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On the Hybrid Power Mean Involving the Two-Term Exponential Sums and Polynomial Character Sums*

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Abstract The main purpose of this paper is using the analytic method and the properties of trigonometric sums and character sums to study the computational problem of one kind hybrid power mean involving two-term exponential sums and polynomial character sums. Then the authors give some interesting calculating formulae for them.

Keywords The two-term exponential sums, Kloosterman sums, Dirichlet character of polynomials, Hybrid power mean, Identity
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1 Introduction

Let q, k and h be three integers with $q \geq 3$ and $k \neq h$. For any integers m and n, the two-term exponential sum $C(m, n, k, h; q)$ is defined by

$$C(m, n, k, h; q) = \sum_{a=1}^{q} \mathrm{e}\Big(\frac{ma^k + na^h}{q}\Big),$$

where $\mathrm{e}(y) = \mathrm{e}^{2\pi \mathrm{i} y}$.

About the properties of $C(m, n, k, h; q)$, some authors had studied it, and obtained many interesting results. For example, Gauss's classical work (see [1]) gave an exact computational formula for $C(1, 0, 2, h; q)$. Han Di [2] studied the asymptotic properties of the hybrid mean value involving the two-term exponential sums and polynomial character sums, and proved the following asymptotic formula:

$$\sum_{m=1}^{p-1} \Big|\sum_{a=1}^{p-1} \mathrm{e}\Big(\frac{ma^k + na}{p}\Big)\Big|^2 \cdot \Big|\sum_{a=1}^{p-1} \chi(ma + \overline{a})\Big|^2 = \begin{cases} 2p^3 + O(|k|p^2), & \text{if } 2 \mid k, \\ 2p^3 + O(|k|p^{\frac{5}{2}}), & \text{if } 2 \nmid k, \end{cases}$$

where p is an odd prime, χ denotes any non-principal even Dirichlet character mod p, and $\overline{a}$ denotes the multiplicative inverse of a mod p. That is, $a\overline{a} \equiv 1 \bmod p$.

Taking $k = -1$ in this theorem, one can deduce the asymptotic formula

$$\sum_{m=1}^{p-1} \Big|\sum_{a=1}^{p-1} \mathrm{e}\Big(\frac{ma + \overline{a}}{p}\Big)\Big|^2 \cdot \Big|\sum_{b=1}^{p-1} \chi(mb + \overline{b})\Big|^2 = 2p^3 + O(p^2).$$

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Recently, Du Xiaoying [3] studied a similar problem, and proved the following conclusion.

Let $p > 3$ be a prime with $(3, p-1) = 1$. Then for any non-principal even character χ mod p, one has the identity

$$\sum_{m=0}^{p-1}\Big|\sum_{a=1}^{p-1}\chi(ma^3+a)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\mathrm{e}\Big(\frac{mb^3+b}{p}\Big)\Big|^2$$
$$=2p(p^2-p-1)-p\Big(2+\Big(\frac{3}{p}\Big)\Big)\sum_{u=1}^{p-1}\overline{\chi}(u)\sum_{a=1}^{p-1}\Big(\frac{(a-1)(a^3-u^2)}{p}\Big),$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre symbol mod p.

From this formula Du Xiaoying [3] deduced the following asymptotic formula:

$$\sum_{m=0}^{p-1}\Big|\sum_{a=1}^{p-1}\chi(ma^3+a)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\mathrm{e}\Big(\frac{mb^3+b}{p}\Big)\Big|^2=2p^3+O(p^{\frac{5}{2}}). \tag{1.1}$$

Some other works related to the two-term exponential sums, Kloosterman sums and polynomial character sums can also be found in [4–13].

Now for any positive integer k, we consider the following hybrid power mean:

$$\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma^k+na}{p}\Big)\Big|^2\cdot\Big|\sum_{a=1}^{p-1}\chi(ma^3+a)\Big|^2. \tag{1.2}$$

We want to know whether there exists an exact computational formula for (1.2).

About this contents, it seems that none had obtained any conclusion, at least we have not seen such a result right now. But the problem is interesting, because it can reveal the profound value distribution properties of the two-term exponential sums and polynomial character sums.

The main purpose of this paper is using the analytic method and the properties of the trigonometric sums and character sums to study this problem, and prove some interesting results.

To complete the proofs of our theorems, we need the following five simple lemmas. Hereinafter, we shall use many properties of the classical Gauss sums, all of them can be found in [1], so they will not be repeated here.

