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Anderson Localization for Jacobi Matrices Associated with High-Dimensional Skew Shifts*

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Abstract In this paper, the authors establish Anderson localization for a class of Jacobi matrices associated with skew shifts on $\mathbb{T}^d$, $d \geq 3$.

Keywords Anderson localization, Jacobi matrices, Skew shifts
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1 Introduction and Main Result

Over the past thirty years, there are many papers on the topic of Anderson localization for lattice Schrödinger operators

$$H = v_n \delta_{nn'} + \Delta, \tag{1.1}$$

where v_n is a quasi-periodic potential, Δ is the lattice Laplacian on $\mathbb{Z}$,

$$\Delta(n, n') = \begin{cases} 1, & |n - n'| = 1, \\ 0, & |n - n'| \neq 1. \end{cases}$$

Anderson localization means that H has pure point spectrum with exponentially localized states $\varphi = (\varphi_n)_{n \in \mathbb{Z}}$,

$$|\varphi_n| < \mathrm{e}^{-c|n|}, \quad |n| \to \infty. \tag{1.2}$$

We may associate the potential v_n to a dynamical system T as follows:

$$v_n = \lambda v(T^n x), \tag{1.3}$$

where v is real analytic on $\mathbb{T}^d$ and T is a shift on $\mathbb{T}^d$:

$$Tx = x + \omega. \tag{1.4}$$

Fix $x = x_0$. If λ is large and ω outside set of small measure, H will satisfy Anderson localization.

The proof of Anderson localization is based on multi-scale analysis and semi-algebraic set theory. In this line, Bourgain and Goldstein [6] proved Anderson localization for Schrödinger

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operators (1.1) with the help of fundamental matrix and Lyapounov exponent. By multi-scale method, Bourgain, Goldstein and Schlag [8] proved Anderson localization for Schrödinger operators on $\mathbb{Z}^2$,

$$H(\omega_1,\omega_2;\theta_1,\theta_2)=\lambda v(\theta_1+n_1\omega_1,\theta_2+n_2\omega_2)+\Delta. \tag{1.5}$$

Later, Bourgain [5] proved Anderson localization for quasi-periodic lattice Schrödinger operators on $\mathbb{Z}^d$, d arbitrary. Recently, using more elaborate semi-algebraic arguments, Bourgain and Kachkovskiy [10] proved Anderson localization for two interacting quasi-periodic particles.

More generally, we can study the long range model

$$H=v(x+n\omega)\delta_{nn'}+S_\phi \tag{1.6}$$

with Δ replaced by a Toeplitz operator

$$S_\phi(n,n')=\widehat{\phi}(n-n'), \tag{1.7}$$

where ϕ is real analytic, and $\widehat{\phi}(n)$ is the Fourier coefficient of ϕ. Bourgain [4] proved Anderson localization for the long-range quasi-periodic operators (1.6). Note that in this case, we cannot use the fundamental matrix formalism as (1.1). Bourgain's method in [4] also permits us to establish Anderson localization for band Schrödinger operators (cf. [9]),

$$H_{(n,s),(n',s')}(\omega,\theta)=\lambda v_s(\theta+n\omega)\delta_{nn'}\delta_{ss'}+\Delta, \tag{1.8}$$

where $\{v_s \mid 1\le s\le b\}$ are real analytic. Recently, this method was used in [13] to prove Anderson localization for quasi-periodic block operators with long-range interactions.

If the transformation T is a skew shift on $\mathbb{T}^2$:

$$T(x_1,x_2)=(x_1+x_2,x_2+\omega), \tag{1.9}$$

using transfer matrix and Lyapounov exponent, Bourgain, Goldstein and Schlag [7] proved Anderson localization for

$$H=\lambda v(T^n x)+\Delta. \tag{1.10}$$

In order to study quantum kicked rotor equation

$$\mathrm{i}\frac{\partial\Psi(t,x)}{\partial t}=a\frac{\partial^2\Psi(t,x)}{\partial x^2}+\mathrm{i}b\frac{\partial\Psi(t,x)}{\partial x}+V(t,x)\Psi(t,x),\quad x\in\mathbb{T}, \tag{1.11}$$

where

$$V(t,x)=\kappa\Big[\sum_{n\in\mathbb{Z}}\delta(t-n)\Big]\cos(2\pi x), \tag{1.12}$$

using multi-scale method, Bourgain [3] proved Anderson localization for the operator

$$W=\phi_{m-n}(T^m x), \tag{1.13}$$

where ϕ_k are trigonometric polynomials and T is a skew shift on $\mathbb{T}^2$.

However, there are few results on high-dimensional skew shifts. When $d\ge 3$, the skew shift $T:\mathbb{T}^d\to\mathbb{T}^d$ is given by

$$(Tx)_i=x_i+x_{i+1},\quad 1\le i\le d-1, \tag{1.14}$$

$$(Tx)_d = x_d + \omega, \quad x = (x_1, \cdots, x_d). \tag{1.15}$$

In [14], Krüger proved positivity of Lyapounov exponents for the Schrödinger operator

$$H = \lambda f((T^n x)_1)\delta_{nn'} + \Delta, \tag{1.16}$$

where T is a skew shift on $\mathbb{T}^d$, d is sufficiently large, and f is a real, nonconstant function on $\mathbb{T}$.

In this paper, we generalize Bourgain's result on skew shifts on $\mathbb{T}^2$ (cf. [3]) to higher dimensional ones on $\mathbb{T}^d$, $d \geq 3$. More precisely, we consider matrices $(A_{mn}(x))_{m,n\in\mathbb{Z}}$, $x \in \mathbb{T}^d$ associated with a skew shift $T : \mathbb{T}^d \to \mathbb{T}^d$ of the form

$$A_{mm}(x) = v(T^m x), \tag{1.17}$$

$$A_{mn}(x) = \phi_{m-n}(T^m x) + \overline{\phi_{n-m}(T^n x)}, \quad m \neq n, \tag{1.18}$$

where

$$v \text{ is a real, nonconstant, trigonometric polynomial,} \tag{1.19}$$

$$\phi_k \text{ is a trigonometric polynomial of degree} < |k|^{C_1}, \tag{1.20}$$

$$\|\phi_k\|_\infty < \gamma \mathrm{e}^{-|k|}. \tag{1.21}$$

We will prove the following result.

