Chin. Ann. Math. Ser. B
41(4), 2020, 559–572
DOI: 10.1007/s11401-020-0217-6

Chinese Annals of Mathematics, Series B

On the Compactness of Grunsky Differential Operators*

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Abstract Grunsky operators play an important role in classical geometric function theory and in the study of Teichmüller spaces. The Grunsky map is known to be holomorphic on the universal Teichmüller space. In this paper the authors deal with the compactness of a Grunsky differential operator. They will give upper and lower estimates of the essential norm of a Grunsky differential operator and discuss when a Grunsky differential operator is a p-Schatten class operator.

Keywords Universal Teichmüller space, Beltrami coefficient, Grunsky operator, Compact operator, p-Schatten class operator, Essential norm
2000 MR Subject Classification 30C62, 32G15

1 Introduction

The universal Teichmüller space T is a complex Banach manifold and can be regarded as a bounded domain of the Banach space of holomorphic functions outside the unit disk which are bounded in the Poincaré metric. Each point in the universal Teichmüller space corresponds to the Schwarzian derivative of a univalent function outside the unit disk. The univalent function theory is thus very useful to study the universal Teichmüller space. Actually, Grunsky inequalities for univalent functions yield a lot of properties of the universal Teichüller space (see [14–15, 29, 35]). Recently, Grunsky operators play an important role in the study of subspaces of the universal Teichmüller space (see [23–28, 30-31]). It is known that the Grunsky map sends holomorphically each point in the universal Teichmüller space to the corresponding Grunsky operator. This fact was used to deal with the compactness of Grunsky operators by Takhtajan-Teo [30] and Shen [23], respectively. In this paper, we will continue to discuss the compactness of the Grunsky differential operators, a topic which was initiated to study in [25] where we discussed the Kobayashi and Carathéodory metrics on the asymptotic Teichmüller space. We will estimate the essential norm of a Grunsky differential operator and discuss when a Grunsky differential operator is a p-Schatten class operator. Main results will be stated in the next section (see Theorems 2.2 and 2.3 below).

2 Preliminaries and Statement of Main Results

In this section, we recall some basic definitions and results on univalent functions and

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*This work was supported by the National Natural Science Foundation of China (No. 11631010).

Teichmüller spaces and state the main results of the paper. For primary references, see Gardiner-Lakic [10], Lehto [16], Nag [18] and Pommenerke [19].

2.1 Universal Teichmüller space

Let M denote the open unit ball of the Banach space $L^\infty(\Delta)$ of essentially bounded measurable functions on the unit disk $\Delta = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$. For $\mu \in M$, let f_μ be the quasiconformal mapping on the extended plane $\widehat{\mathbb{C}}$ with complex dilatation equal to μ in Δ, conformal in $\Delta^* = \widehat{\mathbb{C}} - \overline{\Delta}$, normalized by $f_\mu(z) = z + O(|z|^{-1})$ as $z \to \infty$. We say that μ and ν are equivalent if $f_\mu |_{\Delta^*} = f_\nu \mid \Delta^*$. The equivalence class of μ is denoted by $[\mu]$. The set T of all equivalence classes is called the universal Teichmüller space.

Let Ω be an arbitrary simply connected domain in the extended complex plane $\widehat{\mathbb{C}}$ which is conformally equivalent to the unit disk. Recall that the hyperbolic metric λ_Ω in Ω can be defined by

$$\lambda_\Omega(f(z))|f'(z)| = (1 - |z|^2)^{-1}, \quad z \in \Delta, \tag{2.1}$$

where $f : \Delta \to \Omega$ is any conformal mapping. Let $B(\Omega)$ denote the Banach space of functions ϕ holomorphic in Ω with norm

$$\|\phi\|_{B(\Omega)} = \sup_{z\in\Omega} |\phi(z)|\lambda_\Omega^{-2}(z). \tag{2.2}$$

It is easy to see that a conformal mapping $g : \Omega_1 \to \Omega_2$ induces an isometric isomorphism $\phi \mapsto (\phi \circ g)(g')^2$ from $B(\Omega_2)$ onto $B(\Omega_1)$.

