Chin. Ann. Math. Ser. B
41(4), 2020, 531–546
DOI: 10.1007/s11401-020-0215-8

Chinese Annals of Mathematics, Series B

The Riemann Problem for Chaplygin Gas Flow in a Duct with Discontinuous Cross-Section*

Liancheng DANG[1] Wancheng SHENG[2]

Abstract The fluid flows in a variable cross-section duct are nonconservative because of the source term. Recently, the Riemann problem and the interactions of the elementary waves for the compressible isentropic gas in a variable cross-section duct were studied. In this paper, the Riemann problem for Chaplygin gas flow in a duct with discontinuous cross-section is studied. The elementary waves include rarefaction waves, shock waves, delta waves and stationary waves.

Keywords Duct flow, Chaplygin gas equations, Nonconservative, Riemann problem, Delta wave, Stationary wave
2000 MR Subject Classification 17B40, 17B50

1 Introduction

When the cross-section area $a(x)$ dose not change rapidly, a duct flow of an isothermal fluid in a nozzle can be described as a one dimensional flow:

$$\begin{cases}(a\rho)_t + (a\rho u)_x = 0,\\ (a\rho u)_t + (a\rho u^2 + ap)_x = pa_x,\\ a_t = 0,\end{cases} \tag{1.1}$$

where ρ, u, and p represent the density, the velocity and the pressure of the fluid, respectively. In this paper, the pressure p is given by the state equation

$$p = -\rho^{-1}, \tag{1.2}$$

which was introduced by Chaplygin [4] and Tsien [19] as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics. It is called Chaplygin gas. Brenier [3] studied the one dimensional Riemann problems and obtained the solutions with concentration. In addition, Serre [15] studied the interaction of pressure waves for the 2-D isentropic irrotational Chaplygin gas. He constructively proved the existence of transonic solutions for two cases: Saddle and vortex of 2-D Riemann problem. Guo et al. [6] considered

Manuscript received April 20, 2018. Revised June 27, 2018.
[1]Department of Mathematics, Shanghai University, Shanghai 200444, China.
E-mail:
[2]Corresponding author. Department of Mathematics, Shanghai University, Shanghai 200444, China.
E-mail: mathwcsheng@t.shu.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 11371240, 11771274).

the 2-D Riemann problem for the Chaplygin gas. Lai et al. [10] discussed simple waves for two-dimensional self-similar flow for the Chaplygin gas and found a new type of discontinuity which is a discontinuity supported by a pressure delta function in course of constructing the global solutions, also see [7] for related results. Generally, $a(x)$ is given as a prior, here we view it as a variant which is independent of time (see [12, 14]).

We know that the system (1.1) is not conservative because there is source term which can be seen as nonconservative product (see [5, 11]). The source term plays an important role in numerical approximations for both variable section the nozzle model and the multiphase flow model (see [2, 8–9]). The usual definition of weak solution cannot be applied to the system. A general definition based on the nonconservative product was given in [5].

In 2003, LeFloch et al. [13] solved the Riemann problem of the system (1.1). The Riemann problem of nonisentropic fluid was also studied by Andrianov et al. [1] and Thanh [18]. LeFloch and Thanh divided the (u, ρ) plane by coinciding characteristic curves. In each area, the system (1.1) can be viewed as strictly hyperbolic. LeFloch and Thanh selected a admissible stationary wave relying on the monotone criterion. While Andrianov et al. gave the evolutionary criterion. In 2018, Sheng and Zhang [16] studied the interactions of the elementary waves of isentropic flow in a variable cross-section duct.

In this paper, we study the Riemann problem for Chaplygin gas flows in a nozzle with discontinuous cross-section. In Section 2, characteristic analyses for the system (1.1) with (1.2) are given in the preliminaries. In Section 3, the elementary waves of the system for Chaplygin gas are shown. In Section 4, the Riemann problems of system (1.1) for Chaplygin gas (1.2) are studied.

2 Preliminaries

Denote $U = (u, \rho, a)$. The system (1.1) can be rewritten, when considering a smooth solution, as

$$\partial_t U + A(U)\partial_x U = 0, \tag{2.1}$$

where

$$A = \begin{pmatrix} u & \dfrac{p'(\rho)}{\rho} & 0 \\ \rho & u & \dfrac{\rho u}{a} \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix A has three eigenvalues

$$\lambda_1 = u - c\ , \quad \lambda_2 = 0, \quad \lambda_3 = u + c\ , \tag{2.2}$$

where $c = \sqrt{p'(\rho)} = \frac{1}{\rho}$. The corresponding right eigenvectors are

$$\vec{r}_1 = (-c, \rho, 0)^{\mathrm{t}}, \quad \vec{r}_2 = \Big(-c^2, \rho u, au\Big(\frac{c^2}{u^2} - 1\Big)\Big)^{\mathrm{t}}, \quad \vec{r}_3 = (c, \rho, 0)^{\mathrm{t}}.$$

All the characteristics families are linearly degenerate

$$\nabla\lambda_i(u)\cdot r_i(u)=0,\quad i=1,2,3.$$

The first and the third characteristics may coincide with the second one, so the system is not strictly hyperbolic. More precisely, setting

$$\Gamma_\pm : u=\pm c,$$

we see that

$$\lambda_2=\lambda_1\quad \text{on } \Gamma_+,\quad \lambda_2=\lambda_3\quad \text{on } \Gamma_-.$$

In the (u,ρ)-plane, the curves $\Gamma_\pm$ separate the half-plane $\rho>0$ into three parts. For convenience, we will view them as D_1 (supersonic), D_2 (subsonic) and D_3 (supersonic):

$$D_1=\{(u,\rho)\mid u<-c\,\},\quad D_2=\{(u,\rho)\mid |u|<c\,\},\quad D_3=\{(u,\rho)\mid u>c\,\}.$$

$D_2=D_2^+\bigcup D_2^-$, where

$$D_2^+=\{(u,\rho)\mid 0<u<c\},\quad D_2^-=\{(u,\rho)\mid -c<u\le 0\}.$$

In each of the region, the system is strictly hyperbolic and we have

$$\begin{aligned}\lambda_1<\lambda_3<\lambda_2 &\quad \text{in } D_1,\\ \lambda_1<\lambda_2<\lambda_3 &\quad \text{in } D_2,\\ \lambda_2<\lambda_1<\lambda_3 &\quad \text{in } D_3.\end{aligned}$$

3 The Elementary Waves of System (1.1) for Chaplygin Gas (1.2)

3.1 The rarefaction waves

We look for self-similar solutions $U(\xi)=(u,\rho,a)(\xi),\ \xi=\frac{x}{t}$. The Riemann invariants of each characteristic are

$$\begin{cases}\lambda_1=u-c:\Big\{a,u-\dfrac{1}{\rho}\Big\},\\ \lambda_2=0:\Big\{a\rho u,u^2-\dfrac{1}{\rho^2}\Big\},\\ \lambda_3=u+c:\Big\{a,u+\dfrac{1}{\rho}\Big\}.\end{cases}\tag{3.1}$$