Lemma 1.1 *Let p be an odd prime. Then for any integer n with $(n, p) = 1$, we have the identity*

$$\sum_{a=0}^{p-1}\Big(\frac{a^2+n}{p}\Big)=-1.$$

Proof Note that $\left(\frac{*}{p}\right)=\chi_2=\overline{\chi}_2$. From the properties of Gauss sums we have

$$\sum_{a=0}^{p-1}\Big(\frac{a^2+n}{p}\Big)=\frac{1}{\tau(\chi_2)}\sum_{a=0}^{p-1}\sum_{b=1}^{p-1}\chi_2(b)\mathrm{e}\Big(\frac{b(a^2+n)}{p}\Big)$$
$$=\frac{1}{\tau(\chi_2)}\sum_{b=1}^{p-1}\chi_2(b)\mathrm{e}\Big(\frac{bn}{p}\Big)\sum_{a=0}^{p-1}\mathrm{e}\Big(\frac{ba^2}{p}\Big). \tag{1.3}$$

From [8, Theorem 7.5.4] we know that

$$G(n;p)=\sum_{a=0}^{p-1}\mathrm{e}\Big(\frac{na^2}{p}\Big)=\Big(\frac{n}{p}\Big)G(1;p)=\Big(\frac{n}{p}\Big)\sum_{a=1}^{p-1}\chi_2(a)\mathrm{e}\Big(\frac{a}{p}\Big). \tag{1.4}$$

It is clear that $\chi_2^2=\chi_0$, the principal character mod p. Applying (1.3)–(1.4) and noting the trigonometric sums

$$\sum_{a=0}^{p-1}\mathrm{e}\Big(\frac{ma}{p}\Big)=\begin{cases}p, & \text{if } p\mid m,\\ 0, & \text{if } p\nmid m,\end{cases} \tag{1.5}$$

we have the identity

$$\sum_{a=0}^{p-1}\Big(\frac{a^2+n}{p}\Big)=\frac{1}{\tau(\chi_2)}\sum_{b=1}^{p-1}\chi_2(b)\mathrm{e}\Big(\frac{bn}{p}\Big)\chi_2(b)\tau(\chi_2)=\sum_{b=1}^{p-1}\mathrm{e}\Big(\frac{bn}{p}\Big)=-1.$$

This proves Lemma 1.1.

Lemma 1.2 *Let p be an odd prime and n be any integer with $(n,p)=1$. Then for any positive integer k with $(k,p-1)=d$, we have*

$$\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma^k+na}{p}\Big)\Big|^2=p^2-dp-1.$$

Proof In fact for any integer n with $(n,p)=1$, from (1.5) we can deduce that

$$\begin{aligned}&\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma^k+na}{p}\Big)\Big|^2\\ &=\sum_{m=1}^{p}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma^k+na}{p}\Big)\Big|^2-\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{na}{p}\Big)\Big|^2\\ &=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\mathrm{e}\Big(\frac{n(a-b)}{p}\Big)\sum_{m=1}^{p}\mathrm{e}\Big(\frac{m(a^k-b^k)}{p}\Big)-1\\ &=p\sum_{\substack{a=1\\ a^k\equiv b^k \bmod p}}^{p-1}\sum_{b=1}^{p-1}\mathrm{e}\Big(\frac{n(a-b)}{p}\Big)-1\\ &=p\sum_{\substack{a=1\\ a^k\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\mathrm{e}\Big(\frac{nb(a-1)}{p}\Big)-1\\ &=p(p-1)-1-p\sum_{\substack{a=2\\ a^k\equiv 1 \bmod p}}^{p-1}1.\end{aligned} \tag{1.6}$$

Now let $(k,p-1)=d$. So there exist $d-1$ integers a with $2\le a\le p-1$ such that $a^k\equiv 1 \bmod p$. From (1.6) we may immediately deduce the identity

$$\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma^k+na}{p}\Big)\Big|^2=p(p-1)-1-p(d-1)=p^2-dp-1.$$

This proves Lemma 1.2.