Consider the map $\mathcal{S} : M \to B(\Delta^*)$ defined as $\mathcal{S}(\mu) = S(f_\mu|_{\Delta^*})$, where $S(f)$ denotes the Schwarzian derivative of a locally univalent function f of a domain in the extended plane $\widehat{\mathbb{C}}$, defined as $(f''/f')' - \frac{1}{2}(f''/f')^2$. It is known that $\mathcal{S}$ is a holomorphic split submersion and the differential of $\mathcal{S}$ at μ has the following expression:

$$\mathrm{d}_\mu\mathcal{S}(\nu)(z) = -\frac{6}{\pi}(f_\mu'(z))^2 \iint_\Delta \frac{\nu(\zeta)(\partial f_\mu(\zeta))^2}{(f_\mu(\zeta) - f_\mu(z))^4}\mathrm{d}\xi\mathrm{d}\eta, \quad \nu \in L^\infty(\Delta). \tag{2.3}$$

Now $\mathcal{S}$ descends down to a 1-1 map $\mathcal{B} : T \to B(\Delta^*)$, which is known as the Bers embedding. Via the Bers embedding, T carries a natural complex structure so that the natural projection $\Phi : M \to T$ is a holomorphic split submersion, and $\mathcal{B}$ is a biholomorphic isomorphism from T onto its image.

2.2 Grunsky map

For each $\mu \in M$, its Grunsky coefficients $\alpha_{mn}(\mu)$ are determined from the expression

$$\log \frac{f_\mu(z) - f_\mu(\zeta)}{z - \zeta} = -\sum_{m,n=1}^{\infty} \alpha_{mn}(\mu)z^{-m}\zeta^{-n}, \quad z, \zeta \in \Delta^*, \tag{2.4}$$

where the branch of logarithm is equal to zero for $z = \zeta = \infty$.

We denote as usual by l^2 the Hilbert space of sequences $x = (x_m)$ with the inner product and norm

$$\langle x, y\rangle = \sum_{m=1}^{\infty} x_m\overline{y}_m, \quad \|x\| = \Big(\sum_{m=1}^{\infty} |x_m|^2\Big)^{\frac{1}{2}}. \tag{2.5}$$

Then μ determines the so-called Grunsky operator $G([\mu]) : l^2 \to l^2$ by

$$G([\mu]) : (x_m) \mapsto \Big(\sum_{n=1}^{\infty} \sqrt{mn}\alpha_{mn}(\mu)x_n \Big), \tag{2.6}$$

so that

$$\langle G([\mu])x, \overline{x} \rangle = \sum_{m,n=1}^{\infty} \sqrt{mn}\alpha_{mn}(\mu)x_m x_n, \tag{2.7}$$

$$\|G([\mu])x\|^2 = \sum_{m=1}^{\infty} \Big| \sum_{n=1}^{\infty} \sqrt{mn}\alpha_{mn}(\mu)x_n \Big|^2. \tag{2.8}$$

Since $\alpha_{mn}(\mu) = \alpha_{nm}(\mu)$, Schur's result (see [21]) implies

$$\|G([\mu])\| \doteq \sup_{x \in S_1(l^2)} \|G([\mu])x\| = \sup_{x \in S_1(l^2)} |\langle G([\mu])x, \overline{x} \rangle|, \tag{2.9}$$

where $S_1(l^2)$ is the unit sphere in l^2. A classical result known as the Grunsky inequality says that $\|G([\mu])\| \le \|\mu\|_\infty$, which implies that $G([\mu])$ is a bounded operator with norm strictly less than one. It is well known that Grunsky operator plays an important role in the study of univalent function theory (see [19]) and, as remarked above, in the study of Teichmüller spaces as well (see [14–15, 23–31, 35]).

The Grunsky map G is defined by sending $[\mu]$ to $G([\mu])$. We denote by $\mathcal{L}(l^2)$ the space of all bounded linear operators of l^2 into itself. Then G is a mapping from the universal Teichmüller space T into the unit ball of $\mathcal{L}(l^2)$. In [23] we gave a complete proof of the holomorphy of the Grunsky map, a fact which was asserted and frequently used in the literature (see [14–15]). Furthermore, setting $\widetilde{G} = G \circ \Phi$, we (see [25]) showed that the derivative $\mathrm{d}_\mu \widetilde{G}(\nu)$ at $\mu \in M$ in the direction $\nu \in L^\infty(\Delta)$ is

$$(x_m) \mapsto \Big(\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{x_n}{\sqrt{mn}} \iint_{f_\mu(\Delta)} \Big(\frac{\nu}{1-|\mu|^2} \frac{\partial f_\mu}{\overline{\partial f_\mu}} \Big) \circ f_\mu^{-1} F_m'(\mu) F_n'(\mu) \mathrm{d}u\mathrm{d}v \Big). \tag{2.10}$$