Theorem 1.1 *Consider a lattice operator $H_\omega(x)$ associated to the skew shift $T = T_\omega$ acting on $\mathbb{T}^d$, $d \geq 3$, of the form* (1.17)–(1.21). *Assume $\omega \in \mathrm{DC}$ (Diophantine condition),*

$$\|k\omega\| > c|k|^{-2}, \quad \forall k \in \mathbb{Z} \setminus \{0\}. \tag{1.22}$$

Fix $x_0 \in \mathbb{T}^d$. Then for almost all $\omega \in \mathrm{DC}$ and γ taken sufficiently small in (1.21) (*depending on the initial scale N_0*), *$H_\omega(x_0)$ satisfies Anderson localization.*

We summarize the scheme of the proof. As mentioned above, the transfer matrix and Lyapounov exponent approach is not applicable to the long range case here. We will use the multi-scale method developed in [3, 7]. Our basic strategy is the same as that in [3], but with more complicated computations. First, we need Green's function estimates for $G_{[0,N]}(E, x) = (R_{[0,N]}(H(x) - E)R_{[0,N]})^{-1}$, where R_Λ is the restriction operator to $\Lambda \subset \mathbb{Z}$. We will prove in Section 3 that

$$\|G_{[0,N]}(E, x)\| < \mathrm{e}^{N^{1-}}, \tag{1.23}$$

$$|G_{[0,N]}(E, x)(m, n)| < \mathrm{e}^{-\frac{1}{100}|m-n|}, \quad 0 \leq m, n \leq N, \ |m - n| > \frac{N}{10} \tag{1.24}$$

for $x \notin \Omega_N(E)$, where

$$\operatorname{mes} \Omega_N(E) < \mathrm{e}^{-N^\sigma}, \quad \sigma > 0. \tag{1.25}$$

The main difficulty here is to study the intersection of $\Omega_N(E)$ and skew shift orbits. We need to prove

$$\#\{n = 1, \cdots, M : T^n x \in \Omega_N(E)\} < M^{1-\delta}, \quad \delta > 0, \tag{1.26}$$

where

$$\log\log M \ll \log N \ll \log M. \tag{1.27}$$

To obtain (1.26), we study the ergodic property of skew shifts on $\mathbb{T}^d$ in Section 2.

Next, in Section 4, we use decomposition of semi-algebraic set to estimate

$$\text{mes}\{\omega \in \mathbb{T} : (\omega, T_\omega^j x) \in A,\ \exists\, j \sim M\} < M^{-c}, \quad c > 0,$$

where $x \in \mathbb{T}^d$, $A \subset \mathbb{T}^{d+1}$ is a semi-algebraic set of degree B and measure η, satisfying

$$\log B \ll \log M \ll \log \frac{1}{\eta}.$$

This is a key point to eliminate the energy E in the proof of Anderson localization.

Finally, using Green's function estimates and semi-algebraic set theory, we prove Anderson localization of the operator $H_\omega(x)$ in Section 5 as in [6–7].

We will use the following notations. For positive numbers a, b, $a \lesssim b$ means $Ca \le b$ for some constant $C > 0$. $a \ll b$ means C is large. $a \sim b$ means $a \lesssim b$ and $b \lesssim a$. N^{1-} means $N^{1-\epsilon}$ with some small $\epsilon > 0$. For $x \in \mathbb{T}$, $\|x\| = \inf\limits_{m\in\mathbb{Z}} |x - m|$ for $x = (x_1, \cdots, x_d) \in \mathbb{T}^d$, $\|x\| = \sum\limits_{i=1}^{d} \|x_i\|$.

2 An Ergodic Property of Skew Shifts on $\mathbb{T}^d$

In this section, we prove the following ergodic property of skew shifts on $\mathbb{T}^d$.

Lemma 2.1 *Assume that $\omega \in \text{DC}$, $T = T_\omega$ is the skew shift on $\mathbb{T}^d$, $\epsilon > L^{-\frac{1}{(d+1)2^{d+1}}}$. Then*

$$\#\{n = 1, \cdots, L : \|T^n x - a\| < \epsilon\} < C\epsilon^d L, \quad C = C(d).$$

Proof We assume $a = 0$. Let χ be the indicator function of the ball $B(0, \epsilon)$, $R = \frac{1}{\epsilon}$, and F_R be the Fejer kernel. Then $\chi \le C\epsilon^d \prod\limits_{j=1}^{d} F_R(x_j)$.

Let $f(x) = \prod\limits_{j=1}^{d} F_R(x_j)$. Then

$$\begin{aligned}\sum_{n=1}^{L} \chi(T^n x) &\le C\epsilon^d \sum_{n=1}^{L} f(T^n x) \le C\epsilon^d \sum_{n=1}^{L} \sum_{0 \le |l_j| < R} \widehat{f}(l_1, \cdots, l_d) \mathrm{e}^{2\pi \mathrm{i} \langle T^n x, l\rangle} \\ &\le C\epsilon^d \Big(L + \sum_{0<|k|<\frac{1}{\epsilon}} \Big| \sum_{n=1}^{L} \mathrm{e}^{2\pi \mathrm{i}\langle T^n x, k\rangle} \Big| \Big).\end{aligned}$$

Let

$$S_k = \Big| \sum_{n=1}^{L} \mathrm{e}^{2\pi \mathrm{i} \langle T^n x, k\rangle} \Big|, \quad 0 < |k| < \frac{1}{\epsilon}. \tag{2.1}$$

We only need to prove

$$\sum_{0<|k|<\frac{1}{\epsilon}} S_k \le CL. \tag{2.2}$$

From the skew shift, we have

$$(T^n x)_i = x_i + n x_{i+1} + \cdots + \binom{n}{d-i} x_d + \binom{n}{d-i+1}\omega, \qquad i = 1, \cdots, d,\ x = (x_1, \cdots, x_d). \tag{2.3}$$

If $k_1 = \cdots = k_{d-1} = 0$, then

$$S_k = \Big|\sum_{n=1}^{L} \mathrm{e}^{2\pi \mathrm{i} n k_d \omega}\Big| \le \frac{1}{\|k_d\omega\|} \le C|k_d|^2. \tag{2.4}$$

If $k_1 = \cdots = k_{d-2} = 0,\ k_{d-1} \neq 0$, then $S_k = \big|\sum_{n=1}^{L} \mathrm{e}^{2\pi \mathrm{i} f(n)}\big|$, where $f(n) = \frac{1}{2}n^2 k_{d-1}\omega + cn$, c is independent of n.

