Chin. Ann. Math. Ser. B
41(4), 2020, 511–530
DOI: 10.1007/s11401-020-0214-9

Chinese Annals of Mathematics, Series B

Generalized Exact Boundary Synchronization for a Coupled System of Wave Equations with Dirichlet Boundary Controls*

Yanyan WANG[1]

Abstract This paper deals with the generalized exact boundary synchronization for a coupled system of wave equations with Dirichlet boundary controls in the framework of weak solutions. A necessary and sufficient condition for the generalized exact boundary synchronization is obtained, and some results for its generalized exactly synchronizable states are given.

Keywords Generalized exact boundary synchronization, Coupled system of wave equations, Generalized exactly synchronizable state, Dirichlet boundary control

2000 MR Subject Classification 93B05, 93C20

1 Introduction

Synchronization phenomena can be often found in science, nature, engineering and social life (see [14]). Christiaan Huygens [3] first observed in 1665 the sympathy of two pendulums, and the synchronization was studied systematically from a mathematical point of view since Norbert Wiener's work in 1950s (see [15]). The previous results focused on systems described by ODEs. Recently the research on the synchronization on systems described by PDEs was initiated by Li and Rao [5–6]. They obtained the exact boundary synchronization (by groups) for a coupled system of wave equations with Dirichlet boundary controls in the framework of weak solutions (see [6–7, 9]), and gave an approach to determine its exactly synchronizable states (see [6, 8–9]). Moreover, they have also studied the exact boundary synchronization to this kind of coupled system with various boundary controls for the 1-D case in the framework of classical solutions (see [2, 11]).

In this paper, we will investigate the generalized exact boundary synchronization which covers various important types of exact synchronizations, including the anti-phase synchronization observed by Christiaan Huygens. Referring to the generalized exact boundary synchronization for the 1-D case in the framework of classical solutions (see [12]), the generalized exact boundary synchronization for a coupled system of wave equations in the framework of weak solutions, and the determination of corresponding generalized exactly synchronizable states will be studied in detail.

Manuscript received May 15, 2019. Revised June 9, 2019.
[1]School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: yywang14@fudan.edu.cn
*Project supported by the National Natural Science Foundation of China (No. 11831011).

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and $\text{mes}(\Gamma_1) > 0$. In addition, we assume the following multiplier geometrical condition (see [13]): There exists $x_0 \in \mathbb{R}^n$, such that for $m = x - x_0$ we have

$$(m, \nu) > 0, \quad \forall x \in \Gamma_1; \qquad (m, \nu) \leq 0, \quad \forall x \in \Gamma_0,$$

where ν is the unit outward normal vector and $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^n$.

Let $U = (u^{(1)}, \cdots, u^{(N)})^{\mathrm{T}}$ and $H = (h^{(1)}, \cdots, h^{(M)})^{\mathrm{T}}$ $(M \leq N)$ be the state variable and the boundary control, respectively. Consider the following coupled system of wave equations with Dirichlet boundary controls:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0, +\infty) \times \Omega, \\ U = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ U = DH & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \tag{1.1}$$

with the initial condition

$$t = 0: \ (U, U') = (\hat{U}_0, \hat{U}_1) \quad \text{in } \Omega, \tag{1.2}$$

where $A = (a_{ij}) \in \mathbb{M}^{N \times N}(\mathbb{R})$ is a given coupling matrix with constant elements, and $D \in \mathbb{M}^{N \times M}(\mathbb{R})$ is a full column-rank matrix with constant elements, called the boundary control matrix, which can be used to adjust the collocation of boundary controls flexibly.

Let Θ_p be a given $(N-p) \times N$ $(0 \leq p < N)$ full row-rank matrix, called the generalized synchronization matrix.

Definition 1.1 *System* (1.1) *is generalized exactly boundary synchronizable with respect to* Θ_p*, if there exists a time* $T > 0$*, such that for any given initial data* $(\hat{U}_0, \hat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$*, there exists a boundary control* $H \in L^2_{\text{loc}}(0, +\infty; (L^2(\Gamma_1))^M)$ *with compact support in* $[0, T]$*, such that the corresponding solution* $U = U(t, x) \in C^0_{\text{loc}}(0, +\infty; (L^2(\Omega))^N) \cap C^1_{\text{loc}}(0, +\infty; (H^{-1}(\Omega))^N)$ *to problem* (1.1)–(1.2) *satisfies the following condition*

$$t \geq T: \ \Theta_p U \equiv 0. \tag{1.3}$$

Remark 1.1 Note that $\Theta_p U \equiv 0$ means $U \in \text{Ker}(\Theta_p)$. For another full row-rank matrix $\tilde{\Theta}_p$, if $\text{Ker}(\tilde{\Theta}_p) = \text{Ker}(\Theta_p)$, or equivalently, $\text{Im}(\tilde{\Theta}_p^{\mathrm{T}}) = \text{Im}(\Theta_p^{\mathrm{T}})$, then the generalized exact boundary synchronization with respect to Θ_p is equivalent to that with respect to $\tilde{\Theta}_p$. Hence all the generalized synchronization matrices with the same kernel space can be regarded to be equivalent to each other.

Remark 1.2 Setting

$$\text{Ker}(\Theta_p) = \text{Span}\{\epsilon_1, \cdots, \epsilon_p\}, \tag{1.4}$$

condition (1.3) can be equivalently written as: There exists a vector function $u = (u_1, \cdots, u_p)^{\mathrm{T}}$ of t and x, such that

$$t \geq T: \ U = u_1 \epsilon_1 + \cdots + u_p \epsilon_p = (\epsilon_1, \cdots, \epsilon_p) u, \tag{1.5}$$

where the vector function u, called the corresponding generalized exactly synchronizable state, is a priori unknown, and p is called the group number. The selected basis $\{\epsilon_1, \cdots, \epsilon_p\}$ is called the synchronization basis. The generalized exact boundary synchronization can then be described by (1.4)–(1.5) through the synchronization basis $\{\epsilon_1, \cdots, \epsilon_p\}$.

Remark 1.3 When $p = 0$, the generalized exact boundary synchronization with respect to Θ_0 becomes the exact boundary null controllability, then the corresponding generalized exactly synchronizable state $u = 0$, which is the trivial case. Here and hereafter we will focus on the non-trivial cases $1 < p < N$.

Remark 1.4 For system (1.1), the generalized exact boundary synchronization with respect to Θ_p includes many significant situations.

In the case $p = 1$, denoting $\epsilon = (k_1, k_2, \cdots, k_N)^{\mathrm{T}} \neq 0$, the generalized exact boundary synchronization (1.5) becomes

$$t \geq T: \ U = \epsilon u = (k_1 u, k_2 u, \cdots, k_N u)^{\mathrm{T}}. \tag{1.6}$$

This denotes a kind of phase synchronization admitting different amplitudes. When $N = 2$ and $\epsilon = (1, -1)^{\mathrm{T}}$, this represents the anti-phase synchronization.

(1) Obviously, the generalized exact boundary synchronization with respect to $\epsilon = (1, \cdots, 1)^{\mathrm{T}}$ is the usual exact boundary synchronization (see [6]).

(2) The generalized exact boundary synchronization with respect to

$$\epsilon = (\underbrace{0, \cdots, 0}_{1 \text{ to } m}, \underbrace{1, \cdots, 1}_{(m+1) \text{ to } N})^{\mathrm{T}},$$

where $m \geq 1$, is the exact null controllability and synchronization by 2-groups (see [6, 8]), since condition (1.5) becomes

$$t \geq T: \ U = \epsilon u = (\underbrace{0, \cdots, 0}_{1 \text{ to } m}, \underbrace{u, \cdots, u}_{(m+1) \text{ to } N})^{\mathrm{T}}. \tag{1.7}$$

(3) For the generalized exact boundary synchronization with respect to $\epsilon = (1, 0, \cdots, 0)^{\mathrm{T}}$, since condition (1.5) becomes

$$t \geq T: \ U = \epsilon u = (u^{(1)}, 0, \cdots, 0)^{\mathrm{T}}, \tag{1.8}$$

it is actually the partial null controllability: Only the 2-nd to N-th state variables possess the exact null controllability.

In the general case p $(0 < p < N)$, let

$$0 = n_0 < n_1 < n_2 < \cdots < n_p = N, \tag{1.9}$$

and e_r $(r = 1, \cdots, p)$ be the following column vector in $\mathbb{R}^N$:

$$(e_r)_i = \begin{cases} 1, & n_{r-1} + 1 \leq i \leq n_r, \\ 0, & \text{otherwise}. \end{cases} \tag{1.10}$$

(4) For the generalized exact boundary synchronization with respect to $\{e_1, \cdots, e_p\}$, since condition (1.5) becomes

$$t \geq T: \ U = u_1 e_1 + \cdots + u_p e_p = (\underbrace{u_1, \cdots, u_1}_{1 \text{ to } n_1}, \cdots, \underbrace{u_p, \cdots, u_p}_{(n_{p-1}+1) \text{ to } N})^{\mathrm{T}}, \tag{1.11}$$

it is the usual exact boundary synchronization by p-groups, provided that $n_i - n_{i-1} > 1$ $(i = 1, \cdots, p)$ (see [9]).

(5) When $n_1 = 1$, condition (1.5) becomes

$$t \geq T: \ U = u_1 e_1 + \cdots + u_p e_p = (u_1, \underbrace{u_2, \cdots, u_2}_{2 \text{ to } n_2}, \cdots, \underbrace{u_p, \cdots, u_p}_{(n_{p-1}+1) \text{ to } N})^{\mathrm{T}}, \tag{1.12}$$

then, the first state variable is free, and the 2-nd to N-th state variables possess the exact boundary synchronization by $(p-1)$-groups.