Lemma 1.3 *Let p be an odd prime, r be any integer with $(r,p)=1$. Then we have the identity*

$$\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{mra^3+ma}{p}\Big)\Big|^2 = p^2-2p-1-2p\Big(\frac{-r}{p}\Big)-p\Big(\frac{-3}{p}\Big)-p\Big(\frac{-3r}{p}\Big).$$

Proof From (1.5), Lemma 1.1 and noting that $\left(\frac{r}{p}\right)=\left(\frac{\overline{r}}{p}\right)$ we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{mra^3+ma}{p}\Big)\Big|^2\\
&=\sum_{m=0}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{mra^3+ma}{p}\Big)\Big|^2-(p-1)^2\\
&=p\sum_{\substack{a=1\\ ra^3+a\equiv rb^3+b \bmod p}}^{p-1}\sum_{b=1}^{p-1}1-(p-1)^2\\
&=p\sum_{\substack{a=1\\ (a-1)(rb^2(a^2+a+1)+1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}1-(p-1)^2\\
&=p(p-1)+p\sum_{\substack{a=2\\ (a^2+a+1)\equiv -\overline{r}\overline{b}^2 \bmod p}}^{p-1}\sum_{b=1}^{p-1}1-(p-1)^2\\
&=(p-1)-2p-p\Big(\Big(\frac{-r}{p}\Big)+\Big(\frac{-3r}{p}\Big)\Big)+p\sum_{\substack{a=0\\ (a^2+a+1)\equiv -\overline{r}\overline{b}^2 \bmod p}}^{p-1}\sum_{b=1}^{p-1}1\\
&=-p-1-p\Big(\Big(\frac{-r}{p}\Big)+\Big(\frac{-3r}{p}\Big)\Big)+p\sum_{\substack{a=0\\ (2a+1)^2\equiv -3-4\overline{r}\overline{b}^2 \bmod p}}^{p-1}\sum_{b=1}^{p-1}1\\
&=-p-1-p\Big(\Big(\frac{-r}{p}\Big)+\Big(\frac{-3r}{p}\Big)\Big)+p\sum_{\substack{a=0\\ a^2\equiv -3-4\overline{r}\overline{b}^2 \bmod p}}^{p-1}\sum_{b=1}^{p-1}1\\
&=p\sum_{b=1}^{p-1}\Big(1+\Big(\frac{-3-4\overline{r}\overline{b}^2}{p}\Big)\Big)-p-1-p\Big(\Big(\frac{-r}{p}\Big)+\Big(\frac{-3r}{p}\Big)\Big)\\
&=p^2-2p-1+p\Big(\frac{-\overline{r}}{p}\Big)\sum_{b=1}^{p-1}\Big(\frac{3\overline{4}r+b^2}{p}\Big)-p\Big(\Big(\frac{-r}{p}\Big)+\Big(\frac{-3r}{p}\Big)\Big)\\
&=p^2-2p-1-p\Big(\frac{-r}{p}\Big)\Big(1+\Big(\frac{3\overline{4}r}{p}\Big)\Big)-p\Big(\Big(\frac{-r}{p}\Big)+\Big(\frac{-3r}{p}\Big)\Big)\\
&=p^2-2p-1-2p\Big(\frac{-r}{p}\Big)-p\Big(\frac{-3}{p}\Big)-p\Big(\frac{-3r}{p}\Big).
\end{aligned}$$

This proves Lemma 1.3.

Lemma 1.4 *Let p be an odd prime, χ be any non-principal even character* mod p. *Then* *for any integer m with $(m,p)=1$, if $\chi^3 \neq \chi_0$, we have*

$$\Big|\sum_{a=1}^{p-1}\chi(ma^3+a)\Big|^2 = 2p + \Big(\frac{-m}{p}\Big)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\Big(\frac{(b^3a^2-1)(b-1)}{p}\Big);$$

If $\chi^3 = \chi_0$, we have

$$\Big|\sum_{a=1}^{p-1}\chi(ma^3+a)\Big|^2 = p+1 + \Big(\frac{-m}{p}\Big)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\Big(\frac{(b^3a^2-1)(b-1)}{p}\Big).$$