So

$$S_k^2 = \Big(\sum_{n=1}^{L} \mathrm{e}^{2\pi \mathrm{i} f(n)}\Big)\Big(\sum_{n=1}^{L} \mathrm{e}^{-2\pi \mathrm{i} f(n)}\Big) \lesssim L + \sum_{h=1}^{L-1} \Big|\sum_{n=1}^{L-h} \mathrm{e}^{2\pi \mathrm{i}(f(n+h)-f(n))}\Big|$$
$$\lesssim L + \sum_{h=1}^{L-1} \min\Big(L, \frac{1}{\|h k_{d-1}\omega\|}\Big) \lesssim L + \sum_{m=1}^{|k_{d-1}|L} \min\Big(L, \frac{1}{\|m\omega\|}\Big).$$

Since $\omega \in \mathrm{DC}$, we may find an approximant q of ω satisfying

$$L^{\frac{1}{2}} < q < L. \tag{2.5}$$

Using

$$\#\Big\{M+1 \le n \le M+q : \|n\omega - u\| \le \frac{1}{2q}\Big\} \le 3, \quad \forall M \in \mathbb{Z},\ u \in \mathbb{R},$$

we get

$$\sum_{n=M+1}^{M+q} \min\Big(L, \frac{1}{\|n\omega\|}\Big) \lesssim L + q\log q. \tag{2.6}$$

By (2.5)–(2.6), we have

$$S_k^2 \lesssim \frac{|k_{d-1}|L}{q}(L + q\log q) \lesssim |k_{d-1}|L^{\frac{3}{2}}.$$

Hence

$$S_k \le C|k_{d-1}|^{\frac{1}{2}} L^{\frac{3}{4}}. \tag{2.7}$$

If $k_1 = \cdots = k_{d-3} = 0,\ k_{d-2} \neq 0$, then $S_k = \big|\sum_{n=1}^{L} \mathrm{e}^{2\pi \mathrm{i} g(n)}\big|$, where $g(n) = \frac{1}{6}n^3 k_{d-2}\omega + bn^2 + cn$, b, c is independent of n.

So

$$S_k^2 \lesssim L + \sum_{h_1=1}^{L-1} \Big|\sum_{n=1}^{L-h_1} \mathrm{e}^{2\pi \mathrm{i} g_{h_1}(n)}\Big|, \quad g_{h_1}(n) = g(n+h_1) - g(n).$$

We have

$$\begin{aligned}
S_k^4 &\lesssim L^2 + L\sum_{h_1=1}^{L-1}\Big|\sum_{n=1}^{L-h_1} \mathrm{e}^{2\pi \mathrm{i} g_{h_1}(n)}\Big|^2 \\
&\lesssim L^3 + L\sum_{h_1=1}^{L-1}\sum_{h_2=1}^{L-h_1-1}\Big|\sum_{n=1}^{L-h_1-h_2} \mathrm{e}^{2\pi \mathrm{i}(g_{h_1}(n+h_2)-g_{h_1}(n))}\Big| \\
&\lesssim L^3 + L\sum_{h_1=1}^{L}\sum_{h_2=1}^{L}\min\Big(L, \frac{1}{\|h_1h_2k_{d-2}\omega\|}\Big).
\end{aligned}$$

Using

$$\#\{(h_1,h_2)\in\mathbb{Z}^2 : h_1h_2 = N\} \lesssim N^{0+},$$

we get

$$S_k^4 \lesssim L^3 + L^{1+}\sum_{m=1}^{|k_{d-2}|L^2}\min\Big(L, \frac{1}{\|m\omega\|}\Big) \lesssim L^3 + L^{1+}\frac{|k_{d-2}|L^2}{q}(L+q\log q) \lesssim |k_{d-2}|L^{\frac{7}{2}+}.$$

Hence

$$S_k \le C|k_{d-2}|^{\frac14}L^{\frac78+}. \tag{2.8}$$

Repeating the argument above, we get

$$S_k \le C|k_{d-j}|^{\frac{1}{2^j}}L^{1-\frac{1}{2^{j+1}}+}, \quad k_1=\cdots=k_{d-j-1}=0,\ k_{d-j}\ne 0,\ 2\le j\le d-1. \tag{2.9}$$

By (2.4), (2.7) and (2.9), we have

$$\begin{aligned}
\sum_{0<|k|<\frac{1}{\epsilon}} S_k &\lesssim \sum_{|k_d|<\frac{1}{\epsilon}}|k_d|^2 + \frac{1}{\epsilon}\sum_{|k_{d-1}|<\frac{1}{\epsilon}}|k_{d-1}|^{\frac12}L^{\frac34} + \sum_{j=2}^{d-1}\frac{1}{\epsilon^j}\Big(\sum_{|k_{d-j}|<\frac{1}{\epsilon}}|k_{d-j}|^{\frac{1}{2^j}}L^{1-\frac{1}{2^{j+1}}+}\Big) \\
&\lesssim \Big(\frac{1}{\epsilon}\Big)^3 + \frac{1}{\epsilon}\Big(\frac{1}{\epsilon}\Big)^{\frac32}L^{\frac34} + \sum_{j=2}^{d-1}\Big(\Big(\frac{1}{\epsilon}\Big)^{\frac{1}{2^j}+j+1}L^{1-\frac{1}{2^{j+1}}+}\Big) \lesssim L.
\end{aligned}$$

This proves (2.2) and Lemma 2.1.

Remark 2.1 In the proof of Lemma 2.1, we only need to assume

$$\|k\omega\| > c|k|^{-2}, \quad \forall 0<|k|\le L. \tag{2.10}$$

3 Green's Function Estimates

In this section, we will prove the Green's function estimates by using multi-scale analysis in [3].

We need the following lemma.

Lemma 3.1 (cf. [3, Lemma 3.16]) *Let $A(x)=\{A_{mn}(x)\}_{1\le m,n\le N}$ be a matrix valued function on $\mathbb{T}^d$ such that*

$$A(x) \text{ is self-adjoint for } x\in\mathbb{T}^d, \tag{3.1}$$

$$A_{mn}(x) \text{ is a trigonometric polynomial of degree } < N^{C_1}, \tag{3.2}$$

$$|A_{mn}(x)| < C_2 \mathrm{e}^{-c_2|m-n|}, \tag{3.3}$$

where $c_2, C_1, C_2 > 0$ are constants.

Let $0 < \delta < 1$ be sufficiently small, $M = N^{\delta^6}$, $L_0 = N^{\frac{1}{100}\delta^2}$, $0 < c_3 < \frac{1}{10}c_2$.

Assume that for any interval $I \subset [1, N]$ of size L_0, except for x in a set of measure at most $\mathrm{e}^{-L_0^{\delta^3}}$,

$$\|(R_I A(x) R_I)^{-1}\| < \mathrm{e}^{L_0^{1-}}, \tag{3.4}$$

$$|(R_I A(x) R_I)^{-1}(m,n)| < \mathrm{e}^{-c_3|m-n|}, \quad m, n \in I,\ |m-n| > \frac{L_0}{10}. \tag{3.5}$$

Fix $x \in \mathbb{T}^d$.$n_0 \in [1, N]$ is called a good site if $I_0 = \big[n_0 - \frac{M}{2}, n_0 + \frac{M}{2}\big] \subset [1, N]$,

$$\|(R_{I_0} A(x) R_{I_0})^{-1}\| < \mathrm{e}^{M^{1-}}, \tag{3.6}$$

$$|(R_{I_0} A(x) R_{I_0})^{-1}(m,n)| < \mathrm{e}^{-c_3|m-n|}, \quad m, n \in I_0,\ |m-n| > \frac{M}{10}. \tag{3.7}$$

Denote $\Omega(x) \subset [1, N]$ the set of bad sites. Assume that for any interval $J \subset [1, N]$, $|J| > N^{\frac{\delta}{5}}$, we have $|J \cap \Omega(x)| < |J|^{1-\delta}$.