(6) For the generalized exact boundary synchronization with respect to $\{e_2, \cdots, e_p\}$, since condition (1.5) becomes

$$t \geq T: \ U = u_2 e_2 + \cdots + u_p e_p = (\underbrace{0, \cdots, 0}_{1 \text{ to } n_1}, \underbrace{u_2, \cdots, u_2}_{(n_1+1) \text{ to } n_2}, \cdots, \underbrace{u_p, \cdots, u_p}_{(n_{p-1}+1) \text{ to } N})^{\mathrm{T}}, \tag{1.13}$$

the 1-st to n_1-th state variables possess the exact boundary null controllability and the (n_1+1)-th to N-th state variables possess the exact boundary synchronization by $(p-1)$-groups, then it is the exact null controllability and synchronization by p-groups (see [6, 8]).

A natural question is: For any given generalized synchronization matrix Θ_p, does system (1.1) possess the generalized exact boundary synchronization with respect to Θ_p?

Clearly, when the number of boundary controls $M = N$, system (1.1) is already exactly null controllable, so it is always generalized exactly synchronizable with respect to Θ_p. Therefore, we are interested in the case that there is a lack of boundary controls, that is, the number of boundary controls $M < N$.

In Section 2, we will show that in order to realize the generalized exact boundary synchronization with respect to Θ_p, the minimal number of boundary controls is $M = N - p$. Furthermore, we will show the sufficient and necessary condition satisfied by the coupling matrix A and the boundary control matrix D in this situation. Then we will investigate the corresponding generalized exactly synchronizable state and its determination in Section 3.

2 Generalized Exact Boundary Synchronization

This section will be devoted to get necessary or sufficient conditions for system (1.1) being generalized exactly synchronizable with respect to Θ_p. Before giving the main results, we first show a preliminary conclusion.

Lemma 2.1 *If system (1.1) is generalized exactly synchronizable with respect to Θ_p, then either the coupling matrix A satisfies the following condition of Θ_p-compatibility:*

$$A \operatorname{Ker}(\Theta_p) \subseteq \operatorname{Ker}(\Theta_p), \tag{2.1}$$

namely, $\operatorname{Ker}(\Theta_p)$ *is an invariant subspace of A; or there exists a full row-rank extension matrix of Θ_p*

$$\tilde{\Theta}_{p-1} = \begin{pmatrix} \Theta_p \\ x_p^{\mathrm{T}} \end{pmatrix}_{(N-p+1) \times N}, \tag{2.2}$$

such that system (1.1) is generalized exactly synchronizable with respect to $\tilde{\Theta}_{p-1}$, where x_p^{T} denotes the expanded row.

Proof Noting the generalized exact boundary synchronization (1.3) and (1.5), by applying Θ_p to the coupled equations in (1.1), we have

$$t \geq T: \ \Theta_p AU = \sum_{r=1}^{p} u_r \Theta_p A \epsilon_r \equiv 0. \tag{2.3}$$

When $\Theta_p A\epsilon_r = 0$ $(r = 1, \cdots, p)$, we have (2.1), thus we will focus on the case that $\Theta_p A\epsilon_r$ $(r = 1, \cdots, p)$ are not all equal to zero.

Without loss of generality we assume $\Theta_p A\epsilon_p \neq 0$. Then (2.3) immediately gives $u_p = \sum\limits_{r=1}^{p-1} \alpha_r u_r$, where α_r $(r = 1, \cdots, p-1)$ are some constants. So (1.5) becomes

$$t \geq T: \ U = \sum_{r=1}^{p} u_r \epsilon_r = \sum_{r=1}^{p-1} u_r(\epsilon_r + \alpha_r \epsilon_p) = \sum_{r=1}^{p-1} u_r \hat{\epsilon}_r, \tag{2.4}$$

where $\hat{\epsilon}_r = \epsilon_r + \alpha_r \epsilon_p \in \mathrm{Ker}(\Theta_p)$ for $r = 1, \cdots, p-1$. Since $\dim(\mathrm{Ker}(\Theta_p)) = p$, there exists a non-zero vector x_p in $\mathrm{Ker}(\Theta_p)$, for example $x_p = \epsilon_p/|\epsilon_p|^2 - \sum\limits_{r=1}^{p-1} \alpha_r \epsilon_r/|\epsilon_r|^2$, such that

$$x_p \perp \mathrm{Span}\{\hat{\epsilon}_1, \cdots, \hat{\epsilon}_{p-1}\}. \tag{2.5}$$

Thus, by (2.2), we construct an extension matrix $\tilde{\Theta}_{p-1}$ which is full row-rank because of $x_p \in \mathrm{Ker}(\Theta_p) = (\mathrm{Im}(\Theta_p^{\mathrm{T}}))^{\perp}$. Noting (1.3) and (2.4)–(2.5), we have

$$t \geq T: \ \tilde{\Theta}_{p-1} U \equiv 0, \tag{2.6}$$

which means system (1.1) is also generalized exactly synchronizable with respect to $\tilde{\Theta}_{p-1}$.

Consider the special case that A satisfies the condition of Θ_p-compatibility (2.1). By [12, Lemma 3.3], there exists a unique matrix $\bar{A}_p$ (called the generalized reduced matrix) of order $N-p$, such that

$$\Theta_p A = \bar{A}_p \Theta_p. \tag{2.7}$$

Set $W = \Theta_p U$. Multiplying problem (1.1)–(1.2) by Θ_p, by (2.7) we get the following self-closed reduced system:

$$\begin{cases} W'' - \Delta W + \bar{A}_p W = 0 & \text{in } (0, +\infty) \times \Omega, \\ W = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ W = \Theta_p DH & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \tag{2.8}$$

and

$$t = 0: \ (W, W') = \Theta_p(\hat{U}_0, \hat{U}_1) \quad \text{in } \Omega. \tag{2.9}$$

Lemma 2.2 *Under the assumption that A satisfies the condition of Θ_p-compatibility* (2.1), *the following facts are equivalent:*

(*i*) *System* (1.1) *is generalized exactly synchronizable with respect to* Θ_p.

(*ii*) *Reduced system* (2.8) *is exactly null controllable.*

(*iii*) $\mathrm{rank}(\Theta_p D) = N - p$.

Proof By definition, it is easy to see that (i) and (ii) are equivalent. And the equivalence between (ii) and (iii) follows from the theory of exact boundary controllability given in [6, Theorem 2.2] and [7, Theorem 2].

The above (iii) indicates that if A satisfies the condition of Θ_p-compatibility (2.1), in order to realize the generalized exact boundary synchronization with respect to Θ_p, the required number of boundary controls $\text{rank}(D) \geq \text{rank}(\Theta_p D) = N - p$. Since the set

$$\mathcal{D}_{N-p} = \{D \in \mathbb{M}^{N\times(N-p)}(\mathbb{R}) : \text{rank}(D) = \text{rank}(\Theta_p D) = N - p\} \tag{2.10}$$

is nonempty, we only need $N - p$ boundary controls to realize the generalized exact boundary synchronization with respect to Θ_p in this situation.

For the general case that A may not satisfy the condition of Θ_p-compatibility, we define a full row-rank matrix $\tilde{\Theta}_{\tilde{p}}$ of order $(N - \tilde{p}) \times N$ $(0 \leq \tilde{p} \leq p)$ by

$$\text{Im}(\tilde{\Theta}_{\tilde{p}}^{\text{T}}) = \text{Span}(\Theta_p^{\text{T}}, A^{\text{T}}\Theta_p^{\text{T}}, \cdots, (A^{\text{T}})^{N-1}\Theta_p^{\text{T}}). \tag{2.11}$$

According to Cayley-Hamilton theorem, we have

$$A^{\text{T}}\,\text{Im}(\tilde{\Theta}_{\tilde{p}}^{\text{T}}) \subseteq \text{Im}(\tilde{\Theta}_{\tilde{p}}^{\text{T}}), \tag{2.12}$$

that is, A satisfies the condition of $\tilde{\Theta}_{\tilde{p}}$-compatibility:

$$A\,\text{Ker}(\tilde{\Theta}_{\tilde{p}}) \subseteq \text{Ker}(\tilde{\Theta}_{\tilde{p}}). \tag{2.13}$$

Since each expansion of $\text{Im}(\Theta_p^{\text{T}})$ satisfying (2.12) has to contain the right-hand side of (2.11), $\text{Im}(\tilde{\Theta}_{\tilde{p}}^{\text{T}})$ is the minimal expansion of $\text{Im}(\Theta_p^{\text{T}})$, satisfying (2.12), namely, (2.13). Besides, noting Remark 1.1, without loss of generality we may take

$$\tilde{\Theta}_{\tilde{p}} = \begin{pmatrix} \Theta_p \\ (x_{N-p+1}, \cdots, x_{N-\tilde{p}})^{\text{T}} \end{pmatrix}. \tag{2.14}$$

Apparently, A satisfies the condition of Θ_p-compatibility (2.1) if and only if $\tilde{p} = p$.

Theorem 2.1 *For system* (1.1)*, the generalized exact boundary synchronization with respect to* Θ_p *is equivalent to that with respect to* $\tilde{\Theta}_{\tilde{p}}$*, where* $\tilde{\Theta}_{\tilde{p}}$ *is given by* (2.11)*.*

Proof Noting the generalized exact boundary synchronization (1.3), (2.14) gives immediately the sufficiency. It remains to prove the necessity. Suppose that system (1.1) is generalized exactly synchronizable with respect to Θ_p.

If A satisfies the condition of Θ_p-compatibility (2.1), then $\tilde{\Theta}_{\tilde{p}} = \Theta_p$ which leads to the conclusion directly.