Here $F_n(\mu)$ is the n-th Faber polynomial for μ, which is determined by the following expression:

$$\log \frac{f_\mu(z) - w}{z} = -\sum_{n=1}^{\infty} \frac{1}{n} F_n(\mu)(w) z^{-n}, \quad w \in \mathbb{C}, z \to \infty. \tag{2.11}$$

2.3 Statement of main results

Let $L_0(\Delta)$ be the closed subspace of $L^\infty(\Delta)$ which consists of those functions μ such that $\mu(z) \to 0$ as $|z| \to 1$, and $B_0(\Delta^*)$ be the closed subspace of $B(\Delta^*)$ which consists of those functions ϕ such that $\phi(z)(|z|^2-1)^2 \to 0$ as $|z| \to 1$. Set $M_0 = M \cap L_0(\Delta)$. Then $T_0 = \Phi(M_0)$ is a closed sub-manifold of T, and $\mathcal{B}(T_0) = \mathcal{B}(T) \cap B_0(\Delta^*)$ (see [2, 11, 20]). T_0 is usually called the little universal Teichmüller space. It plays an important role in the recent study on asymptotic Teichmüller spaces (see [6–8, 10–11, 25]).

The points in T_0 can be characterized by means of the compactness of the corresponding Grunsky operators. It was proved respectively by Takhtajan-Teo [30] and Shen [23] that $[\mu] \in T_0$ if and only if $G([\mu])$ is a compact operator. In the infinitesimal setting, we have proved the following result.

Theorem 2.1 (see [25]) *For $\mu \in M$ and $\nu \in L^\infty(\Delta)$, the following conditions are equivalent*:

(1) $\mathrm{d}_\mu \mathcal{S}(\nu) \in B_0(\Delta^*)$;

(2) *There exists some* $\widetilde{\nu} \in L_0(\Delta)$ *with* $\mathrm{d}_\mu \mathcal{S}(\widetilde{\nu}) = \mathrm{d}_\mu \mathcal{S}(\nu)$;

(3) $\mathrm{d}_\mu \widetilde{G}(\nu)$ *is a compact operator.*

Note that (1) $\Leftrightarrow$ (2) was first proved by Earle-Gardiner-Lakic [6]. Theorem 2.1 plays an important role in [25] where we discussed the asymptotic Grunsky operators and the asymptotic Teichmüller space. Here we will continue to discuss the compactness of the Grunsky differential operator $\mathrm{d}_\mu \mathcal{S}(\nu)$ by estimating its essential norm $\|\mathrm{d}_\mu \mathcal{S}(\nu)\|_e$ and prove the following result, from which Theorem 2.1 follows immediately. For simplicity, we fix some notations. $C(\cdot)$, $C_1(\cdot), C_2(\cdot), \cdots$ will denote constants that depend only on the elements put in the brackets and might change from one line to another. The notation $A \asymp B$ means that there is a positive constant C independent of A and B such that $A/C \le B \le CA$. The notation $A \lesssim B$ ($A \gtrsim B$) means that there is a positive constant C independent of A and B such that $A \le CB$ ($A \ge CB$).

Theorem 2.2 *For $\mu \in M$ and $\nu \in L^\infty(\Delta)$, it holds that*

$$\begin{aligned}\|\mathrm{d}_\mu \widetilde{G}(\nu)\|_e &\asymp C_1(\|\mu\|_\infty) \limsup_{|z|\to 1} |\mathrm{d}_\mu \mathcal{S}(\nu)(z)|(|z|^2-1)^2 \\ &\asymp C_2(\|\mu\|_\infty) \inf\{\|\widetilde{\nu}|_{\Delta - E}\|_\infty : \mathrm{d}_\mu \mathcal{S}(\widetilde{\nu}) = \mathrm{d}_\mu \mathcal{S}(\nu), E \subset \Delta \mathit{compact}\}.\end{aligned}$$

Now let $p \ge 1$ be a fixed number. We denote by $\mathcal{L}^p(\Omega)$ the Banach space of all essentially bounded measurable functions μ on Ω with norm

$$\|\mu\|_{\mathcal{L}^p(\Omega)} \doteq \|\mu\|_\infty + \Big(\frac{1}{\pi} \iint_\Omega |\mu(z)|^p \lambda_\Omega^2(z) \mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{p}}. \tag{2.12}$$

Set $\mathcal{M}^p = M \cap \mathcal{L}^p(\Delta)$. Then $T_p = \Phi(\mathcal{M}^p)$ is called the p-integrable Teichmüller space, T_2 is also called the Weil-Petersson Teichmüller space.