Then

$$\|A(x)^{-1}\| < \mathrm{e}^{N^{1-\frac{\delta}{C(d)}}}, \tag{3.8}$$

$$|A(x)^{-1}(m,n)| < \mathrm{e}^{-c_3'|m-n|}, \quad m, n \in [1, N],\ |m-n| > \frac{N}{10} \tag{3.9}$$

except for x in a set of measure at most $\mathrm{e}^{-\frac{N^{\delta^2}}{C(d)}}$, where $C(d)$ is a constant depending on d, and $c_3' > c_3 - (\log N)^{-8}$.

By Lemmas 2.1 and 3.1, we can prove the Green's function estimates.

Proposition 3.1 *Let $T = T_\omega : \mathbb{T}^d \to \mathbb{T}^d$ be the skew shift with frequency ω satisfying*

$$\|k\omega\| > c|k|^{-2}, \quad \forall 0 < |k| \le N. \tag{3.10}$$

$A_{mn}(x)$ is the form (1.17)–(1.21), and γ in (1.21) is small.

Then for all N and energy E,

$$\|G_{[0,N]}(E, x)\| < \mathrm{e}^{N^{1-}}, \tag{3.11}$$

$$|G_{[0,N]}(E, x)(m,n)| < \mathrm{e}^{-\frac{1}{100}|m-n|}, \quad 0 \le m, n \le N,\ |m-n| > \frac{N}{10} \tag{3.12}$$

for $x \notin \Omega_N(E)$, where

$$\operatorname{mes} \Omega_N(E) < \mathrm{e}^{-N^\sigma}, \quad \sigma > 0. \tag{3.13}$$

Proof Since $T^n(x_1, \cdots, x_d) = (x_1 + n x_2 + \cdots + \binom{n}{d-1} x_d + \binom{n}{d}\omega, \cdots, x_d + n\omega)$, $A_{mn}(x)$ is a trigonometric polynomial in x of degree $< (|m| + |n|)^{C_1 + d}$, $\{A_{mn}(x) - E\}_{0 \le m,n \le N}$ satisfy (3.1)–(3.3) with $c_2 = 1$, $C_2 = \gamma$.

First fix any large initial scale N_0 and choose $\gamma = \gamma(N_0)$ small. Using Lojasiewicz's inequality (cf. [3, Section 4]), we get

$$|G_{[0,N_0]}(E,x)(m,n)| < \mathrm{e}^{N_0^{\frac{1}{2}} - \frac{1}{2}|m-n|}, \quad 0 \le m,n \le N_0 \tag{3.14}$$

except for x in a set of measure $< \mathrm{e}^{-cN_0^{\frac{1}{2}}}$.

Then we estabish inductively on the scale N that

$$\begin{aligned} &\operatorname{mes}\{x \in \mathbb{T}^d : |G_{[0,N]}(E,x)(m,n)| > \mathrm{e}^{N^{1-} - c_3|m-n|\chi_{|m-n|>\frac{N}{10}}}, \ \exists\, 0 \le m,n \le N\} \\ &< \mathrm{e}^{-N^{\delta^3}}, \end{aligned} \tag{3.15}$$

where $c_3 > \frac{1}{100}$, $0 < \delta < 1$ is a fixed small number.

(3.14) implies (3.15) for an initial large scale N_0.

Assume that (3.15) holds up to scale $L_0 = N^{\frac{1}{100}\delta^2}$. Since $A_{m+1,n+1}(x) = A_{mn}(Tx)$, we have

$$R_I(A(x)-E)R_I = R_{[0,N]}(A(T^n x) - E)R_{[0,N]}, \quad G_I(E,x) = G_{[0,N]}(E, T^n x), \quad I = [n, n+N].$$

Since T is measure preserving, (3.4)–(3.5) will hold for x outside a set of measure at most $\mathrm{e}^{-L_0^{\delta^3}}$. Denote $\Omega(x) \subset [0,N]$ the set of bad sites with respect to scale $M = N^{\delta^6}$. $n_0 \notin \Omega(x)$ means

$$\begin{aligned} &|G_{[0,M]}(E, T^{n_0 - \frac{M}{2}}x)(m,n)| \\ &= \Big|G_{[n_0 - \frac{M}{2}, n_0 + \frac{M}{2}]}(E,x)\Big(m + n_0 - \frac{M}{2}, n + n_0 - \frac{M}{2}\Big)\Big| \\ &< \mathrm{e}^{M^{1-} - c_3|m-n|\chi_{|m-n|>\frac{M}{10}}}. \end{aligned} \tag{3.16}$$

From the inductive hypothesis, we have

$$\begin{aligned} &|G_{[0,M]}(E,x)(m,n)| \\ &< \mathrm{e}^{M^{1-} - c_3|m-n|\chi_{|m-n|>\frac{M}{10}}}, \quad 0 \le m,n \le M, \ \forall x \notin \Omega, \ \operatorname{mes}\Omega < \mathrm{e}^{-M^{\delta^3}}. \end{aligned} \tag{3.17}$$

By (3.16)–(3.17) and Lemma 3.1, we only need to show that for any $x \in \mathbb{T}^d$, $N^{\frac{\delta}{5}} < L < N$,

$$\#\{1 \le n \le L : T^n x \in \Omega\} < L^{1-\delta}. \tag{3.18}$$

Since $A_{mn}(x)$ is a trigonometric polynomial of degree $< (|m| + |n|)^C$, we can express $G_{[0,M]}(E,x)(m,n)$ as a ratio of determinants to write (3.17) in the form

$$P_{mn}(\cos x_1, \sin x_1, \cdots, \cos x_d, \sin x_d) \le 0, \tag{3.19}$$

where P_{mn} is a polynomial of degree at most M^C. Replacing $\cos, \sin$ by truncated power series, permits us to replace (3.19) by

$$P_{mn}(x_1, \cdots, x_d) \le 0, \quad \deg P_{mn} < M^C. \tag{3.20}$$

So, Ω may be viewed as a semi-algebraic set of degree at most M^C. (For properties of semi-algebraic sets, see Section 4.) When $\epsilon > \mathrm{e}^{-\frac{1}{d}M^{\delta^3}}$, by Corollary 4.1, Ω may be covered by at most $M^C(\frac{1}{\epsilon})^{d-1}\epsilon$-balls. Choosing $\epsilon = L^{-\frac{1}{(d+1)2^{d+1}}} > N^{-1} > \mathrm{e}^{-\frac{1}{d}M^{\delta^3}}$, by (3.10), using Lemma 2.1 and Remark 2.1, we have

$$\#\{1 \le n \le L : T^n x \in \Omega\} < M^C\Big(\frac{1}{\epsilon}\Big)^{d-1}\epsilon^d L < L^{C\delta^5 + 1 - \frac{1}{(d+1)2^{d+1}}} < L^{1-\delta},$$

when δ is small enough.