Proof From the properties of Gauss sums we have

$$\begin{aligned}
&\Big|\sum_{a=1}^{p-1}\chi(ma^3+a)\Big|^2\\
&= \Big|\frac{1}{\tau(\overline{\chi})}\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\overline{\chi}(b)\mathrm{e}\Big(\frac{b(ma^2+1)}{p}\Big)\Big|^2\\
&= \frac{1}{p}\Big|\sum_{b=1}^{p-1}\overline{\chi}(b)\mathrm{e}\Big(\frac{b}{p}\Big)\sum_{a=1}^{p-1}\chi(a)\mathrm{e}\Big(\frac{bma^2}{p}\Big)\Big|^2\\
&= \frac{1}{p}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\overline{\chi}(b\overline{d})\mathrm{e}\Big(\frac{b-d}{p}\Big)\sum_{a=1}^{p-1}\sum_{c=1}^{p-1}\chi(a\overline{c})\mathrm{e}\Big(\frac{mba^2-mdc^2}{p}\Big)\\
&= \frac{1}{p}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\overline{\chi}(b)\mathrm{e}\Big(\frac{d(b-1)}{p}\Big)\sum_{a=1}^{p-1}\chi(a)\sum_{c=1}^{p-1}\mathrm{e}\Big(\frac{mdc^2(ba^2-1)}{p}\Big)\\
&= \frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a\overline{b})\sum_{d=1}^{p-1}\mathrm{e}\Big(\frac{d(b-1)}{p}\Big)\sum_{c=1}^{p-1}\mathrm{e}\Big(\frac{mdc^2(ba^2-1)}{p}\Big)\\
&= \frac{1}{p}\sum_{\substack{a=1\\(a^2b-1,p)=1}}^{p-1}\sum_{b=1}^{p-1}\chi(a\overline{b})\sum_{d=1}^{p-1}\mathrm{e}\Big(\frac{d(b-1)}{p}\Big)\Big(\sum_{c=0}^{p-1}\mathrm{e}\Big(\frac{mdc^2(ba^2-1)}{p}\Big)-1\Big)\\
&\quad + \frac{p-1}{p}\sum_{\substack{a=1\\a^2b\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\chi(a\overline{b})\sum_{d=1}^{p-1}\mathrm{e}\Big(\frac{d(b-1)}{p}\Big).
\end{aligned}\tag{1.7}$$

Since $\chi(-1)=1$, it is easy to prove that

$$\begin{aligned}
&\frac{p-1}{p}\sum_{\substack{a=1\\a^2b\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\chi(a\overline{b})\sum_{d=1}^{p-1}\mathrm{e}\Big(\frac{d(b-1)}{p}\Big)\\
&= \frac{(p-1)^2}{p}\sum_{\substack{a=1\\a^2\equiv 1 \bmod p}}^{p-1}\chi(a) - \frac{p-1}{p}\sum_{\substack{a=1\\a^2b\equiv 1 \bmod p}}^{p-1}\sum_{b=2}^{p-1}\chi(a\overline{b})\\
&= 2(p-1) - \frac{p-1}{p}\sum_{a=1}^{p-1}\chi(a^3)\\
&= \begin{cases} 2(p-1), & \text{if } \chi^3 \neq \chi_0,\\[4pt] \dfrac{p^2-1}{p}, & \text{if } \chi^3 = \chi_0. \end{cases}
\end{aligned}\tag{1.8}$$

For any integer n with $(n,p)=1$, from (1.4) and noting that $G^2(1;p)=\left(\frac{-1}{p}\right)p$ we have

$$\begin{aligned}
&\sum_{\substack{a=1\\(a^2b-1,p)=1}}^{p-1}\sum_{b=1}^{p-1}\chi(a\overline{b})\sum_{d=1}^{p-1}\mathrm{e}\Big(\frac{d(b-1)}{p}\Big)\sum_{c=0}^{p-1}\mathrm{e}\Big(\frac{mdc^2(ba^2-1)}{p}\Big)\\
&=G(1;p)\sum_{\substack{a=1\\(a^2b-1,p)=1}}^{p-1}\sum_{b=1}^{p-1}\chi(a\overline{b})\sum_{d=1}^{p-1}\mathrm{e}\Big(\frac{d(b-1)}{p}\Big)\Big(\frac{md(ba^2-1)}{p}\Big)\\
&=G(1;p)\Big(\frac{m}{p}\Big)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a\overline{b})\Big(\frac{ba^2-1}{p}\Big)\sum_{d=1}^{p-1}\Big(\frac{d}{p}\Big)\mathrm{e}\Big(\frac{d(b-1)}{p}\Big)\\
&=G^2(1;p)\Big(\frac{m}{p}\Big)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a\overline{b})\Big(\frac{ba^2-1}{p}\Big)\Big(\frac{b-1}{p}\Big)\\
&=p\Big(\frac{-m}{p}\Big)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\Big(\frac{(b^3a^2-1)(b-1)}{p}\Big). \qquad (1.9)
\end{aligned}$$