We denote by $\mathcal{B}_p(\Omega)$ the Banach space of functions ϕ holomorphic in Ω with norm

$$\|\phi\|_{\mathcal{B}_p(\Omega)} \doteq \Big(\frac{1}{\pi} \iint_\Omega |\phi(z)|^p \lambda_\Omega^{2-2p}(z) \mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{p}}. \tag{2.13}$$

For a conformal mapping $g : \Omega_1 \to \Omega_2$, the correspondence $\phi \mapsto (\phi \circ g)(g')^2$ again induces an isometric isomorphism from $\mathcal{B}_p(\Omega_2)$ onto $\mathcal{B}_p(\Omega_1)$. Thus, for $1 \le p \le q$, $\mathcal{B}_p(\Omega) \subset \mathcal{B}_q(\Omega) \subset B(\Omega)$, and the inclusion maps are continuous (see [34]).

Under the Bers embedding $\mathcal{B} : T \to B(\Delta^*)$, $\mathcal{B}(T_p) = \mathcal{B}(T) \cap \mathcal{B}_p(\Delta^*)$ (see [4, 12–13, 30]). The points in T_p can also be characterized by means of the corresponding Grunsky operators. It was proved respectively by Takhtajan-Teo [30] and Shen [23] that $[\mu] \in T_2$ if and only if $G([\mu])$ is a Hilbert-Schmidt operator. In fact, in [13] we learned recently, Jones asserted that $[\mu] \in T_p$ if and only if $G([\mu])$ is a p-Schatten class operator. We will prove an analogous result in the infinitesimal setting. In [23–25], we have obtained some partial results, which were used in [25] to prove Theorem 2.1.

Theorem 2.3 *Let $p \ge 2$. Then for $\mu \in M$ and $\nu \in L^\infty(\Delta)$, the following conditions are equivalent*:

(1) $\mathrm{d}_\mu \mathcal{S}(\nu) \in \mathcal{B}_p(\Delta^*)$;

(2) *There exists* $\widetilde{\nu} \in L^\infty(\Delta)$ *with* $\mathrm{d}_\mu \mathcal{S}(\widetilde{\nu}) = \mathrm{d}_\mu \mathcal{S}(\nu)$ *such that* $\widetilde{\nu} \circ f_\mu^{-1} \in \mathcal{L}^p(f_\mu(\Delta))$;
(3) $\mathrm{d}_\mu \widetilde{G}(\nu)$ *is a p-Schatten class operator.*

3 Bers' Reproducing Formula Revisited

In this section, we review a reproducing formula of Bers [3], which plays an important role in Teichmüller theory. It will also be used in the proof of our results.

Let Γ be a closed Jordan curve in the extended complex plane $\widehat{\mathbb{C}}$, and let D and D^* denote the domains interior and exterior to Γ, respectively. Γ is called a quasicircle if it is the image of the unit circle under some quasiconformal mapping of the whole plane. By an anti-quasiconformal reflection about Γ we mean an orientation-reversing homeomorphism h of the entended complex plane such that $h|_\Gamma = id$, $h \circ h = id$, and $\overline{h}$ is quasiconformal. Then we have the following well-known result of Ahlfors [1].

Proposition 3.1 (see [1]) Γ *is a quasicircle if and only if there exists some anti-quasiconformal reflection about* Γ*. When* Γ *passes through* ∞*, there exists an anti-quasiconformal reflection about* Γ *satisfying a uniform Lipschitz condition.*

Now we state the reproducing formula of Bers [3].