This proves (3.18) and Proposition 3.1.

4 Semi-algebraic Sets

We recall some basic facts of semi-algebraic sets. Let $\mathcal{P} = \{P_1, \cdots, P_s\} \subset \mathbb{R}[X_1, \cdots, X_n]$ be a family of real polynomials whose degrees are bounded by d. A semi-algebraic set is given by

$$S = \bigcup_j \bigcap_{l \in L_j} \{\mathbb{R}^n : P_l s_{jl} 0\}, \tag{4.1}$$

where $L_j \subset \{1, \cdots, s\}, s_{jl} \in \{\leq, \geq, =\}$ are arbitrary. We say that S has degree at most sd and its degree is the inf of sd over all representations as in (4.1).

The projection of a semi-algebraic set of $\mathbb{R}^n$ onto $\mathbb{R}^m$ is semi-algebraic.

Proposition 4.1 (cf. [2]) *Let $S \subset \mathbb{R}^n$ be a semi-algebraic set of degree B. Then any projection of S has degree at most B^C, $C = C(n)$.*

We need the following bound on the number of connected components.

Proposition 4.2 (cf. [1]) *Let $S \subset \mathbb{R}^n$ be a semi-algebraic set of degree B. Then the number of connected components of S is bounded by B^C, $C = C(n)$.*

A more advanced part of the theory of semi-algebraic sets is the following triangulation theorem.

Theorem 4.1 (cf. [11]) *For any positive integers r, n, there exists a constant $C = C(n, r)$ with the following property: Any semi-algebraic set $S \subset [0,1]^n$ can be triangulated into $N \lesssim (\deg S + 1)^C$ simplices, where for every closed k-simplex $\Delta \subset S$, there exists a homeomorphism h_Δ of the regular simplex $\Delta^k \subset \mathbb{R}^k$ with unit edge length onto Δ such that $\|D_r h_\Delta\| \leq 1$.*

Corollary 4.1 (cf. [4, Corollary 9.6]) *Let $S \subset [0,1]^n$ be semi-algebraic of degree B. Let $\epsilon > 0$, $\mathrm{mes}_n\, S < \epsilon^n$. Then S may be covered by at most $B^C \left(\frac{1}{\epsilon}\right)^{n-1}$ ϵ-balls.*

Finally, we will make essential use of the following transversality property.

Lemma 4.1 (cf. [5, (1.5)]) *Let $S \subset [0,1]^{n=n_1+n_2}$ be a semi-algebraic set of degree B and*

$$\mathrm{mes}_n\, S < \eta, \quad \log B \ll \log \frac{1}{\eta}, \quad \epsilon > \eta^{\frac{1}{n}}. \tag{4.2}$$

Denote $(x, y) \in [0,1]^{n_1} \times [0,1]^{n_2}$ the product variable.

Then there is a decomposition $S = S_1 \cup S_2$, with S_1 satisfying

$$\mathrm{mes}_{n_1}(\mathrm{Proj}_x\, S_1) < B^C \epsilon \tag{4.3}$$

and S_2 satisfying the transversality property

$$\mathrm{mes}_{n_2}(S_2 \cap L) < B^C \epsilon^{-1} \eta^{\frac{1}{n}} \tag{4.4}$$

for any n_2-dimensional hyperplane L such that $\max\limits_{1 \leq j \leq n_1} |\mathrm{Proj}_L(e_j)| < \frac{\epsilon}{100}$, where $\{e_j \mid 1 \leq j \leq n_1\}$ are x-coordinate vectors.

Now we can prove the following lemma.

Lemma 4.2 *Let $S\subset[0,1]^{d+1}$ be a semi-algebraic set of degree B such that*

$$\operatorname{mes} S<\mathrm{e}^{-B^\sigma},\quad \sigma>0. \tag{4.5}$$

Let M satisfy

$$\log\log M\ll\log B\ll\log M. \tag{4.6}$$

Then for all $x\in\mathbb{T}^d$,

$$\operatorname{mes}\{\omega\in\mathbb{T}:(\omega,T_\omega^j x)\in S,\ \exists\, j\sim M\}<M^{-c},\quad c>0. \tag{4.7}$$

Proof For $x^0=(x_1^0,\cdots,x_d^0)\in\mathbb{T}^d$, we study the intersection of $S\subset[0,1]^{d+1}$ and sets

$$\{(\omega,x_1,\cdots,x_d):\omega\in[0,1]\}, \tag{4.8}$$

where $x_i=(T_\omega^j x^0)_i=x_i^0+jx_{i+1}^0+\cdots+\binom{j}{d-i}x_d^0+\binom{j}{d-i+1}\omega$, $1\le i\le d$ are considered (mod 1).

By (4.5)–(4.6), we have

$$\operatorname{mes}_{d+1}S<\eta=\mathrm{e}^{-B^\sigma},\quad \log B\ll\log M\ll\log\frac{1}{\eta}. \tag{4.9}$$

Take $\epsilon=M^{-1+}$ and apply Lemma 4.1, then $S=S_1\cup S_2$. Since $\operatorname{mes}_1(\operatorname{Proj}_\omega S_1)<B^CM^{-1+}=M^{-1+}$, restriction of ω permits us to replace S by S_2 satisfying

$$\operatorname{mes}_d(S_2\cap L)<B^C\epsilon^{-1}\eta^{\frac{1}{d+1}}<\eta^{\frac{1}{d+2}}, \tag{4.10}$$

whenever L is a d-dimensional hyperplane satisfying $|\operatorname{Proj}_L(e_0)|<\frac{\epsilon}{100}$, where e_0 is the ω-coordinate vector.