$$\begin{aligned}
&\sum_{\substack{a=1\\(a^2b-1,p)=1}}^{p-1}\sum_{b=1}^{p-1}\chi(a\overline{b})\sum_{d=1}^{p-1}\mathrm{e}\Big(\frac{d(b-1)}{p}\Big)\\
&=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a\overline{b})\sum_{d=1}^{p-1}\mathrm{e}\Big(\frac{d(b-1)}{p}\Big)-\sum_{\substack{a=1\\a^2b\equiv1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\chi(a\overline{b})\sum_{d=1}^{p-1}\mathrm{e}\Big(\frac{d(b-1)}{p}\Big)\\
&=(p-1)\sum_{a=1}^{p-1}\chi(a)-\sum_{a=1}^{p-1}\sum_{b=2}^{p-1}\chi(a\overline{b})-(p-1)\sum_{\substack{a=1\\a^2\equiv1 \bmod p}}^{p-1}\chi(a)+\sum_{\substack{a=1\\a^2b\equiv1 \bmod p}}^{p-1}\sum_{b=2}^{p-1}\chi(a\overline{b})\\
&=\begin{cases}-(p+1), & \text{if } \chi^3=\chi_0,\\ -2p, & \text{if } \chi^3\neq\chi_0.\end{cases} \qquad (1.10)
\end{aligned}$$

Combining (1.7)–(1.10) we know that if $\chi^3\neq\chi_0$, then

$$\Big|\sum_{a=1}^{p-1}\chi(ma^3+a)\Big|^2=2p+\Big(\frac{-m}{p}\Big)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\Big(\frac{(b^3a^2-1)(b-1)}{p}\Big). \qquad (1.11)$$

If $\chi^3=\chi_0$, then we have

$$\Big|\sum_{a=1}^{p-1}\chi(ma^3+a)\Big|^2=p+1+\Big(\frac{-m}{p}\Big)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\Big(\frac{(b^3a^2-1)(b-1)}{p}\Big). \qquad (1.12)$$

Now Lemma 1.4 follows immediately from (1.11)–(1.12).

Lemma 1.5 *Let p be an odd prime. Then for any quadratic non-residue r mod p, we have the identities*

$$\sum_{m=1}^{\frac{p-1}{2}}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{rm^2a+\overline{a}}{p}\Big)\Big|^2=\frac{1}{2}p^2-1$$

and

$$\sum_{m=1}^{\frac{p-1}{2}}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{m^2a+\overline{a}}{p}\Big)\Big|^2=\frac{1}{2}(p^2-2p-1).$$

Proof For any integer $1\le m\le p-1$, from the properties of the reduced residue system mod p we have

$$\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{rm^2a+\overline{a}}{p}\Big)=\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{rma+m\overline{a}}{p}\Big). \tag{1.13}$$

From (1.5) and (1.13) we have

$$\begin{aligned}\sum_{m=1}^{\frac{p-1}{2}}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{rm^2a+\overline{a}}{p}\Big)\Big|^2&=\sum_{m=1}^{\frac{p-1}{2}}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{mra+m\overline{a}}{p}\Big)\Big|^2\\&=\frac{1}{2}\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{mra+m\overline{a}}{p}\Big)\Big|^2\\&=\frac{1}{2}p\sum_{\substack{a=1\\ ra+\overline{a}\equiv rb+\overline{b}\bmod p}}^{p-1}\sum_{b=1}^{p-1}1-\frac{1}{2}(p-1)^2\\&=\frac{1}{2}p\sum_{\substack{a=1\\ (a-b)(\overline{a}\overline{b}-r)\equiv 0\bmod p}}^{p-1}\sum_{b=1}^{p-1}1-\frac{1}{2}(p-1)^2.\end{aligned} \tag{1.14}$$

It is clear that if r is a quadratic non-residue mod p, then the congruence equation $a^2-r\equiv 0\bmod p$ has no solution. So we have

$$\sum_{\substack{a=1\\ (a-b)(\overline{a}\overline{b}-r)\equiv 0\bmod p}}^{p-1}\sum_{b=1}^{p-1}1=2(p-1). \tag{1.15}$$

Combining (1.14)–(1.15) we may deduce the first identity of Lemma 1.5.

Similarly, we can also deduce the second identity of Lemma 1.5.

2 Several Theorems

Theorem 2.1 *Let p be an odd prime with $p\equiv 3\bmod 4$, n be any integer with $(n,p)=1$, χ be any non-principal even character* mod p*. Then for any even number $k\neq 0$ and $(k,p-1)=d$, we have the identity*

$$\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma^k+na}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(mb^3+b)\Big|^2$$
$$=\begin{cases}2p(p^2-dp-1), & \text{if } \chi^3\neq\chi_0,\\ (p+1)(p^2-dp-1), & \text{if } \chi^3=\chi_0,\end{cases}$$

where χ_0 denotes the principal character mod p.