We see that, for the rarefaction waves, the cross-section $a(x)$ remains constant, so the system (1.1) degenerates to the gas dynamic equations

$$\begin{cases}\rho_t+(\rho u)_x=0,\\ (\rho u)_t+(\rho u^2+p)_x=0.\end{cases}\tag{3.2}$$

For a given left state (u_0, ρ_0, a_0), we determine the 1-wave and 3-wave rarefaction curves that can be connected on the right by

$$\begin{cases} \overleftarrow{R}_1(U, U_0): \ u - u_0 = \dfrac{1}{\rho} - \dfrac{1}{\rho_0}, \quad \rho < \rho_0, \\ \overrightarrow{R}_3(U, U_0): \ u - u_0 = \dfrac{1}{\rho_0} - \dfrac{1}{\rho}, \quad \rho > \rho_0. \end{cases} \tag{3.3}$$

3.2 The stationary waves

The Rankine-Huguniot relation associated with the third equation of (1.1) is that

$$-\sigma[a] = 0,$$

where $[a] := a_1 - a_0$ is the jump of the cross-section a. And we derive the conclusion as follows.

(1) $\sigma = 0$: The discontinuity speed vanishes. We assume $[a] \neq 0$ and call the stationary contact discontinuity.

(2) $\sigma \neq 0$: It gives that $[a] = 0$. So the cross-section a remains constant across the non-zero speed discontinuity.

Across the stationary contact discontinuity, the Riemann invariants remain constant. From the second equation of (3.1), the right states (u, ρ, a) satisfy

$$S_0(U; U_0): \begin{cases} a_0\rho_0 u_0 = a\rho u, \\ u_0^2 - \dfrac{1}{\rho_0^2} = u^2 - \dfrac{1}{\rho^2}, \end{cases} \tag{3.4}$$

where (u_0, ρ_0, a_0) is the left state. Then, we get

$$S_0(U; U_0): \begin{cases} \rho^2 = \rho_0^2 \dfrac{\left(\dfrac{a_0}{a}\right)^2 \rho_0^2 u_0^2 - 1}{\rho_0^2 u_0^2 - 1}, \\ u = \dfrac{a_0\rho_0 u_0}{a\rho}. \end{cases} \tag{3.5}$$

As shown in [13], the Riemann problem for (1.1) may admit up to a one-parameter family of solutions. This phenomenon can be avoided by requiring Riemann solutions to satisfy an Admissibility Criterion: Monotone condition on the component a.

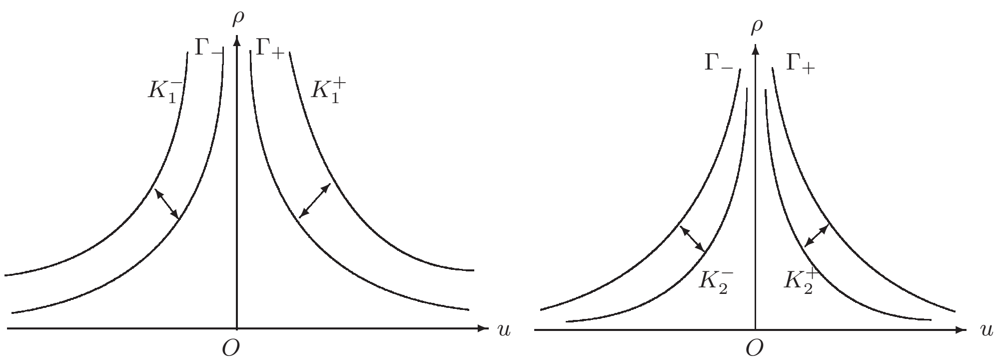

Figure 1 The curves $K_1^{\pm}$ (left) and $K_2^{\pm}$ (right).

From (3.5), we can easily get the following results (see Figure 1).

Lemma 3.1 *There has been a stationary wave $S_0(U;U_0)$, if and only if it satisfies that*

(1) $\rho_0^2u_0^2 > \left(\frac{a}{a_0}\right)^2$ *or* $\rho_0^2u_0^2 < 1$, *when* $a > a_0$;

(2) $\rho_0^2u_0^2 < \left(\frac{a}{a_0}\right)^2$ *or* $\rho_0^2u_0^2 > 1$, *when* $a < a_0$.

See Figure 1, *where* $K_{1,2}^{\pm}$ *denote the curve* $\rho^2u^2 = \left(\frac{a}{a_0}\right)^2$.

Lemma 3.2 *Any stationary wave has to remain in the closure of only one domain* $D_i, i = 1,2,3$.

Proof From the second equation of (3.4), we get

$$\frac{\rho^2u^2-1}{\rho^2} = \frac{\rho_0^2u_0^2-1}{\rho_0^2}. \tag{3.6}$$

Thus, $\rho_0^2u_0^2-1 \gtrless 0$ if and only if $\rho^2u^2-1 \gtrless 0$. From the first equation of (3.4), we know that the sign of u is equivalent to that of u_0.

From the lemma, we get some properties of the stationary curve in (u,ρ)-plane in the following.

Lemma 3.3 *The stationary wave can be viewed as a parameter curve $S_0(U(a);U_0)$ depending only on a with respect to (u,ρ)-plane, and it has the following properties*:

(1) $S_0(U(a);U_0)$ *is strictly increasing* (*decreasing*) *in* u *if* $u<0$ (>0).

(2) $S_0(U(a);U_0)$ *is concave with respect to* u *if* $|u| \le \frac{\sqrt{3}}{3}c$, *or* $|u| > \frac{\sqrt{3}}{3}c$.

(3) *The increasing* (*decreasing*) *velocity and density with respect to* U_0 *lead to the increasing* (*decreasing*) *velocity and density with respect to* U.

(4) $S_0(U(a),U_0)$ *has asymptote* $\rho = 0$ *no matter which region* U_0 *lies in*; *it also has asymptotes* $u = \pm\frac{1}{\rho_0}\sqrt{\rho_0^2u_0^2-1}$ *when* U_0 *lies in* D_3 *or* D_1.

Proof Differentiating the two equations of (3.4), we get

$$\begin{cases} \dfrac{\mathrm{d}a}{a} + \dfrac{\mathrm{d}\rho}{\rho} + \dfrac{\mathrm{d}u}{u} = 0, \\ u\mathrm{d}u + \dfrac{\mathrm{d}\rho}{\rho^3} = 0. \end{cases} \tag{3.7}$$

Then we get

$$\frac{\mathrm{d}\rho}{\mathrm{d}u} = -u\rho^3, \quad \frac{\mathrm{d}u}{\mathrm{d}a} = \frac{u}{a(u^2\rho^2-1)}, \quad \frac{\mathrm{d}\rho}{\mathrm{d}a} = -\frac{u^2\rho^3}{a(u^2\rho^2-1)}.$$

It follows the first statement. For the proof of the second statement, we calculate

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}u^2} = \rho^3(3u^2\rho^2-1).$$

From (3.4), we get

$$u = \frac{a_0\rho_0}{a_1\rho}u_0, \quad \rho = \frac{a_0u_0}{a_1u}\rho_0.$$

Taking derivatives with respect to u_0, ρ_0 respectively, we get

$$\frac{\partial u}{\partial u_0} = \frac{a_0\rho_0}{a_1\rho} > 0, \quad \frac{\partial \rho}{\partial \rho_0} = \frac{a_0 u_0}{a_1 u} > 0.$$

Thus the third statement is proved.