Fixing j, (4.8) can be considered as a subset of $[0,1]^{d+1}$ lying in the union of the parallel d-dimensional hyperplanes

$$Q_{m_1}^{(j)}=\Big[\omega=\frac{x_d}{j}\Big]-\frac{m_1+x_d^0}{j}e_0,\quad |m_1|<M. \tag{4.11}$$

By (4.10), we have

$$\operatorname{mes}_d(S\cap Q_{m_1})<\eta^{\frac{1}{d+2}}. \tag{4.12}$$

Fixing m_1, consider the semi-algebraic set $S\cap Q_{m_1}$ and its intersection with the parallel $(d-1)$-dimensional hyperplanes

$$Q_{m_1,m_2}^{(j)}=Q_{m_1}\bigcap\Big[x_d=\frac{2}{j-1}x_{d-1}-\frac{2}{j-1}\Big(x_{d-1}^0+\frac{j+1}{2}x_d^0+m_2\Big)\Big],\quad |m_2|<M. \tag{4.13}$$

Take $\epsilon=M^{-1+}$ and apply Lemma 4.1 in Q_{m_1}, then $S\cap Q_{m_1}=S_{m_1}^1\cup S_{m_1}^2$, where

$$\operatorname{Proj}_{x_d}S_{m_1}^1 \text{ is a union of at most } B^C \text{ intervals of measure at most } B^CM^{-1+}, \tag{4.14}$$

and by (4.12), we have

$$\operatorname{mes}_{d-1}(S_{m_1}^2\cap Q_{m_1,m_2})<B^CM\eta^{\frac{1}{d(d+2)}}<\eta^{\frac{1}{(d+2)^2}}. \tag{4.15}$$

Fixing m_2, consider the semi-algebraic set $S^2_{m_1} \cap Q_{m_1,m_2}$ and its intersection with the parallel $(d-2)$-dimensional hyperplanes

$$Q^{(j)}_{m_1,m_2,m_3} = Q_{m_1,m_2} \bigcap \Big[x_{d-1} = \frac{3}{j-2} x_{d-2} - \frac{3}{j-2}\Big(x^0_{d-2} + \cdots + \frac{j(j+1)}{6} x^0_d + m_3\Big)\Big],$$

where $|m_3| < M$.

Take $\epsilon = M^{-1+}$ and apply Lemma 4.1 in Q_{m_1,m_2}, then $S^2_{m_1} \cap Q_{m_1,m_2} = S^1_{m_1,m_2} \cup S^2_{m_1,m_2}$, where

$$\mathrm{Proj}_{x_{d-1}} S^1_{m_1,m_2} \text{ is a union of at most } B^C \text{ intervals of measure at most } B^C M^{-1+},$$

and by (4.15), we have

$$\mathrm{mes}_{d-2}(S^2_{m_1,m_2} \bigcap Q_{m_1,m_2,m_3}) < \eta^{\frac{1}{(d+2)^3}}.$$

Repeat the argument above. Fixing $m_i, 2 \le i \le d-1$, consider the semi-algebraic set $S^2_{m_1,\cdots,m_{i-1}} \cap Q_{m_1,\cdots,m_i}$ and its intersection with the parallel $(d-i)$-dimensional hyperplanes

$$\begin{aligned} &Q^{(j)}_{m_1,\cdots,m_{i+1}} \\ &= Q_{m_1,\cdots,m_i} \bigcap \Big[x_{d-i+1} = \frac{i+1}{j-i} x_{d-i} - \frac{i+1}{j-i}\Big(x^0_{d-i} + \cdots + \frac{1}{i+1}\binom{j+1}{i} x^0_d + m_{i+1}\Big)\Big], \end{aligned} \quad (4.16)$$

where $|m_{i+1}| < M$.

Take $\epsilon = M^{-1+}$ and apply Lemma 4.1 in $Q_{m_1,\cdots,m_i}$, then $S^2_{m_1,\cdots,m_{i-1}} \cap Q_{m_1,\cdots,m_i} = S^1_{m_1,\cdots,m_i} \cup S^2_{m_1,\cdots,m_i}$, where

$$\mathrm{Proj}_{x_{d-i+1}} S^1_{m_1,\cdots,m_i} \text{ is a union of at most } B^C \text{ intervals of measure at most } B^C M^{-1+}, \quad (4.17)$$

and

$$\mathrm{mes}_{d-i}(S^2_{m_1,\cdots,m_i} \cap Q_{m_1,\cdots,m_{i+1}}) < \eta^{\frac{1}{(d+2)^{i+1}}}. \quad (4.18)$$

Finally, fixing m_{d-1}, consider the semi-algebraic set $S^2_{m_1,\cdots,m_{d-2}} \cap Q_{m_1,\cdots,m_{d-1}}$ and its intersection with the parallel lines

$$\begin{aligned} &Q^{(j)}_{m_1,\cdots,m_d} \\ &= Q_{m_1,\cdots,m_{d-1}} \bigcap \Big[x_2 = \frac{d}{j-d+1} x_1 - \frac{d}{j-d+1}\Big(x^0_1 + \cdots + \frac{1}{d}\binom{j+1}{d-1} x^0_d + m_d\Big)\Big], \end{aligned} \quad (4.19)$$

where $|m_d| < M$.

Take $\epsilon = M^{-1+}$ and apply Lemma 4.1 in $Q_{m_1,\cdots,m_{d-1}}$, then $S^2_{m_1,\cdots,m_{d-2}} \cap Q_{m_1,\cdots,m_{d-1}} = S^1_{m_1,\cdots,m_{d-1}} \cup S^2_{m_1,\cdots,m_{d-1}}$, where

$$\mathrm{Proj}_{x_2} S^1_{m_1,\cdots,m_{d-1}} \text{ is a union of at most } B^C \text{ intervals of measure at most } B^C M^{-1+}, \quad (4.20)$$

and

$$\mathrm{mes}_1(S^2_{m_1,\cdots,m_{d-1}} \cap Q_{m_1,\cdots,m_d}) < \eta^{\frac{1}{(d+2)^d}}. \quad (4.21)$$

Summing (4.21) over $j, m_1, \cdots, m_d$, the collected contribution in the ω-parameter is less than $M^{-d} M^{d+1} B^C M \eta^{\frac{1}{(d+2)^d}} < \eta^{\frac{1}{(d+2)^{d+1}}}$. So, we only need to consider the contribution of

$S^1_{m_1,\cdots,m_i}$ in (4.17). We just deal with $S^1_{m_1,\cdots,m_{d-1}}$ below, since for other sets, the method is similar.