Proof For any odd prime p, it is clear that if r is a quadratic non-residue mod p, then we have $\left(\frac{r}{p}\right) = -1$ and $\left(\frac{rm}{p}\right) + \left(\frac{m}{p}\right) = 0$. If $p \equiv 3 \bmod 4$, then $\left(\frac{-1}{p}\right) = -1$. From these properties and Lemma 1.4 we have

$$\begin{aligned}
&\Big|\sum_{a=1}^{p-1}\chi(ma^3+a)\Big|^2 + \Big|\sum_{a=1}^{p-1}\chi(-ma^3+a)\Big|^2 \\
&= 4p + \Big(\frac{-m}{p}\Big)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\Big(\frac{(b^3a^2-1)(b-1)}{p}\Big) \\
&\quad + \Big(\frac{m}{p}\Big)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\Big(\frac{(b^3a^2-1)(b-1)}{p}\Big) = 4p. \qquad (2.1)
\end{aligned}$$

Now let $k \geq 2$ be an even number with $(k, p-1) = d$, χ be any non-principal even character mod p, and $\chi^3 \neq \chi_0$. If m passes through a reduced residue system mod p, then $-m$ also passes through a reduced residue system mod p. It is clear that

$$\Big|\sum_{a=1}^{p-1} e\Big(\frac{ma^k+na}{p}\Big)\Big|^2 = \Big|\sum_{a=1}^{p-1} e\Big(\frac{-ma^k-na}{p}\Big)\Big|^2.$$

Thus, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1} e\Big(\frac{ma^k+na}{p}\Big)\Big|^2 \cdot \Big|\sum_{b=1}^{p-1}\chi(mb^3+b)\Big|^2 \\
&= \sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1} e\Big(\frac{(-m)(-a)^k+n(-a)}{p}\Big)\Big|^2 \cdot \Big|\sum_{b=1}^{p-1}\chi((-m)b^3+b)\Big|^2 \\
&= \sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1} e\Big(\frac{-ma^k-na}{p}\Big)\Big|^2 \cdot \Big|\sum_{b=1}^{p-1}\chi(-mb^3+b)\Big|^2 \\
&= \sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1} e\Big(\frac{ma^k+na}{p}\Big)\Big|^2 \cdot \Big|\sum_{b=1}^{p-1}\chi(-mb^3+b)\Big|^2. \qquad (2.2)
\end{aligned}$$

Applying (2.1)–(2.2) and Lemma 1.2 we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1} e\Big(\frac{ma^k+na}{p}\Big)\Big|^2 \cdot \Big|\sum_{b=1}^{p-1}\chi(mb^3+b)\Big|^2 \\
&= \frac{1}{2}\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1} e\Big(\frac{ma^k+na}{p}\Big)\Big|^2 \cdot \Big(\Big|\sum_{b=1}^{p-1}\chi(mb^3+b)\Big|^2 + \Big|\sum_{b=1}^{p-1}\chi(-mb^3+b)\Big|^2\Big) \\
&= 2p \cdot \sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1} e\Big(\frac{ma^k+na}{p}\Big)\Big|^2 = 2p(p^2-dp-1). \qquad (2.3)
\end{aligned}$$

If $\chi^3 = \chi_0$, then from Lemmas 1.2, 1.4 and the method of proving (2.3) we also have

$$\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1} e\Big(\frac{ma^k+na}{p}\Big)\Big|^2 \cdot \Big|\sum_{b=1}^{p-1}\chi(mb^3+b)\Big|^2 = (p+1)(p^2-dp-1). \qquad (2.4)$$

Now Theorem 2.1 follows from (2.3) and (2.4).

Theorem 2.2 *Let p be an odd prime, χ be any non-principal even character* mod p. *Then we have the identity*

$$\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma^3+a}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(mb^3+b)\Big|^2$$
$$=(p+R(\chi)p+1-R(\chi))\Big(p^2-2p-1+2p\Big(\frac{-1}{p}\Big)\Big)$$
$$-p\Big(2\Big(\frac{-1}{p}\Big)+\Big(\frac{-3}{p}\Big)\Big)\Big|\sum_{b=1}^{p-1}\chi(b^3+b)\Big|^2,$$

where $R(\chi)=1$, if $\chi^3\neq\chi_0$ and $R(\chi)=0$, if $\chi^3=\chi_0$. That is, χ is a three order character mod p.