The last result follows from (3.4) directly.

From Lemmas 3.2–3.3, $S_0(U(a);U_0)$ are in the same domain with U_0 (see Figure 2).

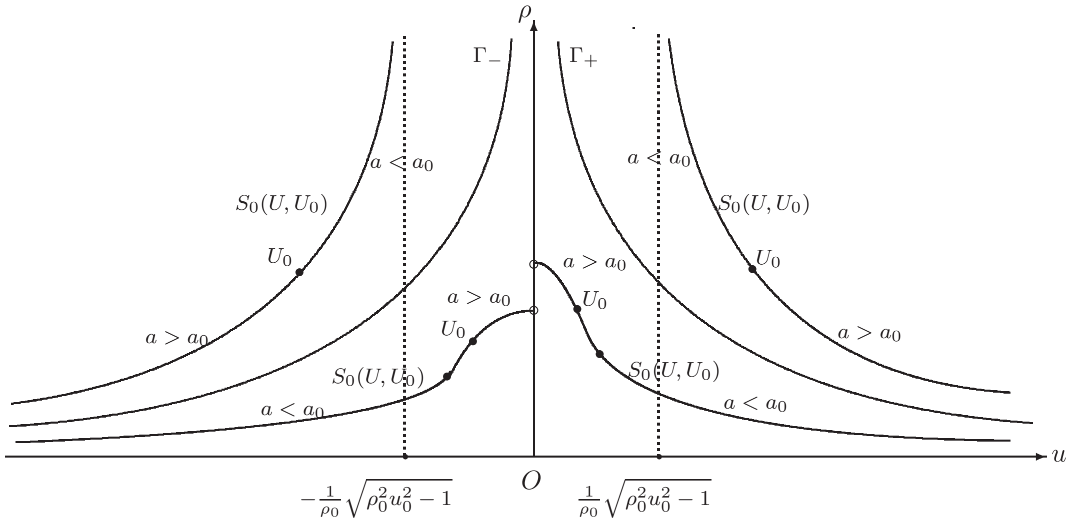

Figure 2 $S_0(U(a), U_0)$ in (u, ρ)-plane.

3.3 The shock waves

For the non-zero speed discontinuity, the cross-section a remains constant. So the left state U_0 and right state U are connected by the Rankine-Hugoniot relations corresponding to (3.2),

$$\begin{cases} -\sigma[\rho] + [\rho u] = 0, \\ -\sigma[\rho u] + [\rho u^2 + p(\rho)] = 0, \end{cases} \tag{3.8}$$

which is equivalent to

$$\sigma_i(U, U_0) = u_0 \mp \frac{1}{\rho_0} = u \mp \frac{1}{\rho}, \quad i = 1, 3.$$

The 1-families and 3-families of discontinuities with non-zero speed connecting a given left to the right are constrained by the Hugoniot set

$$(u - u_0)^2 = \Big(\frac{1}{\rho_0} - \frac{1}{\rho}\Big)^2. \tag{3.9}$$

Shock waves should satisfy the Lax shock conditions (see [11])

$$\lambda_i(U) < \sigma_i(U, U_0) < \lambda_i(U_0), \quad i = 1, 3. \tag{3.10}$$

Using the Lax shock conditions, we get the 1- and 3-shock waves $\overleftarrow{S}_1(U,U_0)$ and $\overrightarrow{S}_3(U,U_0)$ consisting of all right-hand states U by

$$\begin{cases} \overleftarrow{S}_1(U,U_0): u-u_0=\dfrac{1}{\rho}-\dfrac{1}{\rho_0}, & \rho>\rho_0, \\ \overrightarrow{S}_3(U,U_0): u-u_0=\dfrac{1}{\rho_0}-\dfrac{1}{\rho}, & \rho<\rho_0. \end{cases} \tag{3.11}$$

The 1- and 3-shock wave speeds $\sigma_i(U,U_0)$ $(i=1,3)$ may change their signs along the shock curves in the (u,ρ)-plane, more precisely,

$$\sigma_1(U,U_0)\ \begin{cases} <0, & U_0\in D_1, \\ >0, & U_0\in D_2\cup D_3, \end{cases} \qquad \sigma_3(U,U_0)\ \begin{cases} <0, & U_0\in D_1\cup D_2, \\ >0, & U_0\in D_3. \end{cases} \tag{3.12}$$

Let us define the backward and forward wave curves

$$W_1(\rho;U_0)=\begin{cases} \overleftarrow{R_1}(\rho;U_0), & \rho<\rho_0, \\ \overleftarrow{S_1}(\rho;U_0), & \rho>\rho_0, \end{cases} \qquad W_3(\rho;U_0)=\begin{cases} \overrightarrow{R_3}(\rho;U_0), & \rho>\rho_0, \\ \overrightarrow{S_3}(\rho;U_0), & \rho<\rho_0, \end{cases}$$

and stationary wave

$$W_2(\rho;U_0)=S_0(\rho;U_0).$$

We conclude that the wave curve $W_1(\rho;U_0)$ is strictly decreasing and convex in the (u,ρ)-plane, while the wave curve $W_3(\rho;U_0)$ is strictly increasing and convex.

3.4 The delta waves (δ-waves)

By the results in [6], from (3.3) and (3.11), we conclude that the rarefaction waves and the shock waves are coincident in the phase plane, which correspond to contact discontinuities of the first and the third families. Namely, for a given left state (u_l,ρ_l), the contact discontinuity curves, which are the sets of states that can be connected on the right by a 1-contact discontinuity or a 3-contact discontinuity, are as follows:

$$W_{1,3}: \xi=u\mp\frac{1}{\rho}=u_l\mp\frac{1}{\rho_l}. \tag{3.13}$$

In the phase plane $(\rho>0, u\in R)$, through the point (u_l,ρ_l), we draw curves $W_{1,3}$ and $S_\delta: u+\frac{1}{\rho}=u_l-\frac{1}{\rho_l}$ (see Figure 3). $W_{1,3}$ and S_δ divide the phase plane into five parts I, Ⅱ, Ⅲ, Ⅳ and V.