Otherwise, by Lemma 2.1, there exists a full row-rank extension matrix $\tilde{\Theta}_{p-1} = \begin{pmatrix} \tilde{\Theta}_p \\ x_p^{\text{T}} \end{pmatrix}$, such that system (1.1) is generalized exactly synchronizable with respect to $\tilde{\Theta}_{p-1}$. If $A\,\text{Ker}(\tilde{\Theta}_{p-1}) \not\subseteq \text{Ker}(\tilde{\Theta}_{p-1})$, by Lemma 2.1 we get a full row-rank extension matrix $\tilde{\Theta}_{p-2} = \begin{pmatrix} \tilde{\Theta}_{p-1} \\ x_{p-1}^{\text{T}} \end{pmatrix}$, such that system (1.1) is generalized exactly synchronizable with respect to $\tilde{\Theta}_{p-2}$. And so forth, hence there is an r with $1 \leq r \leq p$, such that the corresponding full row-rank extension matrix

$$\tilde{\Theta}_{p-r} = \begin{pmatrix} \tilde{\Theta}_{p-r+1} \\ x_{p-r+1}^{\text{T}} \end{pmatrix} = \begin{pmatrix} \Theta_p \\ (x_p, x_{p-1}, \cdots, x_{p-r+1})^{\text{T}} \end{pmatrix} \tag{2.15}$$

satisfies $A\,\mathrm{Ker}(\tilde{\Theta}_{p-r}) \subseteq \mathrm{Ker}(\tilde{\Theta}_{p-r})$, and then system (1.1) is generalized exactly synchronizable with respect to $\tilde{\Theta}_{p-r}$. Noting that $\tilde{\Theta}_{\tilde{p}}$ is the minimal extension of Θ_p, satisfying the condition of compatibility, we have $\mathrm{Im}(\tilde{\Theta}_{p-r}^{\mathrm{T}}) \supseteq \mathrm{Im}(\tilde{\Theta}_{\tilde{p}}^{\mathrm{T}})$. Therefore the generalized exact boundary synchronization with respect to $\tilde{\Theta}_{p-r}$ implies that with respect to $\tilde{\Theta}_{\tilde{p}}$.

Remark 2.1 Note that the condition of $\tilde{\Theta}_{\tilde{p}}$-compatibility (2.13) always holds by construction, which allows us to convert the general case into the special case with the condition of compatibility. In fact, Theorem 2.1 shows the necessity of the condition of compatibility: For each given generalized synchronization matrix Θ_p, there exists a generalized synchronization matrix $\tilde{\Theta}_{\tilde{p}}$ satisfying the corresponding condition of $\tilde{\Theta}_{\tilde{p}}$-compatibility (2.13), such that the generalized exact boundary synchronization with respect to $\tilde{\Theta}_{\tilde{p}}$ is equivalent to that with respect to Θ_p. Therefore, without loss of generality we can always assume that the condition of Θ_p-compatibility (2.1) holds when we consider the generalized exact boundary synchronization with respect to Θ_p.

Corollary 2.1 *System* (1.1) *is generalized exactly synchronizable with respect to* Θ_p *if and only if the reduced system of* $W = \tilde{\Theta}_{\tilde{p}}U$*:*

$$\begin{cases} W'' - \Delta W + \bar{A}_{\tilde{p}}W = 0 & \textit{in } (0,+\infty)\times\Omega, \\ W = 0 & \textit{on } (0,+\infty)\times\Gamma_0, \\ W = \tilde{\Theta}_{\tilde{p}}DH & \textit{on } (0,+\infty)\times\Gamma_1 \end{cases} \tag{2.16}$$

is exactly null controllable, where $\bar{A}_{\tilde{p}}$ *is the generalized reduced matrix given by* $\tilde{\Theta}_{\tilde{p}}A = \bar{A}_{\tilde{p}}\tilde{\Theta}_{\tilde{p}}$*. Moreover, this fact holds if and only if*

$$\mathrm{rank}(\tilde{\Theta}_{\tilde{p}}D) = N - \tilde{p}, \tag{2.17}$$

namely, $\mathrm{Ker}(D^{\mathrm{T}}) \cap \mathrm{Span}(\Theta_p^{\mathrm{T}}, A^{\mathrm{T}}\Theta_p^{\mathrm{T}}, \cdots, (A^{\mathrm{T}})^{N-1}\Theta_p^{\mathrm{T}}) = \{0\}$*.*

Proof Thanks to Theorem 2.1, system (1.1) is generalized exactly synchronizable with respect to Θ_p if and only if system (1.1) is generalized exactly synchronizable with respect to $\tilde{\Theta}_{\tilde{p}}$. Noting the condition of $\tilde{\Theta}_{\tilde{p}}$-compatibility (2.13), the conclusion follows from Lemma 2.2.

Theorem 2.2 *If system* (1.1) *is generalized exactly synchronizable with respect to* Θ_p*, then*

$$\mathrm{rank}(\Theta_p D) = N - p, \tag{2.18}$$

in particular,

$$\mathrm{rank}(D) \geq N - p. \tag{2.19}$$

Proof By Corollary 2.1, we have (2.17), namely, $\tilde{\Theta}_{\tilde{p}}D$ is of full rank. Then, noting (2.14), we get $\mathrm{rank}(\Theta_p D) = N - p$, and therefore $\mathrm{rank}(D) \geq N - p$.

Then, in the general case, for the generalized exact boundary synchronization with respect to Θ_p, the required number of boundary controls $\mathrm{rank}(D)$ should be bigger than or equal to $N-p$. Thus, if the number of boundary controls is fewer than $N-p$, there exists no matrix Θ_p such that system (1.1) is generalized exactly synchronizable with respect to Θ_p.

By (2.17), in the general case the required number of boundary controls

$$\mathrm{rank}(D) \geq \mathrm{rank}(\tilde{\Theta}_{\tilde{p}}D) = N - \tilde{p}. \tag{2.20}$$

In the special situation that A does not satisfy the condition of Θ_p-compatibility (2.1), i.e., $\tilde{p} < p$, we need more boundary controls to realize the generalized exact boundary synchronization with respect to Θ_p: $\text{rank}(D) \geq N - \tilde{p} > N - p$. Thus we can easily obtain the following corollary.

Corollary 2.2 *If system* (1.1) *is generalized exactly synchronizable with respect to* Θ_p *under the minimal number of boundary controls:* $\text{rank}(D) = N - p$, *then* A *should satisfy the condition of* Θ_p*-compatibility* (2.1).

Besides, in the following situation, the condition of Θ_p-compatibility (2.1) is also a necessary and sufficient condition to guarantee the generalized exact boundary synchronization with respect to Θ_p.

Corollary 2.3 *Assume that the boundary control matrix* $D \in \mathcal{D}_{N-p}$. *System* (1.1) *is generalized exactly synchronizable with respect to* Θ_p *if and only if* A *satisfies the condition of* Θ_p*-compatibility* (2.1).

Proof Noting that $\text{rank}(D) = N - p$, from Corollary 2.2 the generalized exact boundary synchronization with respect to Θ_p leads to the condition of Θ_p-compatibility (2.1).

Inversely, noting that $\text{rank}(\Theta_p D) = N - p$, by Lemma 2.2, when A satisfies the condition of Θ_p-compatibility (2.1), system (1.1) possesses the generalized exact boundary synchronization with respect to Θ_p.

Now, we consider the generalized exact boundary synchronization from the point of view of an invertible linear transformation. We will show that under an invertible linear transformation of the state variable U, the generalized exact boundary synchronization is actually the partial exact boundary null controllability. More precisely, take the invertible linear transformation

$$X = \begin{pmatrix} \Theta_p \\ (x_{N-p+1}, \cdots, x_N)^{\mathrm{T}} \end{pmatrix}, \tag{2.21}$$

where $x_{N-p+1}, \cdots, x_{N-\tilde{p}}$ are given by (2.14). Applying X to the problem (1.1)–(1.2), it is easy to see that $\tilde{U} = XU$ satisfies the system

$$\begin{cases} \tilde{U}'' - \Delta\tilde{U} + \tilde{A}\tilde{U} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \tilde{U} = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \tilde{U} = \tilde{D}H & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \tag{2.22}$$

with the initial data

$$t = 0: \ (\tilde{U}, \tilde{U}') = X(\hat{U}_0, \hat{U}_1) \ \text{ in } \Omega, \tag{2.23}$$

where $\tilde{A} = XAX^{-1}$ is a matrix similar to the original coupling matrix A (the transformed system (2.22) is then called a similar system of the original system (1.1)), and $\tilde{D} = XD$. Thus we get the following theorem.

Theorem 2.3 *System* (1.1) *is generalized exactly synchronizable with respect to* Θ_p *at the time* $t = T$ *if and only if the former* $N - p$ *state variables of the similar system* (2.22) *are exactly null controllable at the time* $t = T$*; or if and only if the former* $N - \tilde{p}$ *state variables of the similar system* (2.22) *are exactly null controllable at the time* $t = T$.

Proof Noting (2.21), on the one hand, the former $N-p$ state variables of $\tilde{U}=XU$ are exactly $\Theta_p U$, which are required to be zero for time $t \geq T$ according to the generalized exact boundary synchronization (1.3), then we get the first part of the result. On the other hand, the former $N-\tilde{p}$ state variables of $\tilde{U}$ are exactly $\tilde{\Theta}_{\tilde{p}} U$, then the second part of the result follows from Corollary 2.1.