Proposition 3.2 (see [3]) *Let* Γ *be a quasicircle passing through* ∞ *with complementary domains* D *and* D^**. Let* h *be an anti-quasiconformal reflection about* Γ *satisfying a uniform Lipschitz condition. Then for any* $\phi \in B(D^*)$ *it holds that*

$$\phi(z) = -\frac{3}{\pi} \iint_D \frac{(\zeta - h(\zeta))^2}{(\zeta - z)^4} \overline{\partial} h(\zeta) \phi(h(\zeta)) \mathrm{d}\xi \mathrm{d}\eta, \quad z \in D^*. \tag{3.1}$$

In our situation, we need the formula (3.1) for a bounded quasicircle Γ. This can be achieved by means of the conformally natural anti-quasiconformal reflection $j(D)$ associated to the interior domain D introduced by Earle-Nag [9], which is defined by using the conformally natural extension operator of Douady-Earle [5]. As observed by Earle-Gardiner-Lakic [6], $j(D)$ satisfies a uniform Lipschitz condition when Γ passes through ∞, which implies that the formula (3.1) holds when h is replaced by $j(D)$. By the conformal naturality property of the reflection $j(D)$ (see [9, Theorem 1]), we conclude that the formula (3.1) still holds with $h = j(D)$ even when Γ is a bounded quasicircle (see [33] for details). We summarize this as following proposition.

Proposition 3.3 (see [33]) *Let* Γ *be a quasicircle with complementary domains* D *and* D^**, and let* $j = j(D)$ *be the conformally natural anti-quasiconformal reflection to* D*. Then for any* $\phi \in B(D^*)$ *it holds that*

$$\phi(z) = -\frac{3}{\pi} \iint_D \frac{(\zeta - j(\zeta))^2}{(\zeta - z)^4} \overline{\partial} j(\zeta) \phi(j(\zeta)) \mathrm{d}\xi \mathrm{d}\eta, \quad z \in D^*. \tag{3.2}$$

Now we use Proposition 3.3 to establish a fundamental lemma, which will be used to prove Theorems 2.2 and 2.3.

Lemma 3.1 *For any* $\mu \in M(\Delta)$ *and* $\nu \in L^\infty(\Delta)$*, there exists some* $\widetilde{\nu} \in L^\infty(\Delta)$ *with* $\|\widetilde{\nu}\|_\infty \lesssim C(\|\mu\|_\infty) \|\mathrm{d}_\mu \mathcal{S}(\nu)\|_{B(\Delta^*)}$ *such that* $\mathrm{d}_\mu \mathcal{S}(\widetilde{\nu}) = \mathrm{d}_\mu \mathcal{S}(\nu)$*. Furthermore,* $\widetilde{\nu}$ *can be chosen so* *that*

$$\inf\{\|\widetilde{\nu}|_{\Delta-E}\|_\infty : E\subset\Delta\, compact\}\lesssim C(\|\mu\|_\infty)\limsup_{|z|\to 1}|\mathrm{d}_\mu\mathcal{S}(\nu)(z)|(|z|^2-1)^2 \tag{3.3}$$

and $\widetilde{\nu}\circ f_\mu^{-1}\in\mathcal{L}^p(f_\mu(\Delta))$ *if* $\mathrm{d}_\mu\mathcal{S}(\nu)\in\mathcal{B}_p(\Delta^*)$.

Proof Most part of this lemma was obtained by Zhai in [33]. For completeness, we will give a detailed proof here. Set $f=f_\mu$, $g=f^{-1}$, $D=f(\Delta)$, $D^*=f(\Delta^*)$, and $\phi=(\mathrm{d}_\mu\mathcal{S}(\nu)\circ g)(g')^2$. Then $\phi\in B(D^*)$, and $\|\phi\|_{B(D^*)}=\|\mathrm{d}_\mu\mathcal{S}(\nu)\|_{B(\Delta^*)}$. More precisely, since $\lambda_{D^*}(f(z))|f'(z)|=(|z|^2-1)^{-1}$, we have

$$|\phi(f(z))|\lambda_{D^*}^{-2}(f(z))=|\mathrm{d}_\mu\mathcal{S}(\nu)(z)|(|z|^2-1)^2,\quad z\in\Delta^*. \tag{3.4}$$