For a given right state (u_l,ρ_l), according to Figure 3, we can construct Riemann solutions of (3.2) with (1.2) locally. When $(u_r,\rho_r)\in \mathrm{I}\cup \mathrm{II}\cup \mathrm{III}\cup \mathrm{IV}$, the Riemann solution contains a 1-contact discontinuity, a 3-contact discontinuity and a nonvacuum intermediate constant state (u_*,ρ_*), where

$$u_*=\frac{1}{2}\Big(u_r+\frac{1}{\rho_r}\Big)+\frac{1}{2}\Big(u_l-\frac{1}{\rho_l}\Big), \quad \frac{1}{\rho_*}=\frac{1}{2}\Big(u_r+\frac{1}{\rho_r}\Big)-\frac{1}{2}\Big(u_l-\frac{1}{\rho_l}\Big). \tag{3.14}$$

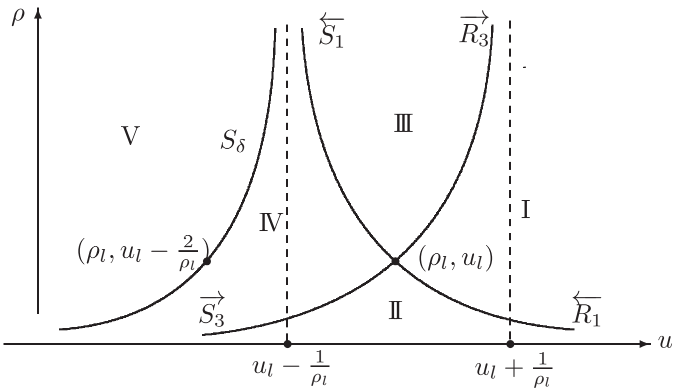

Figure 3 Wave curves for a given left state (u_l, ρ_l) in (u, ρ)-plane.

When $(u_r, \rho_r) \in$ V, a δ-wave, which satisfies the δ-entropy condition

$$u_r + \frac{1}{\rho_r} < \sigma < u_l - \frac{1}{\rho_l}, \tag{3.15}$$

appears in the solution (see [6, 17]),

$$S_\delta : (\rho, u)(x,t) = \begin{cases} (\rho_l, u_l), & x < \sigma t, \\ (\omega(t)\delta(x - \sigma t), \sigma), & x = \sigma t, \\ (\rho_r, u_r), & x > \sigma t, \end{cases} \tag{3.16}$$

where $\omega(t)$ and σ are weight and velocity of delta wave respectively. When $\rho_r \neq \rho_l$,

$$\begin{cases} \omega(t) = \sqrt{\rho_r \rho_l \Big((u_r - u_l)^2 - \Big(\dfrac{1}{\rho_r} - \dfrac{1}{\rho_l} \Big)^2 \Big)} \;\; t, \quad \sigma = \dfrac{\rho_r u_r - \rho_l u_l + \omega'(t)}{\rho_r - \rho_l}, \\ x(t) = \sigma t. \end{cases} \tag{3.17}$$

When $\rho_r = \rho_l$,

$$\begin{cases} \omega(t) = (\rho_l u_l - \rho_r u_r) t, \quad \sigma = \dfrac{1}{2}(u_r + u_l), \\ x(t) = \sigma t = \Big(\dfrac{1}{2}(u_r + u_l) \Big) t. \end{cases} \tag{3.18}$$

Similarly, for a given right state (u_r, ρ_r), in the phase plane $(\rho > 0, u \in R)$, we can also draw curves $W_{1,3}$ and $S_\delta : u - \frac{1}{\rho} = u_r + \frac{1}{\rho_r}$ through the point (u_r, ρ_r) (see Figure 4).

Therefore, the elementary waves of system (1.1) consist of rarefaction waves $\overleftarrow{R_1}(\overrightarrow{R_3})$, shock waves $\overleftarrow{S_1}(\overrightarrow{S_3})$, stationary wave (S_0) and δ-shock wave S_δ.

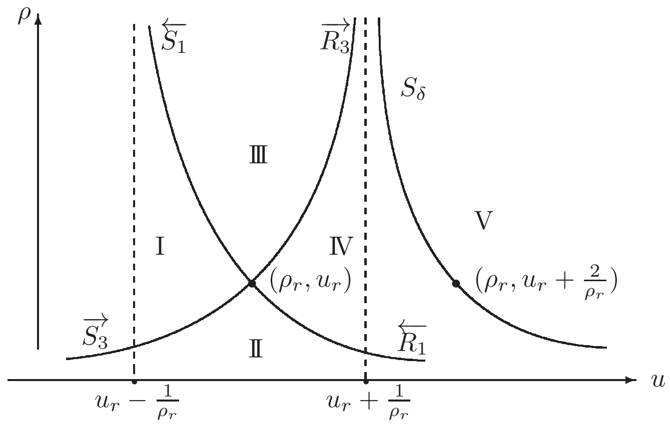

Figure 4 Wave curves for a given right state (u_r, ρ_r) in (u, ρ)-plane.

4 The Riemann Problem (1.1) with (1.2)

In this section we establish the global existence and uniqueness of the Riemann problem for (1.1)–(1.2) with Riemann data

$$(u, \rho, a)|_{t=0} = \begin{cases} U_- = (u_-, \rho_-, a_l), & x < 0, \\ U_+ = (u_+, \rho_+, a_r), & x > 0. \end{cases} \tag{4.1}$$

Without loss of generality (by changing coordinates $x \mapsto -x, u \mapsto -u$, if necessary), we assume for definitiveness in this section that

$$a_l < a_r. \tag{4.2}$$

To construct Riemann solutions of (1.1)–(1.2), we project all the wave curves on the (u, ρ)-plane. Moreover, we will use the following notations:

(i) $W_k(U_l, U_r)$ $(S_k(U_l, U_r), R_k(U_l, U_r))$ denotes the kth-wave (kth-shock, kth-rarefaction wave, respectively) connecting the left-hand state U_l to the right-hand state U_r.

(ii) $S_\delta(U_l, U_r)$ denotes the δ-wave connecting the left-hand state U_l to the right-hand state U_r.

(iii) $W_k(U_l, U_m) \oplus W_n(U_m, U_r)$ indicates that there is a kth-wave from the left-hand state U_l to the middle state U_m, followed by an nth-wave from the middle state U_m to the right-hand state U_r.

(iv) $S_0(U_r, U_l)$ denotes the stationary wave, of which U_l (U_r) is on the left (right) side.

There are nine cases of the Riemann problem, which are $U_- \in D_1$, $U_+ \in D_{1,2,3}$; $U_- \in D_2$, $U_+ \in D_{1,2,3}$; $U_- \in D_3$, $U_+ \in D_{1,2,3}$. Next we discuss the Riemann problems case by case.

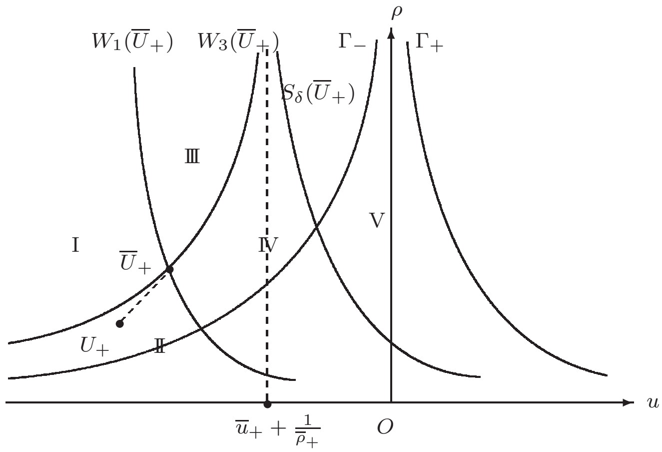

Figure 5 $(u_+, \rho_+) \in D_1, (u_-, \rho_-) \in D_1 \cup D_2$.