Remark 2.2 Theorem 2.3 indicates that: The generalized exact boundary synchronization with respect to Θ_p at the time $t=T$ means the exact boundary null controllability of the former $N-p$ state variables of $\tilde{U}$, which actually leads to the exact boundary null controllability of $N-\tilde{p}$ $(\geq N-p)$ state variables. In other words, the exact boundary null controllability of $p-\tilde{p}$ state variables is hidden here when $\tilde{p}<p$, namely, A does not satisfy the condition of Θ_p-compatibility (2.1). One can see from (3.6) below that this corresponds to the exact boundary null controllability of $p-\tilde{p}$ elements in the generalized exactly synchronizable state u. This also explains that in order to ensure the generalized exact boundary synchronization with respect to Θ_p, the required minimal number of boundary controls should be $N-\tilde{p}$.

3 Generalized Exactly Synchronizable State

Now we consider the generalized exactly synchronizable state when system (1.1) is generalized exactly synchronizable with respect to Θ_p. For this purpose, it is necessary to employ a basis $\{\epsilon_1, \cdots, \epsilon_p\}$ of $\mathrm{Ker}(\Theta_p)$ as the synchronization basis to give the generalized exactly synchronizable state through (1.5).

Under this synchronization basis $\{\epsilon_1, \cdots, \epsilon_p\}$, the condition of Θ_p-compatibility (2.1) is equivalent to

$$A(\epsilon_1, \cdots, \epsilon_p) = (\epsilon_1, \cdots, \epsilon_p)\tilde{A}_p, \tag{3.1}$$

where $\tilde{A}_p = (\alpha_{rs}) \in \mathbb{M}^{p\times p}(\mathbb{R})$ is called the generalized row-sum matrix of A. In the special case that $\{\epsilon_1, \cdots, \epsilon_p\} = \{e_1, \cdots, e_p\}$, the condition of compatibility (3.1) indicates that $\tilde{A}_p$ is the row-sum matrix of A, namely,

$$\alpha_{rs} = \sum_{j=n_{s-1}+1}^{n_s} a_{ij}, \quad 1 \leq r, s \leq p, \quad n_{r-1}+1 \leq i \leq n_r, \tag{3.2}$$

where $n_0, \cdots, n_p$ and $e_1, \cdots, e_p$ are given by (1.9) and (1.10), respectively.

For the special case that A satisfies the condition of Θ_p-compatibility (2.1), when system (1.1) is generalized exactly synchronizable with respect to Θ_p, we have (1.5). Plugging (1.5) into system (1.1), and noting $\mathrm{Ker}(\Theta_p) = \mathrm{Span}\{\epsilon_1, \cdots, \epsilon_p\}$, by (3.1) we can obtain that the generalized exactly synchronizable state $u = (u_1, \cdots, u_p)^{\mathrm{T}}(t,x)$ satisfies

$$\begin{cases} u'' - \Delta u + \tilde{A}_p u = 0 & \text{in } (T, +\infty) \times \Omega, \\ u = 0 & \text{on } (T, +\infty) \times \Gamma, \\ t = T: \ (u, u') = (u_T, u'_T) & \text{in } \Omega. \end{cases} \tag{3.3}$$

Thereby, the evolution of the generalized exactly synchronizable state u for $t \geq T$ is entirely determined by its values (u_T, u'_T) at the synchronized time $t=T$.

Theorem 3.1 *Suppose that A satisfies the condition of Θ_p-compatibility* (2.1), *and that* $\mathrm{rank}(\Theta_p D) = N-p$. *Then the attainable set of the values (u_T, u'_T) at the time $t=T$ of the* *generalized exactly synchronizable state* u *is the whole space* $(L^2(\Omega))^p \times (H^{-1}(\Omega))^p$ *as the initial data* $(\hat{U}_0, \hat{U}_1)$ *of system* (1.1) *vary in the space* $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$.

Proof First of all, as a result of Lemma 2.2, system (1.1) is generalized exactly synchronizable with respect to Θ_p. Next, we show that the attainable set of the values (u_T, u'_T) is the whole space $(L^2(\Omega))^p \times (H^{-1}(\Omega))^p$.

For any given $(u_T, u'_T) \in (L^2(\Omega))^p \times (H^{-1}(\Omega))^p$, by solving the backward problem (3.3) with homogeneous Dirichlet Boundary conditions on the time interval $[0, T]$, we can get the initial values of the solution $(u, u')(0) = (u_0, u_1)$. From this, we set the initial data of system (1.1) to be $(\hat{U}_0, \hat{U}_1) = (\epsilon_1, \cdots, \epsilon_p)(u_0, u_1)$, and the boundary control to be $H \equiv 0$, then, by the condition of Θ_p-compatibility (3.1), the solution to the corresponding problem (1.1)–(1.2) is $U = (\epsilon_1, \cdots, \epsilon_p)u$, which is actually generalizable exactly synchronized with respect to Θ_p. Moreover, its values at the time $t = T$ are $(U, U')(T) = (\epsilon_1, \cdots, \epsilon_p)(u_T, u'_T)$. Thus, the corresponding generalized exactly synchronizable state u satisfies $(u, u')(T) = (u_T, u'_T)$.

Corollary 3.1 *A satisfies the condition of Θ_p-compatibility* (2.1) *if and only if there exist an initial data $(\hat{U}_0, \hat{U}_1)$ and a boundary control H which realize the generalized exact boundary synchronization with respect to Θ_p for system* (1.1), *such that the p elements of the corresponding generalized exactly synchronizable state $u = (u_1, \cdots, u_p)^{\mathrm{T}}$ are linearly independent.*

Proof Suppose that A satisfies the condition of Θ_p-compatibility (2.1). Given $u_T = (u_{T1}, \cdots, u_{Tp})^{\mathrm{T}}$ whose p elements are linearly independent, by solving problem (3.3) and the backward problem (3.3) with homogeneous Dirichlet boundary conditions on the time interval $[0, T]$, respectively, we get its solution $u = u^*(t, x)$ for all $t \geq 0$.

Then, setting the initial data $(\hat{U}_0, \hat{U}_1) = (\epsilon_1, \cdots, \epsilon_p)(u^*, u^{*\prime})(0)$, and the boundary control $H \equiv 0$, by the condition of Θ_p-compatibility (3.1), the solution to problem (1.1)–(1.2) is $U = (\epsilon_1, \cdots, \epsilon_p)u^*$, hence this system is generalized exactly synchronizable with respect to Θ_p, and the p elements of its corresponding generalized exactly synchronizable state u^* are linearly independent at least at the time $t = T$.

In turn, if the p elements of the corresponding generalized exactly synchronizable state $u = (u_1, \cdots, u_p)^{\mathrm{T}}$ are linearly independent, then by (2.3) we have $\Theta_p A \epsilon_r = 0$ $(r = 1, \cdots, p)$, which means exactly the condition of Θ_p-compatibility (2.1).

Now consider the general case that A may not satisfy the condition of Θ_p-compatibility (2.1). When system (1.1) is generalized exactly synchronizable with respect to Θ_p, by Theorem 2.1 it is also generalized exactly synchronizable with respect to $\tilde{\Theta}_{\tilde{p}}$, where $\tilde{\Theta}_{\tilde{p}}$ is given by (2.11). Let $\{\tilde{\epsilon}_1, \cdots, \tilde{\epsilon}_{\tilde{p}}\}$ be a basis of $\mathrm{Ker}(\tilde{\Theta}_{\tilde{p}})$. The corresponding generalized exactly synchronizable state $\tilde{u} = (\tilde{u}_1, \cdots, \tilde{u}_{\tilde{p}})^{\mathrm{T}}$ with respect to $\tilde{\Theta}_{\tilde{p}}$ can be given by

$$t \geq T: \ U = \tilde{u}_1 \tilde{\epsilon}_1 + \cdots + \tilde{u}_{\tilde{p}} \tilde{\epsilon}_{\tilde{p}} = (\tilde{\epsilon}_1, \cdots, \tilde{\epsilon}_{\tilde{p}}) \tilde{u}. \tag{3.4}$$

Noting (2.11), we have $\mathrm{Im}(\tilde{\Theta}_{\tilde{p}}^{\mathrm{T}}) \supseteq \mathrm{Im}(\Theta_p^{\mathrm{T}})$, so

$$\mathrm{Ker}(\tilde{\Theta}_{\tilde{p}}) = \mathrm{Span}\{\tilde{\epsilon}_1, \cdots, \tilde{\epsilon}_{\tilde{p}}\} \subseteq \mathrm{Ker}(\Theta_p) = \mathrm{Span}\{\epsilon_1, \cdots, \epsilon_p\}.$$

Therefore, there exists a unique full column-rank matrix Q of order $p \times \tilde{p}$, such that

$$(\tilde{\epsilon}_1, \cdots, \tilde{\epsilon}_{\tilde{p}}) = (\epsilon_1, \cdots, \epsilon_p) Q. \tag{3.5}$$

Then it follows from (3.4) and (1.5) that the corresponding generalized exactly synchronizable state u with respect to Θ_p can be linearly expressed by the generalized exactly synchronizable state $\tilde{u}$ with respect to $\tilde{\Theta}_{\tilde{p}}$ as follows:

$$u = Q\tilde{u}. \tag{3.6}$$

Thus, by the condition of $\tilde{\Theta}_{\tilde{p}}$-compatibility (2.13), the properties of the generalized exactly synchronizable state $\tilde{u}$ with respect to $\tilde{\Theta}_{\tilde{p}}$ can be obtained from the results acquired in the former case with the condition of compatibility, and finally the properties of the generalized exactly synchronizable state u with respect to Θ_p come by (3.6).

Remark 3.1 When A does not possess the condition of Θ_p-compatibility (2.1), by (3.6) the generalized exactly synchronizable state u with respect to Θ_p has only $\tilde{p}$ $(< p)$ linearly independent components. This is indeed a generalized exact boundary synchronization by fewer groups (the generalized exact boundary synchronization with respect to $\tilde{\Theta}_{\tilde{p}}$), consequently, more boundary controls are demanded.