Thus, ϕ satisfies (3.2) with j being the conformally natural anti-quasiconformal reflection to D. Recall that Earle-Nag [9] showed there exists some positive constant $C(\|\mu\|_\infty)$ such that

$$\frac{1}{C(\|\mu\|_\infty)}\le|j(w)-w|^2\lambda_D(w)\lambda_{D^*}(j(w))\le C(\|\mu\|_\infty),\quad w\in D, \tag{3.5}$$

$$\frac{1}{C(\|\mu\|_\infty)}\le|j(w)-w|^2\lambda_{D^*}^2(w)|\overline{\partial}j(j(w))|\le C(\|\mu\|_\infty),\quad w\in D^*. \tag{3.6}$$

Now we define

$$\widetilde{\nu}(z)=\frac{1}{2}(f(z)-j(f(z)))^2\overline{\partial}j(f(z))\phi(j(f(z))(1-|\mu(z)|^2)\frac{\overline{\partial f(z)}}{\partial f(z)},\quad z\in\Delta. \tag{3.7}$$

It follows from (3.4) and (3.6) that

$$|\widetilde{\nu}(z)|\lesssim C(\|\mu\|_\infty)|\phi(j(f(z))|\lambda_{D^*}^{-2}(j(f(z))=C(\|\mu\|_\infty)|\mathrm{d}_\mu\mathcal{S}(\nu)(\widehat{z})|(|\widehat{z}|^2-1)^2 \tag{3.8}$$

for $\widehat{z}=g(j(f(z)))$. Thus, $\widetilde{\nu}\in L^\infty(\Delta)$ with $\|\widetilde{\nu}\|_\infty\lesssim C(\|\mu\|_\infty)\|\mathrm{d}_\mu\mathcal{S}(\nu)\|_{B(\Delta^*)}$, and (3.3) holds.

Now we suppose that $\mathrm{d}_\mu\mathcal{S}(\nu)\in\mathcal{B}_p(\Delta^*)$. Then $\phi\in\mathcal{B}_p(D^*)$, and $\|\phi\|_{\mathcal{B}_p(D^*)}=\|\mathrm{d}_\mu\mathcal{S}(\nu)\|_{\mathcal{B}_p(\Delta^*)}$. It follows from (3.5), (3.6) and (3.8) that

$$\begin{aligned}\iint_D|\widetilde{\nu}(g(w))|^p\lambda_D^2(w)\mathrm{d}u\mathrm{d}v&\lesssim C_1(\|\mu\|_\infty)\iint_D|\phi(j(w))|^p\lambda_{D^*}^{-2p}(j(w))\lambda_D^2(w)\mathrm{d}u\mathrm{d}v\\&\lesssim C_2(\|\mu\|_\infty)\iint_D|\phi(j(w))|^p\lambda_{D^*}^{-2p-2}(j(w))|j(w)-w|^{-4}\mathrm{d}u\mathrm{d}v\\&\lesssim C_3(\|\mu\|_\infty)\iint_D|\phi(j(w))|^p\lambda_{D^*}^{2-2p}(j(w))|\overline{\partial}j(w)|^2\mathrm{d}u\mathrm{d}v\\&\lesssim C_4(\|\mu\|_\infty)\iint_{D^*}|\phi(w)|^p\lambda_{D^*}^{2-2p}(w)\mathrm{d}u\mathrm{d}v.\end{aligned}$$

Therefore, $\widetilde{\nu}\circ g\in\mathcal{L}^p(D)$, that is, $\widetilde{\nu}\circ f_\mu^{-1}\in\mathcal{L}^p(f_\mu(\Delta))$.

It remains to prove that $\mathrm{d}_\mu\mathcal{S}(\widetilde{\nu})=\mathrm{d}_\mu\mathcal{S}(\nu)$. By (2.3) and (3.2) we have

$$\begin{aligned}\mathrm{d}_\mu\mathcal{S}(\widetilde{\nu})(z)&=-\frac{6}{\pi}(f'(z))^2\iint_\Delta\frac{\widetilde{\nu}(\zeta)(\partial f(\zeta))^2}{(f(\zeta)-f(z))^4}\mathrm{d}\xi\mathrm{d}\eta\\&=-\frac{3}{\pi}(f'(z))^2\iint_D\frac{(\zeta-j(\zeta))^2}{(\zeta-f(z))^4}\overline{\partial}j(\zeta)\phi(j(\zeta))\mathrm{d}\xi\mathrm{d}\eta\\&=\phi(f(z))(f'(z))^2\\&=\mathrm{d}_\mu\mathcal{S}(\nu)(z).\end{aligned}$$

This finishes the proof of Lemma 3.1.