Case 1 $(u_+, \rho_+) \in D_1$ and $(u_-, \rho_-) \in D_1 \cup D_2$.

For a given right state $(u_+, \rho_+) \in D_1$, from (3.13), we know that the wave speeds of both W_1 and W_3 through the point (u_+, ρ_+) are negative. Besides, if there is a δ-wave $S_\delta(U_l, U_r)$, where $U_r \in D_1$, $U_l \in D_{1,2}$, from the δ-entropy condition (3.15), we have $\sigma_\delta < 0$ obviously (σ_δ is from (3.17) or (3.18)). Therefore, there must be a stationary wave $S_0(U_+, \overline{U}_+)$ connecting U_+ to a state $\overline{U}_+$ firstly.

Thus, the solutions to the Riemann problem (1.1)–(1.2) can be constructed as follows (see Figure 5):

(1) When $U_- \in \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}$, the solution is

$$W_1(U_-, U_*) \oplus W_3(U_*, \overline{U}_+) \oplus S_0(\overline{U}_+, U_+);$$

(2) when $U_- \in \mathrm{V} \backslash D_3$, the solution is

$$S_\delta(U_-, \overline{U}_+) \oplus S_0(\overline{U}_+, U_+).$$

Case 2 $(u_-, \rho_-) \in D_1$ and $(u_+, \rho_+) \in D_2 \cup D_3$.

From (3.13), we know that the wave speed of W_1 through the point (u_-, ρ_-) are negative. While, the wave speed of W_3 through the point (u_+, ρ_+) are positive. Noticing Lemma 3.2, we can construct $W_1(U_-, U_*)$ connecting U_- to a state $U_* \in D_2$ firstly. Besides, from Figure 3, we know that there is no delta wave in this case.

We give the following notation: $W_1^{\pm}(U_-)$ is the part of the $W_1(U_-)$ for $u \gtrless 0$ in the region $D_2^{\pm}$. $S_0^{\pm}(U_-)$ are made of the right states reached by $S_0(U, U_0)$, where $U_0 \in W_1^{\pm}(U_-)$. $S_0 = S_0^+ \bigcup S_0^-$.

Then the curve S_0 has some properties shown in the following lemma.

Lemma 4.1 *The curve S_0 has the properties*:

(1) S_0^+ *is monotone decreasing, while S_0^- has an extreme point, before which the curve is increasing and after which decreasing*;

(2) *S_0 has two asymptotes $\rho = 0$ and $u = 0$;*

(3) *S_0^+ lies below the curve $W_1^+(U_-)$;*

(4) *$W_3(U_+)$ intersects S_0 at only one point.*

Proof (1) Assume that $(u_0, \rho_0) \in W_1(U_-)$, (u, ρ) is the right state reached by $S_0(U, U_0)$. Then, we have

$$\begin{cases} u_0 - \dfrac{1}{\rho_0} = u_- - \dfrac{1}{\rho_-}, \\ a_r \rho u = a_l \rho_0 u_0, \\ u^2 - \dfrac{1}{\rho^2} = u_0^2 - \dfrac{1}{\rho_0^2}. \end{cases} \tag{4.3}$$

We view u_0, ρ, u as the functions of ρ_0. Now we differentiate the two sides of the above three equations with respect to ρ_0. Then we get

$$\begin{cases} u_0' = -\dfrac{1}{\rho_0^2}, \\ a_r u \rho' + a_r \rho u' = a_l u_0 + a_l \rho_0 u_0', \\ u u' + \dfrac{\rho'}{\rho^3} = u_0 u_0' + \dfrac{1}{\rho_0^3}. \end{cases} \tag{4.4}$$

Then we get

$$\frac{\mathrm{d}u}{\mathrm{d}\rho} = -\frac{u}{\rho^3} \frac{\rho_0 + \rho^2 u_0}{u_0 + \rho_0 u^2} \tag{4.5}$$

Therefore we have $\frac{\mathrm{d}u}{\mathrm{d}\rho} < 0$, when $(u_0, \rho_0) \in D_2^+$. It follows that S_0^+ is monotone decreasing.

When $(u_0, \rho_0) \in D_2^-$, from Lemma 3.3, $u_0 + \rho_0 u^2 < u_0 + \rho_0 u_0^2 = u_0(1 + \rho_0 u_0) < 0$. Next, we determine the sign of the factor $\rho^2 u_0 + \rho_0$.

From (3.5), we get

$$\rho^2 u_0 + \rho_0 = \frac{\rho_0}{\rho_0^2 u_0^2 - 1} \Big(\frac{a_l^2}{a_r^2} \rho_0^3 u_0^3 + \rho_0^2 u_0^2 - \rho_0 u_0 - 1 \Big). \tag{4.6}$$

Setting $t = \rho_0 u_0 \in (-1, 0)$, $G(t) = \frac{a_l^2}{a_r^2} t^3 + t^2 - t - 1$, then we have $G'(t) = 3\frac{a_l^2}{a_r^2} t^2 + 2t - 1$, $G'(-1) = 3\frac{a_l^2}{a_r^2} - 3 < 0$, $G'(0) = -1 < 0$, so $G'(t) < 0,\ t \in (-1, 0)$. While, $G(-1) = 1 - \frac{a_l^2}{a_r^2} > 0,\ G(0) = -1$. Thus, there is only one solution to the equation $G(t) = 0$, which means that there is an extreme point at the curve S_0^-.

(2) When $u_0 \rho_0 \to -1$, i.e., $u_0^2 - \frac{1}{\rho_0^2} \to 0$, from (3.4), we know that $u^2 - \frac{1}{\rho^2} \to 0$. If ρ is finite, then u is finite, which contradicts $a_r \rho u = a_l \rho_0 u_0$. So $u \to 0^-, \rho \to +\infty$ and $\rho u \to -\frac{a_l}{a_r}$; when $\rho_0 \to 0^+, u_0 \to +\infty$, from $u_0 - \dfrac{1}{\rho_0} = u_- - \dfrac{1}{\rho_-}$, we know that $\rho_0 u_0 \to 1$; from $a_r \rho u = a_l \rho_0 u_0$, we get $\rho u \to \frac{a_l}{a_r}$; from $u^2 - \frac{1}{\rho^2} = u_0^2 - \frac{1}{\rho_0^2}$, it holds that $u^2 - \frac{1}{\rho^2} \to 0$. So, $u \to +\infty, \rho \to 0^+$.