Remark 3.2 In particular, if system (1.1) is generalized exactly synchronizable with respect to Θ_p, and $\tilde{p} = 0$ in (2.11), then the corresponding generalized exactly synchronizable state $u \equiv 0$, which means system (1.1) is actually exactly null controllable.

4 Estimation of Generalized Exactly Synchronizable States

Now we will give an estimate on generalized exactly synchronizable states when system (1.1) is generalized exactly synchronizable under the minimal number of boundary controls, namely, $D \in \mathcal{D}_{N-p}$. For this purpose, we firstly give the following proposition.

Proposition 4.1 *$D \in \mathcal{D}_{N-p}$ if and only if $\mathrm{Ker}(D^{\mathrm{T}}) = (\mathrm{Im}(D))^{\perp}$ and $\mathrm{Ker}(\Theta_p)$ are bi-orthonormal.*

Proof When $D \in \mathcal{D}_{N-p}$, we prove that $(\mathrm{Im}(D))^{\perp}$ and $\mathrm{Ker}(\Theta_p)$ are bi-orthonormal. Noting $(\mathrm{Im}(D))^{\perp}$ and $\mathrm{Ker}(\Theta_p)$ have the same dimension, it suffices to prove

$$\mathrm{Im}(D) \cap \mathrm{Ker}(\Theta_p) = \{0\}. \tag{4.1}$$

In fact, for each $x\ (\neq 0) \in \mathrm{Im}(D)$, there exists $y \neq 0$, such that $x = Dy$. Noting that $D \in \mathcal{D}_{N-p}$ claims that $\Theta_p D$ is invertible, it follows from $y \neq 0$ that $\Theta_p x \neq 0$, then $x \notin \mathrm{Ker}(\Theta_p)$.

Inversely, when $\mathrm{Ker}(D^{\mathrm{T}})$ and $\mathrm{Ker}(\Theta_p)$ are bi-orthonormal, we have $\mathrm{rank}(D) = \mathrm{rank}(\Theta_p) = N - p$, so it suffices to verify $\mathrm{rank}(\Theta_p D) = N - p$. Suppose that $\mathrm{rank}(\Theta_p D) < N - p$, then there exists a vector $y = (y_1, \cdots, y_{N-p})^{\mathrm{T}} \neq 0$, such that $\Theta_p D y = 0$. Since $\mathrm{rank}(D) = N - p$, the $N - p$ columns $d_1, \cdots, d_{N-p}$ of $D = (d_1, \cdots, d_{N-p})$ are linearly independent, hence

$$Dy = y_1 d_1 + \cdots + y_{N-p} d_{N-p} \neq 0. \tag{4.2}$$

Taking $x = Dy$, we have $\Theta_p x = \Theta_p D y = 0$, then there exists $x\ (\neq 0) \in \mathrm{Im}(D) \cap \mathrm{Ker}(\Theta_p)$. This contradicts with (4.1) which is an equivalent form of the bi-orthonormality between $(\mathrm{Im}(D))^{\perp}$ and $\mathrm{Ker}(\Theta_p)$. Thus we get $D \in \mathcal{D}_{N-p}$.

Then similarly to [8, Theorem 4.1] in the case of exact boundary synchronization, we obtain the following theorem.

Theorem 4.1 *Assume that A satisfies the condition of Θ_p-compatibility (2.1), then for any given $D \in \mathcal{D}_{N-p}$, system (1.1) is generalized exactly synchronizable with respect to Θ_p, and each generalized exactly synchronizable state u satisfies the following estimate:*

$$\|(u,u')-(\varphi,\varphi')\|(T)_{(H_0^1(\Omega))^p\times(L^2(\Omega))^p} \le c_T\|\Theta_p(\hat{U}_0,\hat{U}_1)\|_{(L^2(\Omega))^{N-p}\times(H^{-1}(\Omega))^{N-p}}, \tag{4.3}$$

where c_T is a positive constant only depending on the synchronization time T, but independent of the initial data; function φ is a solution to the following problem with homogeneous boundary condition:

$$\begin{cases} \varphi''-\Delta\varphi+\tilde{A}_p\varphi=0 & \text{in } (0,+\infty)\times\Omega,\\ \varphi=0 & \text{on } (0,+\infty)\times\Gamma,\\ t=0:\ (\varphi,\varphi')=(y_1,\cdots,y_p)^{\mathrm{T}}(\hat{U}_0,\hat{U}_1) & \text{in } \Omega, \end{cases} \tag{4.4}$$

where $\tilde{A}_p$ is given by (3.1), and $\{y_1,\cdots,y_p\}$ constitutes a basis of $\mathrm{Ker}(D^{\mathrm{T}})$, which is bi-orthonormal to the synchronization basis $\{\epsilon_1,\cdots,\epsilon_p\}$.

Proof By Proposition 4.1, it follows from $D\in\mathcal{D}_{N-p}$ that $\mathrm{Ker}(D^{\mathrm{T}})$ is bi-orthonormal to $\mathrm{Ker}(\Theta_p)$, then $\mathrm{Ker}(D^{\mathrm{T}})$ has a basis $\{y_1,\cdots,y_p\}$, which is bi-orthonormal to the basis $\{\epsilon_1,\cdots,\epsilon_p\}$ of $\mathrm{Ker}(\Theta_p)$.

Let $\psi=(y_1,\cdots,y_p)^{\mathrm{T}}U$. Noting (1.5), we have

$$t\ge T:\ \psi=(y_1,\cdots,y_p)^{\mathrm{T}}(\epsilon_1,\cdots,\epsilon_p)u=u. \tag{4.5}$$

Since $(y_1,\cdots,y_p)^{\mathrm{T}}D=0$, multiplying problem (1.1)–(1.2) by $(y_1,\cdots,y_p)^{\mathrm{T}}$, we get

$$\begin{cases} \psi''-\Delta\psi+\tilde{A}_p\psi=f & \text{in } (0,+\infty)\times\Omega,\\ \psi=0 & \text{on } (0,+\infty)\times\Gamma,\\ t=0:\ (\psi,\psi')=(y_1,\cdots,y_p)^{\mathrm{T}}(\hat{U}_0,\hat{U}_1) & \text{in } \Omega, \end{cases} \tag{4.6}$$

where $f=\tilde{A}_p\psi-(y_1,\cdots,y_p)^{\mathrm{T}}AU=-FU$ with $F=(y_1,\cdots,y_p)^{\mathrm{T}}A-\tilde{A}_p(y_1,\cdots,y_p)^{\mathrm{T}}$.

By (3.1), $F^{\mathrm{T}}\in(\mathrm{Span}\{\epsilon_1,\cdots,\epsilon_p\})^{\perp}=(\mathrm{Ker}(\Theta_p))^{\perp}=\mathrm{Im}(\Theta_p^{\mathrm{T}})$, so there exists $Y\in\mathbb{M}^{(N-p)\times p}(\mathbb{R})$, such that $F^{\mathrm{T}}=\Theta_p^{\mathrm{T}}Y$, namely, $F=Y^{\mathrm{T}}\Theta_p$. Therefore

$$f=-Y^{\mathrm{T}}\Theta_pU. \tag{4.7}$$

Observe that problem (4.6) and problem (4.4) share the same initial data and the same boundary condition, by the well-posedness, noting (4.5) and (4.7), there exists a positive constant c_1 such that

$$\|(u,u')-(\varphi,\varphi')\|^2(T)_{(H_0^1(\Omega))^p\times(L^2(\Omega))^p}\le c_1\int_0^T\|\Theta_pU(\tau)\|^2_{(L^2(\Omega))^{N-p}}d\tau.$$

Finally, denote $W=\Theta_pU$, then by the null controllability of reduced system (2.8), there exists a positive constant c_2 such that

$$\int_0^T\|\Theta_pU(\tau)\|^2_{(L^2(\Omega))^{N-p}}d\tau\le c_2\|\Theta_p(\hat{U}_0,\hat{U}_1)\|^2_{(L^2(\Omega))^{N-p}\times(H^{-1}(\Omega))^{N-p}}.$$

Thus we complete the proof.

Remark 4.1 Inequality (4.3) says that, when the initial data $(\hat{U}_0, \hat{U}_1)$ are in a neighbourhood of $\mathrm{Ker}(\Theta_p)$, the corresponding generalized synchronizable state u can be approximately estimated by the solution φ to problem (4.4).