4 Proof of Theorem 2.2

We first recall a general formula due to Shapiro [22] for the essential norm $\|L\|_e$ of a linear operator L on a Hilbert space H.

Proposition 4.1 (see [22]) *Suppose L is a bounded linear operator on a Hilbert space H. Let $\{K_n\}$ be a sequence of compact self-adjoint operators on H, and write $R_n = \mathrm{I} - K_n$ with I being the identity operator. Suppose $\|R_n\| = 1$ for each n, and $\|R_n x\| \to 0$ for each $x \in H$. Then $\|L\|_e = \lim_{n\to\infty} \|LR_n\|$.*

Now we use Proposition 4.1 to obtain an upper estimate to a bounded operator L on the Hilbert space l^2. For simplicity, we call a sequence $(x^{(j)})$ in l^2 to be degenerating if $\|x^{(j)}\| \le 1$, and $(x^{(j)})$ converges to zero weakly. Then we have the following basic result.

Proposition 4.2 *For any $L \in \mathcal{L}(l^2)$, it holds that*

$$\|L\|_e = \sup_{(x^{(j)})} \limsup_{j\to\infty} \|Lx^{(j)}\|, \tag{4.1}$$

where the supremum is taken from all degenerating sequences.

Proof We can obtain $\|L\|_e \ge \sup_{(x^{(j)})} \limsup_{j\to\infty} \|Lx^{(j)}\|$ by a standard discussion (see [25, Lemma 7.3]). We apply Proposition 4.1 to obtain the other direction. To do so, we borrow some discussion from Shapiro [22] (see also [32]). For each $n \ge 1$, we define K_n by $K_n x = (x_1, x_2, \cdots, x_n, 0, 0, \cdots)$ for $x = (x_m) \in l^2$. Then K_n satisfies the conditions in Proposition 4.1, which implies that $\|L\|_e = \lim_{n\to\infty} \|LR_n\|$ with $R_n = \mathrm{I} - K_n$. Then for each n there exists $x^{(n)} \in S(l^2)$ such that

$$\|L\|_e \le \limsup_{n\to\infty} \|LR_n x^{(n)}\|.$$

Now it is easy to see that $(R_n x^{(n)})$ is a degenerating sequence, which implies that $\|L\|_e \le \sup_{(x^{(j)})} \limsup_{j\to\infty} \|Lx^{(j)}\|$ as required.

Lemma 4.1 *For any $\mu \in M(\Delta)$ and $\nu \in L^\infty(\Delta)$, we have*

$$\|\mathrm{d}_\mu \widetilde{G}(\nu)\|_e \ge \frac{1}{6} \limsup_{|z|\to 1} |\mathrm{d}_\mu \mathcal{S}(\nu)(z)|(|z|^2 - 1)^2.$$

Proof We need some close relation between Schwarzian derivatives, Grunsky coefficients and Grunsky operators. We use some discussion form [23–25]. Consider a subset of l^2 as follows: For any finite $z \in \Delta^*$, set $x_z = (x_m)$ with $x_m = \sqrt{m}(|z|^2 - 1)z^{-(m+1)}$. Then

$$\|x_z\|^2 = \sum_{m=1}^{\infty} |x_m|^2 = (|z|^2 - 1)^2 \sum_{m=1}^{\infty} m|z|^{-2(m+1)} = 1.$$

Noting that for each fixed m, $x_m \to 0$ as $|z| \to 1$, we conclude that (x_z) converges to zero weakly as $|z| \to 1$.

Differentiating both sides of the equation (2.4) partially with respect to z and ζ yields the relation

$$\frac{f_\mu'(z) f_\mu'(\zeta)}{[f_\mu(z) - f_\mu(\zeta)]^2} - \frac{1}{(z - \zeta)^2} = -\sum_{m,n=1}^{\infty} mn\alpha_{mn}(\mu) z^{-(m+1)} \zeta^{-(n+1)}, \quad z, \zeta \in \Delta^*.$$

Letting $\zeta \to z$, we get

$$\mathcal{S}(\mu)(z) = -6\sum_{m,n=1}^{\infty} mn\alpha_{mn}(\mu)z^{-(m+n+2)} = -6\frac{\langle \widetilde{G}(\mu)x_z, \overline{x}_z\rangle}{(|z|^2-1)^2}.$$