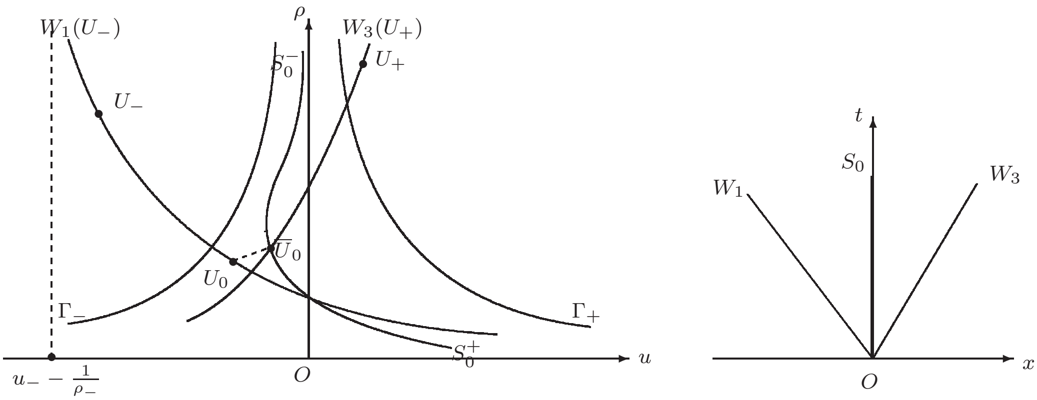

Figure 6 $(u_-,\rho_-)\in D_1$, $(u_+,\rho_+)\in D_2\cup D_3$.

(3) The slope of the curve $W_1^+(U_-): u-\dfrac{1}{\rho}=u_- -\dfrac{1}{\rho_-}$ at any point (u,ρ) is

$$\frac{\mathrm{d}\rho}{\mathrm{d}u}=-\rho^2, \tag{4.7}$$

and the slope of the curve S_0^+ at any point (u,ρ) is

$$\frac{\mathrm{d}\rho}{\mathrm{d}u}=-\rho^2\frac{\rho}{u}\frac{u_0+\rho_0u^2}{\rho_0+\rho^2u_0}. \tag{4.8}$$

The value of (4.7) is larger than that of (4.8), which is sufficient from $\rho>\rho_0, u_0>u$. It follows the result.

(4) Providing that $W_3(U_+)$ intersects S_0 at two points (ρ_i,u_i) $(i=1,2)$, where (ρ_i,u_i) $(i=1,2)$ are the states reached by $S_0(U,U_0)$ using $(\rho_{i0},u_{i0})\in W_1(U_-)$ $(i=1,2)$, respectively. It holds that

$$u_{10}-\frac{1}{\rho_{10}}=u_{20}-\frac{1}{\rho_{20}},\quad u_1+\frac{1}{\rho_1}=u_2+\frac{1}{\rho_2}. \tag{4.9}$$

Thus

$$\frac{\rho_{10}u_{10}-1}{\rho_{10}}=\frac{\rho_{20}u_{20}-1}{\rho_{20}}=\frac{\rho_{10}u_{10}-\rho_{20}u_{20}}{\rho_{10}-\rho_{20}}, \tag{4.10}$$

and

$$\frac{\rho_1u_1+1}{\rho_1}=\frac{\rho_2u_2+1}{\rho_2}=\frac{\rho_1u_1-\rho_2u_2}{\rho_1-\rho_2}=\frac{a_l}{a_r}\frac{\rho_{10}u_{10}-\rho_{20}u_{20}}{\rho_1-\rho_2}. \tag{4.11}$$

From (4.10)–(4.11), we get

$$\frac{\rho_1u_1+1}{\rho_1}=\frac{a_l}{a_r}\frac{\rho_{10}-\rho_{20}}{\rho_1-\rho_2}\frac{\rho_{10}u_{10}-1}{\rho_{10}}. \tag{4.12}$$

The left side of the above equation is positive, and the third factor of the right side is negative, so the second factor must be negative, but this contradicts $\rho_{\rho_0}>0$ (see Lemma 4.1(1)).

Therefore, from Lemma 4.1, the solutions of the Riemann problem (1.1)–(1.2) can be constructed as follows (see Figure 6):

$$W_1(U_-, U_0) \oplus S_0(U_0, \overline{U}_0) \oplus W_3(\overline{U}_0, U_+).$$

Case 3 $(u_-, \rho_-) \in D_2$ and $(u_+, \rho_+) \in D_2 \cup D_3$.

The solutions to the Riemann problem (1.1)–(1.2) can be constructed as follows (see Figure 7):

$$W_1(U_-, U_0) \oplus S_0(U_0, \overline{U}_0) \oplus W_3(\overline{U}_0, U_+).$$

The analysis is similar to Case 2.

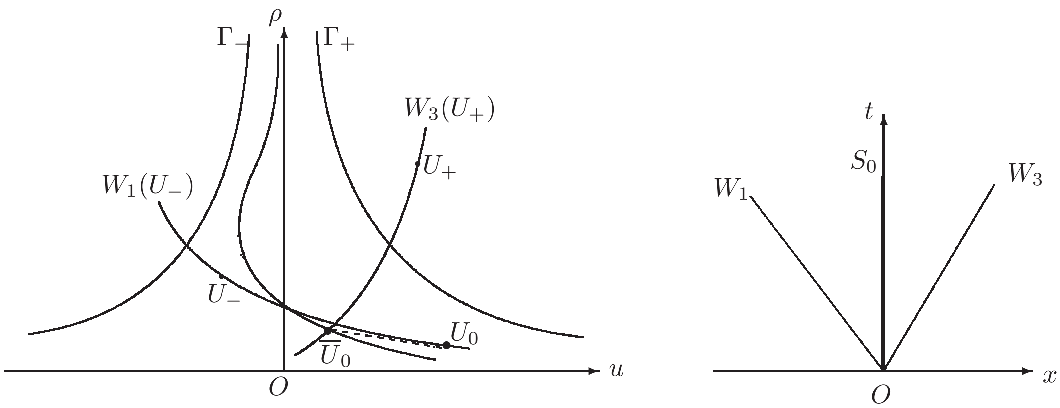

Figure 7 $(u_-, \rho_-) \in D_2$, $(u_+, \rho_+) \in D_2 \cup D_3$.

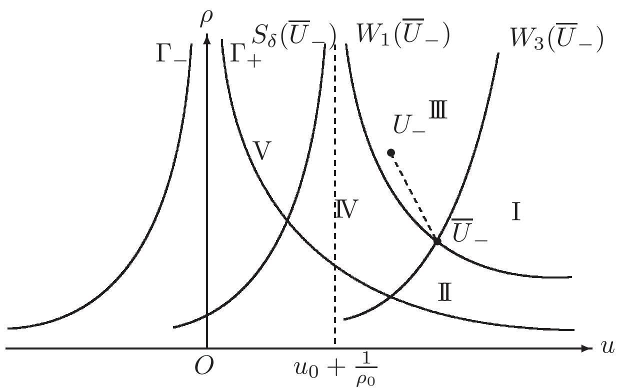

Figure 8 $(u_-, \rho_-) \in D_3$ and $\rho_- u_- > \dfrac{a_r}{a_l}$, $(u_+, \rho_+) \in D_2 \cup D_3$.