Proof Let r be a quadratic non-residue mod p. It is clear that $1^2,\ 2^2,\ \cdots,\ \left(\frac{p-1}{2}\right)^2$, $r1^2,\ r2^2,\ \cdots,\ r\left(\frac{p-1}{2}\right)^2$ pass through a reduced residue system mod p. So if $\chi^3\neq\chi_0$, then from (2.1) and Lemma 1.3 we have

$$\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma^3+a}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(mb^3+b)\Big|^2$$
$$=\sum_{m=1}^{\frac{p-1}{2}}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{m^2a^3+a}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(m^2b^3+b)\Big|^2$$
$$+\sum_{m=1}^{\frac{p-1}{2}}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{rm^2a^3+a}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(rm^2b^3+b)\Big|^2$$
$$=\frac{1}{2}\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{\overline{m}a^3+\overline{m}a}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(\overline{m}b^3+\overline{m}b)\Big|^2$$
$$+\frac{1}{2}\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{r\overline{m}a^3+\overline{m}a}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(r\overline{m}b^3+\overline{m}b)\Big|^2$$
$$=\frac{1}{2}\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma^3+ma}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(b^3+b)\Big|^2$$
$$+\frac{1}{2}\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{rma^3+ma}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(rb^3+b)\Big|^2$$
$$=\frac{1}{2}\Big[p^2-2p-1-2p\Big(\Big(\frac{-1}{p}\Big)+\Big(\frac{-3}{p}\Big)\Big)\Big]\Big|\sum_{b=1}^{p-1}\chi(b^3+b)\Big|^2$$
$$+\frac{1}{2}\Big[p^2-2p-1+2p\Big(\frac{-1}{p}\Big)\Big]\Big|\sum_{b=1}^{p-1}\chi(rb^3+b)\Big|^2$$
$$=\frac{1}{2}\Big[p^2-2p-1+2p\Big(\frac{-1}{p}\Big)\Big]\Big(\Big|\sum_{b=1}^{p-1}\chi(b^3+b)\Big|^2+\Big|\sum_{b=1}^{p-1}\chi(rb^3+b)\Big|^2\Big)$$
$$-p\Big(2\Big(\frac{-1}{p}\Big)+\Big(\frac{-3}{p}\Big)\Big)\Big|\sum_{b=1}^{p-1}\chi(b^3+b)\Big|^2$$

$$= 2p^3 - 4p^2 - 2p + 4p^2\Big(\frac{-1}{p}\Big) - p\Big(2\Big(\frac{-1}{p}\Big) + \Big(\frac{-3}{p}\Big)\Big)\Big|\sum_{b=1}^{p-1}\chi(b^3+b)\Big|^2. \tag{2.5}$$

If $\chi^3 = \chi_0$, then from Lemma 1.4 and the method of proving (2.5) we also have

$$\begin{aligned}&\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma^3+a}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(mb^3+b)\Big|^2\\ &= (p+1)\Big(p^2-2p-1+2p\Big(\frac{-1}{p}\Big)\Big)\\ &\quad - p\Big(2\Big(\frac{-1}{p}\Big)+\Big(\frac{-3}{p}\Big)\Big)\Big|\sum_{b=1}^{p-1}\chi(b^3+b)\Big|^2.\end{aligned} \tag{2.6}$$

Combining (2.5)–(2.6) we may immediately deduce Theorem 2.2.

Theorem 2.3 *Let p be an odd prime, χ be any non-principal even character* mod p. *Then we have the identity*

$$\begin{aligned}&\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma+\overline{a}}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(mb^3+b)\Big|^2\\ &= \begin{cases} 2p(p^2-1) - p\Big|\sum\limits_{b=1}^{p-1}\chi(b^3+b)\Big|^2, & \text{if } \chi^3 \neq \chi_0,\\ (p+1)(p^2-1) - p\Big|\sum\limits_{b=1}^{p-1}\chi(b^3+b)\Big|^2, & \text{if } \chi^3 = \chi_0.\end{cases}\end{aligned}$$

Proof It is clear that for any integer m with $(m,p)=1$, from the properties of the reduced residue system mod p we have

$$\Big|\sum_{b=1}^{p-1}\chi(m^2b^3+b)\Big|^2 = \Big|\sum_{b=1}^{p-1}\chi(\overline{m}b+\overline{m}b)\Big|^2 = \Big|\sum_{b=1}^{p-1}\chi(b^3+b)\Big|^2 \tag{2.7}$$

and

$$\Big|\sum_{b=1}^{p-1}\chi(rm^2b^3+b)\Big|^2 = \Big|\sum_{b=1}^{p-1}\chi(r\overline{m}b^3+\overline{m}b)\Big|^2 = \Big|\sum_{b=1}^{p-1}\chi(rb^3+b)\Big|^2. \tag{2.8}$$