Remark 4.2 By Theorem 4.1, when A possesses the condition of Θ_p-compatibility (2.1), as long as $D \in \mathcal{D}_{N-p}$, namely, using the minimal number of boundary controls to realize the generalized exact boundary synchronization with respect to Θ_p for system (1.1), we can construct a solution φ to problem (4.4) such that for all admissible applied boundary controls, the corresponding generalized synchronizable state u satisfies the estimate (4.3). While, when A does not satisfy the condition of Θ_p-compatibility (2.1), Theorem 4.1 can be applied to the enlarged matrix $\tilde{\Theta}_{\tilde{p}}$ given by (2.11), thus, as long as D satisfies

$$\mathrm{rank}(D) = \mathrm{rank}(\tilde{\Theta}_{\tilde{p}} D) = N - \tilde{p},$$

system (1.1) still possesses the generalized exact boundary synchronization with respect to Θ_p, and the corresponding generalized synchronizable state u satisfies an estimate like (4.3):

$$\|(u, u') - Q(\tilde{\varphi}, \tilde{\varphi}')\|(T)_{(H_0^1(\Omega))^p \times (L^2(\Omega))^p} \leq c \|\tilde{\Theta}_{\tilde{p}}(\hat{U}_0, \hat{U}_1)\|_{(L^2(\Omega))^{N-\tilde{p}} \times (H^{-1}(\Omega))^{N-\tilde{p}}}, \tag{4.8}$$

where, c is a positive constant only depending on T, but independent of the initial data; the matrix Q is given by (3.5), provided that $\{\tilde{\epsilon}_1, \cdots, \tilde{\epsilon}_{\tilde{p}}\}$ is a basis of $\mathrm{Ker}(\tilde{\Theta}_{\tilde{p}})$, and $\tilde{\varphi}$ satisfies the following problem with homogeneous boundary condition:

$$\begin{cases} \tilde{\varphi}'' - \Delta \tilde{\varphi} + \tilde{A}_{\tilde{p}} \tilde{\varphi} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \tilde{\varphi} = 0 & \text{on } (0, +\infty) \times \Gamma, \\ t = 0 : (\tilde{\varphi}, \tilde{\varphi}') = (\tilde{y}_1, \cdots, \tilde{y}_{\tilde{p}})^{\mathrm{T}} (\hat{U}_0, \hat{U}_1) & \text{in } \Omega, \end{cases} \tag{4.9}$$

where $\tilde{A}_{\tilde{p}}$ is given by

$$A(\tilde{\epsilon}_1, \cdots, \tilde{\epsilon}_{\tilde{p}}) = (\tilde{\epsilon}_1, \cdots, \tilde{\epsilon}_{\tilde{p}}) \tilde{A}_{\tilde{p}},$$

and $\{\tilde{y}_1, \cdots, \tilde{y}_{\tilde{p}}\}$ as a basis of $\mathrm{Ker}(D^{\mathrm{T}})$ is bi-orthonormal to $\{\tilde{\epsilon}_1, \cdots, \tilde{\epsilon}_{\tilde{p}}\}$.

5 Determination of Generalized Exactly Synchronizable States

Assume that system (1.1) is generalized exactly synchronizable with respect to Θ_p. In general, generalized exactly synchronizable states u depend not only on the initial data $(\hat{U}_0, \hat{U}_1)$ but also on applied boundary controls H. In what follows, we are interested in under what conditions the generalized synchronizable states u will be determined only by the initial data but independent of applied boundary controls H.

We will first give an extension of $\mathrm{Ker}(\Theta_p)$, and then define a synchronization decomposition of system (1.1) by separating it into the controllable part and the synchronizable part. In this way, by studying the synchronizable part, we will give the condition under which the generalized exactly synchronizable state is independent of applied boundary controls.

5.1 An extension of $\mathrm{Ker}(\Theta_p)$

Proposition 5.1 *There exists a minimal extension* $\mathrm{Span}\{\epsilon_1, \cdots, \epsilon_p, \cdots, \epsilon_q\}$ $(p \le q \le N)$ *of* $\mathrm{Ker}(\Theta_p) = \mathrm{Span}\{\epsilon_1, \cdots, \epsilon_p\}$, *which is an invariant subspace of* A, *and admits a bi-orthonormal space* $\mathrm{Span}\{\eta_1, \cdots, \eta_q\}$, *being invariant for* A^{T}, *namely,*

$$A\,\mathrm{Span}\{\epsilon_1, \cdots, \epsilon_q\} \subseteq \mathrm{Span}\{\epsilon_1, \cdots, \epsilon_q\}, \tag{5.1}$$

$$A^{\mathrm{T}}\,\mathrm{Span}\{\eta_1, \cdots, \eta_q\} \subseteq \mathrm{Span}\{\eta_1, \cdots, \eta_q\} \tag{5.2}$$

and

$$(\eta_1, \cdots, \eta_q)^{\mathrm{T}}(\epsilon_1, \cdots, \epsilon_q) = I_q. \tag{5.3}$$

Here and hereafter, I_q *stands for the identity matrix of order* q (*instead,* I *denotes the identity matrix of order* N).

Proposition 5.2 *Conditions* (5.1)–(5.3) *are equivalent to that* $\mathrm{Span}\{\epsilon_1, \cdots, \epsilon_q\}$ *is an invariant subspace of* A *and admits a supplement* $\mathrm{Span}\{x_{q+1}, \cdots, x_N\}$ *which is also invariant for* A.

From Propositions 5.1–5.2, we have the following proposition.

Proposition 5.3 *The coupling matrix* A *satisfies the condition of* Θ_p*-strong compatibility:*

$$\begin{cases} \mathrm{Ker}(\Theta_p) = \mathrm{Span}\{\epsilon_1, \cdots, \epsilon_p\} \text{ is an invariant subspace of } A, \\ A^{\mathrm{T}} \text{ admits an invariant subspace } \mathrm{Span}\{\eta_1, \cdots, \eta_p\} \\ \quad \text{which is bi-orthonormal to } \mathrm{Span}\{\epsilon_1, \cdots, \epsilon_p\}, \end{cases} \tag{5.4}$$

in other words, $\mathrm{Ker}(\Theta_p)$ *is an invariant subspace of* A *and admits a supplement also invariant for* A, *if and only if* $q = p$ *in Proposition 5.1 or equivalently in Proposition 5.2.*

The proof of Propositions 5.2–5.1 will be given in what follows, respectively.

Proof of Proposition 5.2 By (5.3), taking a basis $x_{q+1}, \cdots, x_N$ in $(\mathrm{Span}\{\eta_1, \cdots, \eta_q\})^{\perp}$, the matrix

$$X = (\epsilon_1, \cdots, \epsilon_q, x_{q+1}, \cdots, x_N) \tag{5.5}$$

is invertible, furthermore, the former q rows of X^{-1} are exactly $(\eta_1, \cdots, \eta_q)^{\mathrm{T}}$. let $X^{-\mathrm{T}} = (X^{-1})^{\mathrm{T}} = (\eta_1, \cdots, \eta_q, y_{q+1}, \cdots, y_N)$. According to (5.1), we have

$$AX = X \begin{pmatrix} \tilde{A}_q & l \\ 0 & \tilde{A}_{N-q} \end{pmatrix}, \tag{5.6}$$

where $\tilde{A}_q$ and $\tilde{A}_{N-q}$ are square matrices of order q and $N-q$, respectively, and l is a matrix of order $q \times (N-q)$. By taking the transpose of the above formula and then multiplying both sides by $X^{-\mathrm{T}}$, we have

$$A^{\mathrm{T}}X^{-\mathrm{T}} = X^{-\mathrm{T}} \begin{pmatrix} \tilde{A}_q^{\mathrm{T}} & 0 \\ l^{\mathrm{T}} & \tilde{A}_{N-q}^{\mathrm{T}} \end{pmatrix}, \tag{5.7}$$

whose former q columns say that $A^{\mathrm{T}}(\eta_1, \cdots, \eta_q) = (\eta_1, \cdots, \eta_q)\tilde{A}_q^{\mathrm{T}} + (y_{q+1}, \cdots, y_N)l^{\mathrm{T}}$. Then we have $(y_{q+1}, \cdots, y_N)l^{\mathrm{T}} = 0$ from (5.2), therefore

$$l = 0. \tag{5.8}$$

Inserting this into (5.6) and taking the last $N-q$ columns, we get

$$A(x_{q+1},\cdots,x_N)=(x_{q+1},\cdots,x_N)\tilde{A}_{N-q}, \tag{5.9}$$

which declares that the supplement $\mathrm{Span}\{x_{q+1},\cdots,x_N\}$ of $\mathrm{Span}\{\epsilon_1,\cdots,\epsilon_q\}$ is also an invariant subspace of A.

On the other hand, since $\mathrm{Span}\{\epsilon_1,\cdots,\epsilon_q\}$ and its supplement $\mathrm{Span}\{x_{q+1},\cdots,x_N\}$ are invariant subspaces of A, $X=(\epsilon_1,\cdots,\epsilon_q,x_{q+1},\cdots,x_N)$ is invertible, and A can be diagonalized by blocs as

$$AX=X\begin{pmatrix}\tilde{A}_q & 0\\ 0 & \tilde{A}_{N-q}\end{pmatrix}, \tag{5.10}$$

where $\tilde{A}_q$ and $\tilde{A}_{N-q}$ are square matrices of order q and $N-q$, respectively. By taking the transpose of the above formula and then multiplying both sides by $X^{-\mathrm{T}}$, we get

$$A^{\mathrm{T}}X^{-\mathrm{T}}=X^{-\mathrm{T}}\begin{pmatrix}\tilde{A}_q^{\mathrm{T}} & 0\\ 0 & \tilde{A}_{N-q}^{\mathrm{T}}\end{pmatrix}. \tag{5.11}$$

Denoting the former q columns of $X^{-\mathrm{T}}$ as $(\eta_1,\cdots,\eta_q)$, we have

$$A^{\mathrm{T}}(\eta_1,\cdots,\eta_q)=(\eta_1,\cdots,\eta_q)\tilde{A}_q^{\mathrm{T}}, \tag{5.12}$$

which means that $\mathrm{Span}\{\eta_1,\cdots,\eta_q\}$ is an invariant subspace of A^{T}. Besides, by $X^{-1}X=I$, the bi-orthonormality (5.3) holds.

Proof of Proposition 5.1 Firstly, there exists a basis $\{\xi_1,\cdots,\xi_q,\cdots,\xi_N\}$ of $\mathbb{R}^N$, such that A can be transformed to its real Jordan form (see [1]) by

$$A(\xi_1,\cdots,\xi_q,\xi_{q+1},\cdots,\xi_N)=(\xi_1,\cdots,\xi_q,\xi_{q+1},\cdots,\xi_N)\begin{pmatrix}J_q & 0\\ 0 & J_{N-q}\end{pmatrix} \tag{5.13}$$

and

$$\mathrm{Span}\{\xi_1,\cdots,\xi_q\}\supseteq\mathrm{Span}\{\epsilon_1,\cdots,\epsilon_p\}. \tag{5.14}$$

Obviously $q\geq p$, and when $q=N$, (5.13)–(5.14) hold. Therefore there exists a minimum of q, denoted still by q, satisfying both (5.13) and (5.14). Then the corresponding $\mathrm{Span}\{\xi_1,\cdots,\xi_q\}$ is a minimal extension of $\mathrm{Span}\{\epsilon_1,\cdots,\epsilon_p\}$, such that it and its supplement $\mathrm{Span}\{\xi_{q+1},\cdots,\xi_N\}$ are all invariant for A. By Proposition 5.2, $\mathrm{Span}\{\xi_1,\cdots,\xi_q\}$ is a minimal extension satisfying (5.1)–(5.3).