Case 4 $(u_-, \rho_-) \in D_3$, $\rho_- u_- > \dfrac{a_r}{a_l}$ and $(u_+, \rho_+) \in D_2 \cup D_3$.

For a given left state $(u_-, \rho_-) \in D_3$, from (3.13), we know that the wave speeds of both W_1 and W_3 through the point (u_-, ρ_-) are positive. Besides, if there is a δ-wave $S_\delta(U_l, U_r)$, where $U_l \in D_3$, $U_r \in D_{2,3}$, from the δ-entropy condition (3.15), we have $\sigma_\delta > 0$ obviously (σ_δ is from (3.17) or (3.18)). Therefore, there must be a stationary wave $S_0(\overline{U}_-, U_-)$ connecting U_- to a state $\overline{U}_-$ firstly. And also from Lemma 3.1, we know that (u_-, ρ_-) must satisfy $\rho_- u_- > \dfrac{a_r}{a_l}$.

Thus, the solutions to the Riemann problem (1.1)–(1.2) can be constructed as follows (see Figure 8):

(1) When $U_+ \in \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}$, the solution is

$$S_0(U_-, \overline{U}_-) \oplus W_1(\overline{U}_-, U_*) \oplus W_3(U_*, U_+);$$

(2) when $U_+ \in \mathrm{V} \backslash D_1$, the solution is

$$S_0(U_-, \overline{U}_-) \oplus S_\delta(\overline{U}_-, U_+).$$

Case 5 $(u_-, \rho_-) \in D_3$ and $(u_+, \rho_+) \in D_1$.

In this case, from (3.13), we know that the wave speed of W_1 through the point (u_-, ρ_-) is positive, while the wave speed of W_3 through the point (u_+, ρ_+) is negative. In this case, δ-wave is needed.

By simple calculations, from (3.17)–(3.18), we get the sign of δ-wave speed as in the following lemma.

Lemma 4.2 *For a given state* $(u_r, \rho_r) \in D_1$, *when* $(u_l, \rho_l) \in D_3$, *there is a delta wave* $S_\delta(U_l, U_r)$, *and* $\sigma_\delta(U_l, U_r) \gtreqless 0$ (σ_δ *is denoted by* (3.17) *or* (3.18)) *if and only if* (u_l, ρ_l) *satisfies that when* $\rho_l \neq \rho_r$,

$$\rho_l^2 u_l^2 \gtreqless 1 + \frac{\rho_l}{\rho_r}(\rho_r^2 u_r^2 - 1); \tag{4.13}$$

when $\rho_l = \rho_r$,

$$u_l \gtreqless -u_r. \tag{4.14}$$

From Lemma 4.2 and δ-entropy condition (3.15), we know that if there is a $S_\delta(U_-, U_0)$ ($S_\delta(U_1, U_+)$) satisfying $\sigma_\delta < 0$ ($\sigma_\delta > 0$), then it must hold that $(u_0, \rho_0) \in D_1$ ($(u_1, \rho_1) \in D_3$). And noticing Lemma 3.2, we know that in the same case the Riemann solutions cannot contain two delta waves, the speeds of which are negative and positive, respectively. Besides, the Riemann solutions cannot contain delta waves and W_1 or W_3 in the same case because the wave speeds of both W_1 and W_3 through $(u_+, \rho_+) \in D_1$ are negative and through $(u_-, \rho_-) \in D_3$ are positive. Therefore, the Riemann solutions only contain stationary wave and one delta wave. One case is that the delta wave speed is negative, the other is that the delta wave speed is positive.

Subcase 1 $\sigma_\delta < 0$. For a given (u_+, ρ_+), from (3.4) and Lemma 4.2, we have

$$\begin{cases} a_r \rho_+ u_+ = a_l \rho_0 u_0, \\ u_+^2 - \dfrac{1}{\rho_+^2} = u_0^2 - \dfrac{1}{\rho_0^2}, \\ \rho_-^2 u_-^2 < 1 + \dfrac{\rho_-}{\rho_0}(\rho_0^2 u_0^2 - 1). \end{cases} \tag{4.15}$$

Then we get

$$\rho_-^2 u_-^2 < 1 + \frac{\rho_-}{\rho_+}\sqrt{(\rho_+^2 u_+^2 - 1)\Big(\Big(\frac{a_r}{a_l}\Big)^2 \rho_+^2 u_+^2 - 1\Big)}. \tag{4.16}$$

Denoting $J_1(U_+) : \rho^2 u^2 = 1 + \frac{\rho}{\rho_+}\sqrt{(\rho_+^2 u_+^2 - 1)\big(\big(\frac{a_r}{a_l}\big)^2 \rho_+^2 u_+^2 - 1\big)}$, we conclude that the curve $J_1(U_+)$ lies in the right of Γ_+ and is strictly decreasing in the (u, ρ)-plane. Therefore, when $U_+ \in D_1$ and $U_- \in D_3$ lies in the left region of the curve $J_1(U_+)$, the solution is (see Figure 9)

$$S_\delta(U_-, U_0) \oplus S_0(U_0, U_+).$$

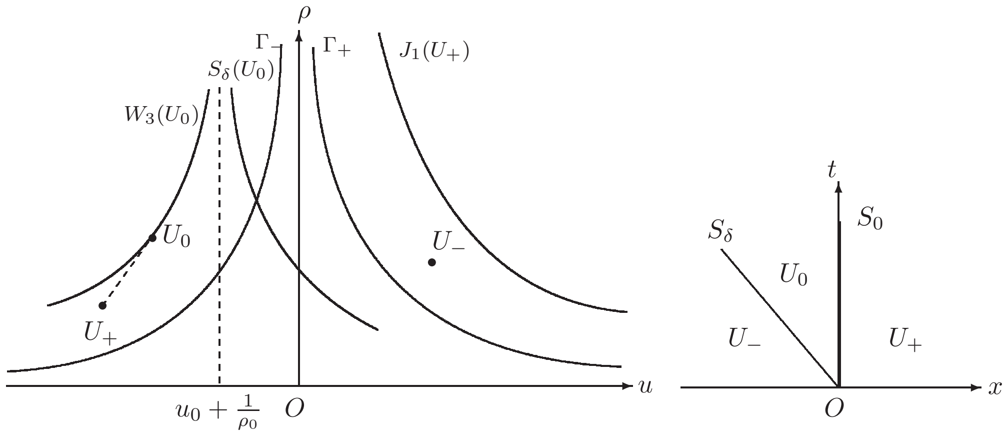

Figure 9 Delta wave speed is negative.