If (4.7) fails, we have

$$\begin{aligned}\sum_{j\sim M,|m_1|,\cdots,|m_d|<M} \text{mes}[\text{Proj}_\omega \text{Proj}_{x_2}(S^1_{m_1,\cdots,m_{d-1}} \cap Q^{(j)}_{m_1,\cdots,m_d})] > M^{0-},\\ \sum_{j\sim M,|m_1|,\cdots,|m_d|<M} \text{mes}[\text{Proj}_{x_2}(S^1_{m_1,\cdots,m_{d-1}} \cap Q^{(j)}_{m_1,\cdots,m_d})] > M^{d-1-}.\end{aligned} \tag{4.22}$$

So, there is a set $J \subset \mathbb{Z} \cap [j \sim M]$, $|J| > M^{1-}$ such that for each $j \in J$, there are at least M^{d-1-} values of $(m_1,\cdots,m_{d-1})$ satisfying

$$\sum_{|m_d|<M} \text{mes}[\text{Proj}_{x_2}(S^1_{m_1,\cdots,m_{d-1}} \cap Q^{(j)}_{m_1,\cdots,m_d})] > M^{-1}. \tag{4.23}$$

By (4.20), $S^1_{m_1,\cdots,m_{d-1}} \cap Q^{(j)}_{m_1,\cdots,m_d} \neq \emptyset$ for at most M^{0+} values of m_d. Hence

$$\max_{m_d} \text{mes}_1(S \cap Q^{(j)}_{m_1,\cdots,m_d}) > M^{0-}. \tag{4.24}$$

For fixed j,

$$Q^{(j)}_{m_1,\cdots,m_d} // \xi_j // \Big(1, \binom{j}{d}, \cdots, j\Big)^{\text{T}}, \quad \|\xi_j\| = 1. \tag{4.25}$$

Denote S_x the intersection of S and the d-dimensional hyperplane $[x' = x]$. From (4.24), to each $(m_1,\cdots,m_{d-1})$ we can associate some m_d, such that

$$\int_0^1 \#\{|m_1|,\cdots,|m_{d-1}| < M \mid S_x \cap Q_{m_1,\cdots,m_d} \neq \emptyset\} \text{d}x > M^{d-1-}. \tag{4.26}$$

If $\text{mes}_d S_x < \eta^{\frac{1}{2}}$, then $S_x \cap Q_{m_1,\cdots,m_d} \neq \emptyset$ implies $\text{dist}(Q_{m_1,\cdots,m_d}, \partial S_x) < \eta^{\frac{1}{2d}}$, where ∂S_x is a union of at most B^C connected $(d-1)$-dimensional algebraic set of degree at most B^C. From (4.26), it follows that there is a fixed $(d-1)$-dimensional algebraic set $\Gamma = \Gamma^{(j)}$ of degree at most B^C such that for $x \in [0,1]$ in a set of measure $> M^{0-}$, there are at least M^{d-1-} $\frac{1}{M}$-separated points that are $\eta^{\frac{1}{2d}}$-close to both ∂S_x and $\Gamma + x\xi_j$. Hence $(\Gamma + x\xi_j) \cap S_{\eta_1}$ (η_1-neighborhood of S, $\eta_1 = 2\eta^{\frac{1}{2d}}$) contains at least M^{d-1-} $\frac{1}{M}$-separated points. So, $\text{mes}_{d-1}((\Gamma + x\xi_j) \cap S_{\eta_1}) > M^{0-}$.

The hypercylinder $\mathcal{C}^{(j)} = t\xi_j + \Gamma^{(j)}$ satisfies

$$\text{mes}_d(\mathcal{C}^{(j)} \cap S_{\eta_1}) > M^{0-}. \tag{4.27}$$

By Corollary 4.1, we have

$$\text{mes}_{d+1} S_{\eta_1} < B^C \eta_1. \tag{4.28}$$

Since (4.27) holds for all $j \in J$, by (4.27)–(4.28), we have

$$\sum_{j_1,\cdots,j_{d+1} \in J} \text{mes}_{d+1}\Big[\bigcap_{1 \le i \le d+1} \mathcal{C}^{(j_i)}_{\eta_1}\Big] > \eta_1 M^{d+1-}.$$

So, there are distinct $j_1,\cdots,j_{d+1} \sim M$ such that

$$\text{mes}_{d+1}\Big[\bigcap_{1 \le i \le d+1} \mathcal{C}^{(j_i)}_{\eta_1}\Big] > \eta_1 M^{0-}. \tag{4.29}$$

By (4.25), using Vandermonde determinant, we have

$$\det[\xi_{j_1}, \cdots, \xi_{j_{d+1}}] \neq 0 \tag{4.30}$$

for distinct $j_1, \cdots, j_{d+1}$. So, the vectors $\xi_{j_1}, \cdots, \xi_{j_{d+1}}$ are not in any d-dimensional hyperplane. Since $\log M \ll \log \frac{1}{\eta_1}$, this leads to a contradiction to (4.29).

This proves Lemma 4.2.

5 Proof of Anderson Localization

In this section, we give the proof of Anderson localization as in [6].

By application of the resolvent identity, we have the following lemma.

Lemma 5.1 (cf. [4, Lemma 10.33]) *Let $I \subset \mathbb{Z}$ be an interval of size N and $\{I_\alpha\}$ be subintervals of size $M \ll N$. Assume that*

(i) *if $k \in I$, then there is some α such that $\left[k - \frac{M}{4}, k + \frac{M}{4}\right] \cap I \subset I_\alpha$.*

(ii) *For all α,*

$$\|G_{I_\alpha}\| < \mathrm{e}^{M^{1-}}, \quad |G_{I_\alpha}(n_1, n_2)| < \mathrm{e}^{-c_0|n_1-n_2|}, \quad n_1, n_2 \in I_\alpha,\ |n_1 - n_2| > \frac{M}{10}.$$

Then

$$|G_I(n_1, n_2)| < \mathrm{e}^{-(c_0-)|n_1-n_2|}, \quad n_1, n_2 \in I,\ |n_1 - n_2| > \frac{N}{10}.$$

Let $T = T_\omega$ be the skew shift on $\mathbb{T}^d$ with frequency ω satisfying

$$\|k\omega\| > c|k|^{-2}, \quad \forall k \in \mathbb{Z} \setminus \{0\}. \tag{5.1}$$

Fix $x_0 \in \mathbb{T}^d$,

$$H(x_0)(m, m) = v(T^m x_0), \tag{5.2}$$
$$H(x_0)(m, n) = \phi_{m-n}(T^m x_0) + \overline{\phi_{n-m}(T^n x_0)}, \quad m \neq n \tag{5.3}$$

with v and ϕ_k satisfying (1.19)–(1.21) and γ taken small enough. Then we have the following theorem.