5.2 A synchronization decomposition

For the bi-orthonormal systems $\{\epsilon_1,\cdots,\epsilon_q\}$ and $\{\eta_1,\cdots,\eta_q\}$ given by Proposition 5.1, we may define a projection operator

$$P=\sum_{i=1}^{q}\epsilon_i\otimes\eta_i, \tag{5.15}$$

where $(\epsilon\otimes\eta)U=(U,\eta)\epsilon=\eta^{\mathrm{T}}U\epsilon,\ \forall U\in\mathbb{R}^N$.

Proposition 5.4 *The projection operator P has the following properties:*

$$P\epsilon_k = \epsilon_k, \quad \eta_k^{\rm T} P = \eta_k^{\rm T}, \quad 1 \le k \le q, \tag{5.16}$$

$$\mathrm{Ker}(P) = (\mathrm{Span}\{\eta_1, \cdots, \eta_q\})^{\perp},\ \mathrm{Im}(P) = \mathrm{Span}\{\epsilon_1, \cdots, \epsilon_q\} \tag{5.17}$$

and

$$AP = PA. \tag{5.18}$$

Proof (5.16)–(5.17) follow directly from definition (5.15). Noting

$$PU = \sum_{i=1}^{q} \epsilon_i\ (\eta_i^{\rm T} U) = (\epsilon_1, \cdots, \epsilon_q)(\eta_1, \cdots, \eta_q)^{\rm T} U,\ \forall U \in \mathbb{R}^N,$$

P can also be written as a matrix:

$$P = (\epsilon_1, \cdots, \epsilon_q)(\eta_1, \cdots, \eta_q)^{\rm T}. \tag{5.19}$$

In view of (5.1)–(5.3) satisfied by the bi-orthonormal systems $\{\epsilon_1, \cdots, \epsilon_q\}$ and $\{\eta_1, \cdots, \eta_q\}$, there exists a square matrix $\tilde{A}_q$ of order q, such that

$$A(\epsilon_1, \cdots, \epsilon_q) = (\epsilon_1, \cdots, \epsilon_q)\tilde{A}_q,\ A^{\rm T}(\eta_1, \cdots, \eta_q) = (\eta_1, \cdots, \eta_q)\tilde{A}_q^{\rm T},$$

which leads to (5.18) by (5.19).

The solution $U = U(t, x)$ to problem (1.1)–(1.2) can be divided into two parts: $U = U_c + U_s$, where

$$U_c = (I - P)U, \quad U_s = PU \tag{5.20}$$

satisfying the following problems deduced by multiplying problem (1.1)–(1.2) by $(I - P)$ and P, respectively,

$$\begin{cases} U_c'' - \Delta U_c + AU_c = 0 & \text{in } (0, +\infty) \times \Omega, \\ U_c = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ U_c = (I - P)DH & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0: \ (U_c, U_c') = (I - P)(\hat{U}_0, \hat{U}_1) & \text{in } \Omega \end{cases} \tag{5.21}$$

and

$$\begin{cases} U_s'' - \Delta U_s + AU_s = 0 & \text{in } (0, +\infty) \times \Omega, \\ U_s = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ U_s = PDH & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0: \ (U_s, U_s') = P(\hat{U}_0, \hat{U}_1) & \text{in } \Omega. \end{cases} \tag{5.22}$$

Noting (5.16), the generalized exact boundary synchronization (1.5) becomes

$$t \ge T: \quad U_c = 0,\ U_s = U = (\epsilon_1, \cdots, \epsilon_p)u, \tag{5.23}$$

hence, U_c should be exactly null controllable (called the controllable part of U), while, U_s generalized exactly synchronizable with respect to Θ_p (called the synchronizable part of U).

5.3 Determination of generalized exactly synchronizable states

If the synchronizable part U_s is independent of applied boundary controls, by (5.23) the generalized exactly synchronizable state u is independent of applied boundary controls, hence can be determined only by the initial data.

Similarly to the situation on the exact boundary synchronization (see [8, Theorem 3.1]), without the condition of Θ_p-compatibility (2.1), we have the following theorem.

Theorem 5.1 *(i) If system* (1.1) *possesses the generalized exact boundary synchronization with respect to Θ_p, and its synchronizable part U_s is independent of applied boundary controls H, then A satisfies the condition of Θ_p-strong compatibility* (5.4)*, moreover, $D \in \mathcal{D}_{N-p}$ satisfies $PD = 0$.*

(ii) If A satisfies the condition of Θ_p-strong compatibility (5.4)*, then there exists a boundary control matrix $D \in \mathcal{D}_{N-p}$, satisfying $PD = 0$, such that system* (1.1) *is generalized exactly synchronizable with respect to Θ_p, and its synchronizable part U_s and then its generalized exactly synchronizable state u are independent of applied boundary controls H.*

Proof (i) Let H_1 and H_2 be two admissible boundary controls to realize the generalized exact boundary synchronization with respect to Θ_p for system (1.1). By Corollary 2.1, it is equivalent to say that both H_1 and H_2 realize the exact boundary null controllability of the reduced system (2.16). Then, applying [8, Theorem 2.2], there exists an $\varepsilon > 0$ small enough, such that the values of $H_1 - H_2$ on $(T - \varepsilon, T) \times \Gamma_1$ can be arbitrarily chosen.

If the synchronizable part U_s is independent of applied boundary controls H_1 and H_2, then we have $PD(H_1 - H_2) = 0$ on $(0, T) \times \Gamma_1$, thus we get

$$PD = 0, \tag{5.24}$$

namely, $\mathrm{Im}(D) \subseteq \mathrm{Ker}(P)$. Recalling (2.19), the dimension of $\mathrm{Im}(D)$ is equal to $\mathrm{rank}(D) \geq N-p$, and the dimension of $\mathrm{Ker}(P)$ is equal to $N - q$, so $N - p \leq N - q$, then $q = p$, and

$$\mathrm{rank}(D) = N - p. \tag{5.25}$$

Besides, by Theorem 2.2 we have $\mathrm{rank}(\Theta_p D) = N - p$. Noting that D is assumed to be full rank, so $D \in \mathcal{D}_{N-p}$.

(ii) Let $D \in \mathbb{M}^{N\times(N-p)}(\mathbb{R})$ be defined by

$$\mathrm{Im}(D) = \mathrm{Span}\{\eta_1, \cdots, \eta_p\}^{\perp}. \tag{5.26}$$

We will show that D is the desired boundary control matrix. Obviously, we have that $\mathrm{rank}(D) = N - p$, and $(\eta_1, \cdots, \eta_p)^{\mathrm{T}} D = 0$. Thanks to the condition of Θ_p-strong compatibility (5.4), we get

$$PD = \sum_{i=1}^{p} (\eta_i^{\mathrm{T}} D)\epsilon_i = 0. \tag{5.27}$$

If $\mathrm{rank}(\Theta_p D) = N - p$, then by Lemma 2.2 we get the generalized exact boundary synchronization of system (1.1) with respect to Θ_p, and by (5.27) we get the independence of the synchronizable part U_s and then of the generalized exactly synchronizable state u with respect to applied boundary controls H.

It remains to verify $\text{rank}(\Theta_p D) = N - p$. For any given $x \in \text{Ker}(\Theta_p D)$, $Dx \in \text{Ker}(\Theta_p) = \text{Span}\{\epsilon_1, \cdots, \epsilon_p\}$, which declares that $Dx = \sum_{i=1}^{p} k_i \epsilon_i$, in which $k_i = \eta_i^{\mathrm{T}} Dx = 0$ $(i = 1, \cdots, p)$, hence $Dx = 0$, namely, $x \in \text{Ker}(D)$. Thus we get $\text{Ker}(\Theta_p D) \subseteq \text{Ker}(D)$, then

$$\text{Ker}(\Theta_p D) = \text{Ker}(D). \tag{5.28}$$

Therefore, we have $\text{rank}(\Theta_p D) = \text{rank}(D) = N - p$.

Remark 5.1 Theorem 5.1 offers the following necessary and sufficient condition:

$$\begin{cases} A \text{ satisfies the condition of } \Theta_p\text{-strong compatibility (5.4)}, \\ D \in \mathcal{D}_{N-p} \text{ satisfies } PD = 0 \end{cases} \tag{5.29}$$

for the generalized exact boundary synchronization of system (1.1) with respect to Θ_p, and the independence of the synchronizable part U_s with respect to applied boundary controls H. However, this is only sufficient for the independence of the corresponding generalized exactly synchronizable state u with respect to applied boundary controls H, but not necessary. In fact, the failure of its necessity can be deduced by Remark 3.2 where the generalized exactly synchronizable state $u \equiv 0$ independent of applied boundary controls H, but A does not satisfy the condition of Θ_p-strong compatibility (5.4).

If A satisfies the condition of Θ_p-strong compatibility (5.4), then the evolution of the generalized exactly synchronizable state u with respect to t can be easily obtained, provided that the boundary control matrix D is suitably given.

