon}$, $u(x, y)$, $v(x, y)$, $p(x, y)\cdots\in C^{1+\varepsilon}$.

Moreover, for a corresponding straight wedge, the boundary condition (3.3) reduces to the following form:

on $y=f_0(x)\equiv f'(0)x$:

$$v=f'(0)u. \tag{3.3$'$}$$

It is well known that the unperturbed problem (3.1), (3.2), (3.3)$'$, (3.4)—(3.6) has a global solution: the shock is a straight line $y=y_0(x)\equiv \mathrm{tg}\,\beta_0\cdot x$ and the state behind the shock is uniform, $(u, v)=(u_1, v_1)$, $p=p_1$, $\cdots$. Thus, according to remark 1 of theorem 1, a continuity argument shows the following:

Theorem 3. *Under the assumptions of theorem 2, let $R>0$ be given, if the C^2 norm of $f(x)-f_0(x)$ is sufficiently small, then the solution in theorem 2 exists on an angular region containing the disk of radius R between the wedge and the shock and this solution is close to the corresponding unperturbed solution (precisely, the C^1 norm of $(u-u_1, v-v_1, p-p_1, \cdots)$ and the C^2 norm of $y(x)-y_0(x)$ are sufficiently small).*

If $f(x)\in C^{2+\varepsilon}$, from theorem 3 we obtain D. G. Schaeffer's result.

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关于一个自由边界问题

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摘 要

本文利用 [4]—[6] 中关于拟线性双曲型方程组自由边界问题的一般性结论，在一般的(非等熵)平面定常流动的情形，对 D. G. Schaeffer 关于超音速气流对尖缘平面翼型的绕流问题利用 Nash-Moser 隐函数存在定理所得到的结果，给出了一个简单而直接的证明，在证明中不需要有关 Hölder 连续性的附加假设.