Theorem 5.1 *For almost all ω satisfying* (5.1)*, the lattice operator $H_\omega(x_0)$ satisfies Anderson localization.*

Proof By Shnol's theorem (cf. [12]), to establish Anderson localization, it suffices to show that if $\xi = (\xi_n)_{n\in\mathbb{Z}}, E \in \mathbb{R}$ satisfy

$$\xi_0 = 1, \quad |\xi_n| < C|n|, \quad |n| \to \infty, \tag{5.4}$$
$$H(x_0)\xi = E\xi, \tag{5.5}$$

then

$$|\xi_n| < \mathrm{e}^{-c|n|}, \quad |n| \to \infty. \tag{5.6}$$

Let $M = N^{C_0},\ L = M^C$. Denote $\Omega \subset \mathbb{T}^d$ the set of x such that

$$|G_{[-M,M]}(E, x)(m, n)| < \mathrm{e}^{M^{1-} - \frac{1}{100}|m-n|\chi_{|m-n|>\frac{M}{10}}}$$

fails for some $|m|, |n| \le M$. It was shown in Section 3 that

$$\#\{1 \le |n| \le L : T^n x_0 \in \Omega\} < L^{1-\delta}.$$

So, we may find an interval $I \subset [0, L]$ of size M, such that

$$T^{n_0} x_0 \notin \Omega, \quad \forall n_0 \in I \cup (-I).$$

Hence

$$|G_{[n_0-M,n_0+M]}(E, x_0)(m, n)| < \mathrm{e}^{M^{1-} - \frac{1}{100}|m-n|\chi_{|m-n|>\frac{M}{10}}}, \quad m, n \in [n_0 - M, n_0 + M]. \tag{5.7}$$

By (5.4)–(5.5) and (5.7), we have

$$\begin{aligned} |\xi_{n_0}| &\le \sum_{n' \in [n_0-M,n_0+M], n'' \notin [n_0-M,n_0+M]} \mathrm{e}^{M^{1-} - \frac{1}{100}|n_0-n'|\chi_{|n_0-n'|>\frac{M}{10}}} \mathrm{e}^{-|n'-n''|} |\xi_{n''}| \\ &< \mathrm{e}^{-\frac{M}{200}}. \end{aligned} \tag{5.8}$$

Denoting j_0 the center of I, we have

$$1 = |\xi_0| \le \|G_{[-j_0,j_0]}(x_0, E)\| \|R_{[-j_0,j_0]} H(x_0) R_{\mathbb{Z}\setminus[-j_0,j_0]} \xi\|. \tag{5.9}$$

By (5.4) and (5.8), we have for $|n| \le j_0$,

$$|(H(x_0) R_{\mathbb{Z}\setminus[-j_0,j_0]}\xi)_n| \le \sum_{|n'|>j_0} \mathrm{e}^{-|n-n'|} |\xi_{n'}| < \mathrm{e}^{-\frac{M}{400}} + \sum_{|n'|>j_0+\frac{M}{2}} \mathrm{e}^{-|n-n'|}|\xi_{n'}| < \mathrm{e}^{-\frac{M}{500}}. \tag{5.10}$$

By (5.9)–(5.10), we have

$$\|G_{[-j_0,j_0]}(x_0, E)\| > \mathrm{e}^N. \tag{5.11}$$

So there is some j_0, $|j_0| < N_1 = N^{C_1}$ (C_1 is a sufficiently large constant), such that by (5.11)

$$\mathrm{dist}(E, \mathrm{spec}\, H_{[-j_0,j_0]}(x_0)) < \mathrm{e}^{-N}. \tag{5.12}$$

Denote $\Omega(E) \subset \mathbb{T}^d$ the set of x such that

$$|G_{[-N,N]}(E, x)(m, n)| < \mathrm{e}^{N^{1-} - \frac{1}{100}|m-n|\chi_{|m-n|>\frac{N}{10}}}$$

fails for some $|m|, |n| \le N$. Let $\mathcal{E}_\omega = \bigcup_{|j|\le N_1} \mathrm{spec}\, H_{[-j,j]}(x_0)$. It follows from (5.12) that if $x \notin \bigcup_{E' \in \mathcal{E}_\omega} \Omega(E')$, then

$$|G_{[-N,N]}(E, x)(m, n)| < \mathrm{e}^{N^{1-} - \frac{1}{100}|m-n|\chi_{|m-n|>\frac{N}{10}}}, \quad |m|, |n| \le N. \tag{5.13}$$

Consider the set $S = S_N \subset \mathbb{T}^{d+1} \times \mathbb{R}$ of (ω, x, E'), where

$$\|k\omega\| > c|k|^{-2}, \quad \forall 0 < |k| \le N, \tag{5.14}$$

$$x \in \Omega(E'), \tag{5.15}$$

$$E' \in \mathcal{E}_\omega. \tag{5.16}$$

By (5.14)–(5.16),

$$\mathrm{Proj}_{\mathbb{T}^{d+1}} S \text{ is a semi-algebraic set of degree } < N^C, \tag{5.17}$$

and by Proposition 3.1,

$$\mathrm{mes}(\mathrm{Proj}_{\mathbb{T}^{d+1}} S) < \mathrm{e}^{-\frac{1}{2}N^\sigma}. \tag{5.18}$$

Let $N_2 = \mathrm{e}^{(\log N)^2}$,

$$\mathcal{B}_N = \{\omega \in \mathbb{T} : (\omega, T^j x_0) \in \mathrm{Proj}_{\mathbb{T}^{d+1}} S_N,\ \exists\, |j| \sim N_2\}. \tag{5.19}$$

By (5.17)–(5.19), using Lemma 4.2, $\mathrm{mes}\,\mathcal{B}_N < N_2^{-c},\ c > 0$. Let

$$\mathcal{B} = \bigcap_{N_0} \bigcup_{N > N_0} \mathcal{B}_N. \tag{5.20}$$

Then by Borel-Cantelli theorem, $\mathrm{mes}\,\mathcal{B} = 0$. We restrict $\omega \notin \mathcal{B}$.

If $\omega \notin \mathcal{B}_N$, we have for all $|j| \sim N_2,\ (\omega, T^j x_0) \notin \mathrm{Proj}_{\mathbb{T}^{d+1}} S_N$, by (5.13),

$$|G_{[j-N, j+N]}(E, x_0)(m,n)| < \mathrm{e}^{N^{1-} - \frac{1}{100}|m-n| \chi_{|m-n| > \frac{N}{10}}}. \tag{5.21}$$

Let $\Lambda = \bigcup_{\frac{1}{4}N_2 < j < 2N_2} [j-N, j+N] \supset \big[\frac{1}{4}N_2, 2N_2\big]$. By Lemma 5.1, we deduce from (5.21) that

$$|G_\Lambda(E, x_0)(m,n)| < \mathrm{e}^{-\frac{1}{200}|m-n|}, \quad |m-n| > \frac{N_2}{10}, \tag{5.22}$$

and therefore

$$|\xi_j| < \mathrm{e}^{-\frac{1}{1000}|j|}, \quad \frac{1}{2}N_2 \le j \le N_2. \tag{5.23}$$

Since $\omega \notin \mathcal{B}$, by (5.20), there is some $N_0 > 0$, such that for all $N \ge N_0$, $\omega \notin \mathcal{B}_N$. So (5.23) holds for $j \in \bigcup_{N \ge N_0} \big[\frac{1}{2}\mathrm{e}^{(\log N)^2}, \mathrm{e}^{(\log N)^2}\big] = \big[\frac{1}{2}\mathrm{e}^{(\log N_0)^2}, \infty\big)$. This proves (5.6) for $j > 0$, similarly for $j < 0$. Hence Theorem 5.1 follows.

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