If $\chi^3 \neq \chi_0$, then from (2.1), (2.7)–(2.8) and Lemma 1.5 we have

$$\begin{aligned}&\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma+\overline{a}}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(mb^3+b)\Big|^2\\ &= \sum_{m=1}^{\frac{p-1}{2}}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{m^2a+\overline{a}}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(m^2b^3+b)\Big|^2\\ &\quad + \sum_{m=1}^{\frac{p-1}{2}}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{rm^2a+\overline{a}}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(rm^2b^3+b)\Big|^2\\ &= \sum_{m=1}^{\frac{p-1}{2}}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{m^2a+\overline{a}}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(b^3+b)\Big|^2\end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=1}^{\frac{p-1}{2}}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{rm^2a+\overline{a}}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(rb^3+b)\Big|^2\\
&=\frac{1}{2}(p^2-2p-1)\Big|\sum_{b=1}^{p-1}\chi(b^3+b)\Big|^2+\frac{1}{2}(p^2-1)\Big|\sum_{b=1}^{p-1}\chi(rb^3+b)\Big|^2\\
&=\frac{1}{2}(p^2-1)\Big(\Big|\sum_{b=1}^{p-1}\chi(b^3+b)\Big|^2+\Big|\sum_{b=1}^{p-1}\chi(rb^3+b)\Big|^2\Big)-p\Big|\sum_{b=1}^{p-1}\chi(b^3+b)\Big|^2\\
&=2p(p^2-1)-p\Big|\sum_{b=1}^{p-1}\chi(b^3+b)\Big|^2. \qquad (2.9)
\end{aligned}
$$

If $\chi^3=\chi_0$, then from Lemma 1.4 and the method of proving (2.9) we have

$$
\begin{aligned}
&\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma+\overline{a}}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(mb^3+b)\Big|^2\\
&=(p+1)(p^2-1)-p\Big|\sum_{b=1}^{p-1}\chi(b^3+b)\Big|^2. \qquad (2.10)
\end{aligned}
$$

From (2.9)–(2.10) we may immediately deduce Theorem 2.3.

3 Two Corollaries

If p is an odd prime with $(3,p-1)=1$, then for any non-principal even character $\chi \bmod p$, one has $\chi^3\neq\chi_0$. So from Theorems 2.2–2.3 we can deduce the following corollary.

Corollary 3.1 *Let p be an odd prime with $(3,p-1)=1$. Then for any non-principal even character $\chi \bmod p$, we have the asymptotic formulae*

$$\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma^3+a}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(mb^3+b)\Big|^2=2p^3+O(p^2)$$

and

$$\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma+\overline{a}}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(mb^3+b)\Big|^2=2p^3+O(p^2).$$

It is clear that the first result in Corollary 3.1 improves the Du Xiaoying's work. That is, our result is much better than asymptotic formula (1.1).

If $3\mid(p-1)$, and $\chi \bmod p$ satisfies $\chi^3=\chi_0$ and $\chi\neq\chi_0$, then noting the estimate for character sums

$$\Big|\sum_{b=1}^{p-1}\chi(b^3+b)\Big|^2=\Big|\sum_{b=1}^{p-1}\chi(b^2+1)\Big|^2=p+O(\sqrt{p}),$$

from Theorems 2.2–2.3 we may immediately deduce the following corollary.

Corollary 3.2 *Let p be any odd prime. Then for any three order character $\chi \bmod p$, we have the asymptotic formulae*

$$\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma^3+a}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(mb^3+b)\Big|^2=p^2\Big(p-1-\Big(\frac{-3}{p}\Big)\Big)+O(p^{\frac{3}{2}})$$

and

$$\sum_{m=1}^{p-1}\Big|\sum_{a=1}^{p-1}\mathrm{e}\Big(\frac{ma+\overline{a}}{p}\Big)\Big|^2\cdot\Big|\sum_{b=1}^{p-1}\chi(mb^3+b)\Big|^2=p^3+O(p^{\frac{3}{2}}).$$

Note: Our Theorem 2.2 is much better than the corresponding result in [3].

In fact in Theorem 2.1, we only consider the case $p\equiv 3 \bmod 4$. Does there exist a similar formula for the case $p\equiv 1 \bmod 4$? This is an open problem.

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