Corollary 5.1 *If A satisfies the condition of Θ_p-strong compatibility* (5.4), *and $D \in \mathcal{D}_{N-p}$ satisfies $PD = 0$, then system* (1.1) *is generalized exactly synchronizable with respect to Θ_p, and the corresponding generalized exactly synchronizable state u is determined by*

$$u = \phi \ (t \geq T), \tag{5.30}$$

where $\phi = (\eta_1, \cdots, \eta_p)^{\mathrm{T}} U$ is the solution of

$$\begin{cases} \phi'' - \Delta\phi + \tilde{A}_p \phi = 0 & \text{in } (0, +\infty) \times \Omega, \\ \phi = 0 & \text{on } (0, +\infty) \times \Gamma, \\ t = 0: \ (\phi, \phi') = (\eta_1, \cdots, \eta_p)^{\mathrm{T}} (\hat{U}_0, \hat{U}_1) & \text{in } \Omega, \end{cases} \tag{5.31}$$

in which $\tilde{A}_p$ is defined by (3.1).

Proof Multiplying (1.1)–(1.2) by $(\eta_1, \cdots, \eta_p)^{\mathrm{T}}$, the conclusion can be easily obtained.

Theorem 5.2 *Suppose that A satisfies the condition of Θ_p-strong compatibility* (5.4), *and that system* (1.1) *is generalized exactly synchronizable with respect to Θ_p. If the boundary control matrix D is modified as*

$$\tilde{D} = D - (\epsilon_1, \cdots, \epsilon_p)(\eta_1, \cdots, \eta_p)^{\mathrm{T}} D, \tag{5.32}$$

then the corresponding system is still generalized exactly synchronizable with respect to Θ_p, moreover, the corresponding generalized exactly synchronizable state u still satisfies (5.30) *and is independent of applied boundary controls H.*

Proof Since $\mathrm{Span}\{\eta_1, \cdots, \eta_p\}$ and $\mathrm{Span}\{\epsilon_1, \cdots, \epsilon_p\} = \mathrm{Ker}(\Theta_p)$ are bi-orthonormal, the matrix

$$X = \begin{pmatrix} \Theta_p \\ (\eta_1, \cdots, \eta_p)^{\mathrm{T}} \end{pmatrix} \tag{5.33}$$

is invertible. $\tilde{D}$ defined by (5.32) satisfies

$$\Theta_p \tilde{D} = \Theta_p D, \ (\eta_1, \cdots, \eta_p)^{\mathrm{T}} \tilde{D} = 0, \tag{5.34}$$

then

$$X\tilde{D} = \begin{pmatrix} \Theta_p \tilde{D} \\ 0 \end{pmatrix} = \begin{pmatrix} \Theta_p D \\ 0 \end{pmatrix}. \tag{5.35}$$

Thus

$$\mathrm{rank}(\tilde{D}) = \mathrm{rank}(X\tilde{D}) = \mathrm{rank}(\Theta_p \tilde{D}) = \mathrm{rank}(\Theta_p D). \tag{5.36}$$

Since the condition of Θ_p-strong compatibility (5.4) implies the condition of Θ_p-compatibility (2.1), by Lemma 2.2 the generalized exact boundary synchronization of system (1.1) induces

$$\mathrm{rank}(\Theta_p \tilde{D}) = \mathrm{rank}(\Theta_p D) = N - p, \tag{5.37}$$

then we still have the generalized exact boundary synchronization of system (1.1) with boundary control matrix $\tilde{D}$. Thus we get the conclusion by employing Corollary 5.1.

Now for practical convenience, we give the following results on condition (5.29). Firstly, we give a sufficient condition to the condition of Θ_p-strong compatibility (5.4).

Remark 5.2 By Proposition 5.3, we have the following sufficient condition to the condition of Θ_p-strong compatibility (5.4): Both $\mathrm{Ker}(\Theta_p) = \mathrm{Span}\{\epsilon_1, \cdots, \epsilon_p\}$ and its supplement $\mathrm{Im}(\Theta_p^{\mathrm{T}})$ are invariant for A, in other words, A can be diagonalized by blocs in the following way:

$$A(\epsilon_1, \cdots, \epsilon_p, \Theta_p^{\mathrm{T}}) = (\epsilon_1, \cdots, \epsilon_p, \Theta_p^{\mathrm{T}}) \begin{pmatrix} \tilde{A}_p & 0 \\ 0 & \hat{A} \end{pmatrix}, \tag{5.38}$$

where $\hat{A} = (\Theta_p \Theta_p^{\mathrm{T}})^{-1} \Theta_p A \Theta_p^{\mathrm{T}}$, or equivalently, $\mathrm{Ker}(\Theta_p)$ is an invariant subspace of both A and A^{T}. Particularly, when A is symmetric or antisymmetric, the condition of Θ_p-compatibility (2.1) induces the corresponding condition of Θ_p-strong compatibility (5.4).

Proposition 5.5 *If A satisfies the condition of Θ_p-strong compatibility* (5.4)*, then*
(i) $PD = 0$ is equivalent to $(\eta_1, \cdots, \eta_p)^{\mathrm{T}} D = 0$;
(ii) if $D \in \mathcal{D}_{N-p}$, then $PD = 0$ is equivalent to

$$\mathrm{Ker}(D^{\mathrm{T}}) = \mathrm{Span}\{\eta_1, \cdots, \eta_p\}. \tag{5.39}$$

Proof (i) According to Proposition 5.1, the condition of Θ_p-strong compatibility (5.4) implies $q = p$, then by (5.19), $PD = 0$ is just $(\epsilon_1, \cdots, \epsilon_p)(\eta_1, \cdots, \eta_p)^{\mathrm{T}} D = 0$, which is equivalent to $(\eta_1, \cdots, \eta_p)^{\mathrm{T}} D = 0$ by the linear independence of $\epsilon_1, \cdots, \epsilon_p$.
(ii) By (i), we have that $PD = 0$ is equivalent to

$$\mathrm{Span}\{\eta_1, \cdots, \eta_p\} \subseteq \mathrm{Ker}(D^{\mathrm{T}}). \tag{5.40}$$

Since $D \in \mathcal{D}_{N-p}$, we have $\mathrm{rank}(D) = N - p$, which means $\dim(\mathrm{Ker}(D^{\mathrm{T}})) = p$, hence (5.40) becomes (5.39).

Remark 5.3 Condition (5.29) is equivalent to that

$$\begin{cases} A \text{ satisfies the condition of } \Theta_p\text{-compatibility (2.1)}, \\ \mathrm{Ker}(D^{\mathrm{T}}) \text{ is invariant for } A^{\mathrm{T}} \text{ and bi-orthonormal to } \mathrm{Ker}(\Theta_p). \end{cases} \tag{5.41}$$

Remark 5.4 The main results in this paper still hold for a coupled system of wave equations with Neumann boundary controls (see [4, 10]).

Acknowledgements The author would like to thank Professors Tatsien Li and Bopeng Rao for the valuable discussions.

References

[1] Gohberg, I., Lancaster, P. and Rodman, L., Invariant Subspaces of Matrices with Applications, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, Inc., New York, 1986.

[2] Hu, L., Li, T. T. and Rao, B. P., Exact boundary synchronization for a coupled system of 1-D wave equations with coupled boundary conditions of dissipative type, *Commun. Pure Appl. Anal.*, **13**(2), 2014, 881–901.

[3] Huygens. C., Oeuvres Complètes, Vol. 15, Swets & Zeitlinger, Amsterdam, 1967.

[4] Li, T. T., Lu, X. and Rao, B. P., Exact boundary synchronization for a coupled system of wave equations with Neumann boundary controls, *Chin. Ann. Math. Ser. B*, **39**(2), 2018, 233–252.

[5] Li, T. T. and Rao, B. P., Synchronisation exacte d'un système couplé d'équations des ondes par des contrôles frontières de Dirichlet, *C. R. Math. Acad. Sci. Paris*, **350**(15-16), 2012, 767–772.

[6] Li, T. T. and Rao, B. P., Exact synchronization for a coupled system of wave equations with Dirichlet boundary controls, *Chin. Ann. Math. Ser. B*, **34**(1), 2013, 139–160.

[7] Li, T. T. and Rao, B. P., A note on the exact synchronization by groups for a coupled system of wave equations, *Math. Methods Appl. Sci.*, **38**(2), 2015, 241–246.

[8] Li, T. T. and Rao, B. P., On the exactly synchronizable state to a coupled system of wave equations, *Port. Math.*, **72**(2-3), 2015, 83–100.

[9] Li, T. T. and Rao, B. P., Exact synchronization by groups for a coupled system of wave equations with Dirichlet boundary controls, *J. Math. Pures Appl. (9)*, **105**(1), 2016, 86–101.

[10] Li, T. T. and Rao, B. P., Exact boundary controllability for a coupled system of wave equations with Neumann boundary controls, *Chin. Ann. Math. Ser. B*, **38**(2), 2017, 473–488.

[11] Li, T. T., Rao, B. P. and Hu, L., Exact boundary synchronization for a coupled system of 1-D wave equations, *ESAIM Control Optim. Calc. Var.*, **20**(2), 2014, 339–361.

[12] Li, T. T., Rao, B. P. and Wei, Y. M., Generalized exact boundary synchronization for a coupled system of wave equations, *Discrete Contin. Dyn. Syst.*, **34**(7), 2014, 2893–2905.

[13] Lions, J.-L., Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués, Vol. 1, Masson, Paris, 1988.

[14] Pikovsky, A., Rosenblum, M. and Kurths J., Synchronization: A Universal Concept in Nonlinear Sciences, Vol. 12, Cambridge Nonlinear Science Series, Cambridge University Press, Cambridge, 2001.

[15] Wiener, N., Cybernetics, or Control and Communication in the Animal and the Machine, 2nd ed., The M.I.T. Press, Cambridge, Mass.; John Wiley & Sons, Inc., New York-London, 1961.