Subcase 2 $\sigma_\delta > 0$. For a given (u_-, ρ_-), from Lemma 3.1, (u_-, ρ_-) must satisfy $\rho_- u_- > \dfrac{a_r}{a_l}$. From (3.4) and Lemma 4.2, we have

$$\begin{cases} a_r \rho_1 u_1 = a_l \rho_- u_-, \\ u_1^2 - \dfrac{1}{\rho_1^2} = u_-^2 - \dfrac{1}{\rho_-^2}, \\ \rho_+^2 u_+^2 < 1 + \dfrac{\rho_+}{\rho_1}(\rho_1^2 u_1^2 - 1). \end{cases} \tag{4.17}$$

Then, we get

$$\rho_+^2 u_+^2 < 1 + \frac{\rho_+}{\rho_-}\sqrt{(\rho_-^2 u_-^2 - 1)\Big[\Big(\frac{a_l}{a_r}\Big)^2 \rho_-^2 u_-^2 - 1\Big]}. \tag{4.18}$$

Denoting $J_2(U_-) : \rho^2 u^2 = 1 + \frac{\rho}{\rho_-}\sqrt{(\rho_-^2 u_-^2 - 1)\big[\big(\frac{a_l}{a_r}\big)^2 \rho_-^2 u_-^2 - 1\big]}$, we conclude that the curve $J_2(U_-)$ lies in the left of Γ_- and is strictly increasing in the (u, ρ)-plane. Therefore, when $U_- \in D_3$ and $\rho_- u_- > \dfrac{a_r}{a_l}$, $U_+ \in D_1$ lies in the right region of the curve $J_2(U_-)$, the solution is (see Figure 10)

$$S_0(U_-, U_1) \oplus S_\delta(U_1, U_+).$$

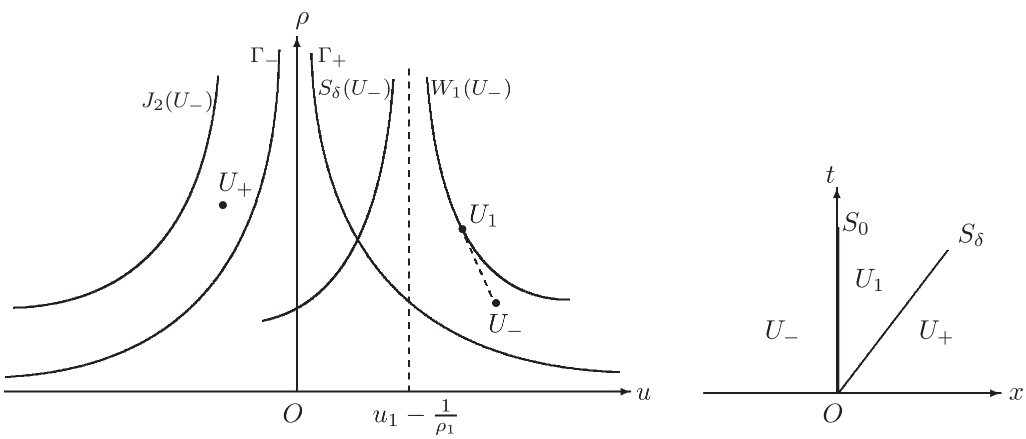

Figure 10 Delta wave speed is positive.

In summary, we get the Riemann solutions for the system (1.1) − (1.2) constructively.

Theorem 4.1 *For the Riemann problem for Chaplygin gas flows to the system* (1.1) − (1.2) *in a duct with discontinuous cross-section, there exists a unique global solution with any given Riemann initial data* (4.1)*, which can be established constructively.*

References

[1] Andrianov, N. and Warnecke, G., On the solution to the Riemann problem for the compressible duct flow, *SIAM J. Appl. Math.*, **64**, 2004, 878–901.

[2] Baer, M. R. and Nunziato, J. W., A two-phase mixture theory for the deflagration-todetonation transition (DDT) in reactive granular materials, *Int. J. Multiphase Flows*, **12**, 1986, 861–889.

[3] Brenier, Y., Solutions with concentration to the Riemann problem for one-dimensional Chaplygin gas equations, *J. Math. Fluid Mech.*, **7**, 2005, 326–331.

[4] Chaplygin, S., On gas jets, *Sci. Mem. Moscow Univ. Math. Phys.*, **21**, 1904, 1–121.

[5] Dal Maso, G., LeFloch, P. G. and Murat, F., Definition and weak stability of nonconservative products, *J. Math. Pures Appl.*, **74**(9), 1995, 483–548.

[6] Guo, L. H., Sheng, W. C. and Zhang, T., The two-dimensional Riemann problem for isentropic Chaplygin gas dynamic system, *Commun. Pure Appl. Anal.*, **9**(2), 2010, 431–458.

[7] Kong, D. X. and Wang, Y. Z., Global existence of smooth solutions to two-dimensional compressible isentropic Euler equations for Chaplygin gases, *Sci. China Math.*, **53**(3), 2010, 719–738.

[8] Kroner, D., LeFloch, P. G. and Thanh, M. D., The minimum entropy principle for fluid flows in a nozzle with discontinuous cross-section, *M2AN Math. Model Numer. Anal.*, **42**, 2008, 425–442.

[9] Kroner, D. and Thanh, M. D., Numerical solutions to compressible flows in a nozzle with variable cross-section, *SIAM J. Numer. Anal.*, **43**, 2005, 796–824.

[10] Lai, G., Sheng, W. C. and Zheng, Y. X., Simple waves and pressure delta waves for a Chaplygin gas in multi-dimensions, *Discrete Contin. Dyn. Syst.*, **31**(2), 2011, 489–523.

[11] Lax, P. D., Shock waves and entropy, Contributions to Functional Analysis, Zarantonello, E. A. (ed.), Academic Press, New York, 1971, 603–634.

[12] LeFloch, P. G., Shock Waves for Nonlinear Hyperbolic Systems in Nonconservative Form, Institute for Mathematics and its Application, Univ. of Minnesota, **593**, preprint, 1989.

[13] LeFloch, P. G. and Thanh, M. D., The Riemann problem for fluid flows in a nozzle with discontinuous cross-section, *Commun. Math. Sci.*, **1**, 2003, 763–797.

[14] Marchesin, D. and Paes-Leme, P. J., A Riemann problem in gas dynamics with bifurcation, *Comput. Math. Appl. Part A*, **12**, 1986, 433–455.

[15] Serre, D., Multidimensional shock interaction for a Chaplygin gas, *Arch. Ration. Mech. Anal.*, **191**, 2009, 539–577.

[16] Sheng, W. C. and Zhang, Q. L., Interaction of the elementary waves of isentropic flow in a variable cross-section duct, *Commun. Math. Sci.,* **16**(6), 2018, 1659–1684.

[17] Sheng, W. C. and Zhang, T., The Riemann problem for transportation equations in gas dynamics, *Mem. Amer. Math. Soc.*, **137**(654), 1999, viii+77pp.

[18] Thanh, M. D., The Riemann problem for a nonisentropic fluid in a nozzle with discontinuous cross-sectional area, *SIAM J. Appl. Math.*, **69**, 2009, 1501–1519.

[19] Tsien, H. S., Two dimensional subsonic flow of compressible fluids, *J. Aeronaut. Sci.*, **6**, 1939, 399–407.