Consequently,

$$\mathrm{d}_\mu\mathcal{S}(\nu)(z) = -6\sum_{m,n=1}^{\infty} mn\mathrm{d}_\mu\alpha_{mn}(\nu)z^{-(m+n+2)} = -6\frac{\langle \mathrm{d}_\mu\widetilde{G}(\nu)x_z, \overline{x}_z\rangle}{(|z|^2-1)^2}. \tag{4.2}$$

By Proposition 4.2, we conclude by (4.2) that

$$\begin{aligned}\|\mathrm{d}_\mu\widetilde{G}(\nu)\|_e &\geq \sup_{(x^{(j)})}\limsup_{j\to\infty}|\langle \mathrm{d}_\mu\widetilde{G}(\nu)x^{(j)}, \overline{x^{(j)}}\rangle| \\ &\geq \limsup_{|z|\to 1}|\langle \mathrm{d}_\mu\widetilde{G}(\nu)x_z, \overline{x}_z\rangle| \\ &= \frac{1}{6}\limsup_{|z|\to 1}|\mathrm{d}_\mu\mathcal{S}(\nu)(z)|(|z|^2-1)^2.\end{aligned}$$

To obtain an upper estimate of $\|\mathrm{d}_\mu\widetilde{G}(\nu)\|_e$, we use an operator introduced in [24]. Let $\mu \in M$ be given with Faber polynomials $F_n(\mu)$. It was proved in [24] that $\sum_{n=1}^{\infty}\frac{F_n'^{\,2}}{n}$ converges absolutely and locally uniformly in $f_\mu(\Delta)$ and thus represents an analytic function in $f_\mu(\Delta)$. More precisely, it holds that

$$\sum_{n=1}^{\infty}\frac{|F_n'(\mu)(w)|^2}{n} \leq \frac{C(\mu)}{d^2(w, \partial f_\mu(\Delta))} \asymp C(\mu)\lambda^2_{f_\mu(\Delta)}(w), \quad w \in f_\mu(\Delta), \tag{4.3}$$

where $d(w, \partial f_\mu(\Delta)) = \inf\{|\zeta - w| : \zeta \in \partial f_\mu(\Delta)\}$. Consequently, for each fixed $x \in l^2$, the function

$$P_\mu x = \sum_{n=1}^{\infty}\frac{x_n}{\sqrt{n}}F_n'(\mu) \tag{4.4}$$

converges absolutely and locally uniformly in $f_\mu(\Delta)$ and thus represents an analytic function in $f_\mu(\Delta)$. It was proved in [24] that

$$\frac{1}{\pi}\iint_{f_\mu(\Delta)}|P_\mu x(w)|^2\mathrm{d}u\mathrm{d}v = \|x\|^2 - \|G([\mu])x\|^2. \tag{4.5}$$

It was also proved there that $\{P_\mu x^{(j)}\}$ converges to zero locally uniformly in $f_\mu(\Delta)$ when $(x^{(j)})$ converges to zero weakly.

Using the operator P_μ, the Grunsky differential operator (2.10) has the following simple expression

$$\mathrm{d}_\mu\widetilde{G}(\nu)(x_m) = \Big(\frac{1}{\sqrt{m}\pi}\iint_{f_\mu(\Delta)}\Big(\frac{\nu}{1-|\mu|^2}\frac{\overline{\partial f_\mu}}{\partial f_\mu}\Big)\circ f_\mu^{-1}F_m'(\mu)P_\mu x\mathrm{d}u\mathrm{d}v\Big). \tag{4.6}$$

Therefore,

$$\langle \mathrm{d}_\mu\widetilde{G}(\nu)x, \overline{x}\rangle = \frac{1}{\pi}\iint_{f_\mu(\Delta)}\Big(\frac{\nu}{1-|\mu|^2}\frac{\overline{\partial f_\mu}}{\partial f_\mu}\Big)\circ f_\mu^{-1}(P_\mu x)^2\mathrm{d}u\mathrm{d}v, \tag{4.7}$$

$$\|\mathrm{d}_\mu\widetilde{G}(\nu)x\|^2 = \frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m}\Big|\iint_{f_\mu(\Delta)}\Big(\frac{\nu}{1-|\mu|^2}\frac{\overline{\partial f_\mu}}{\partial f_\mu}\Big)\circ f_\mu^{-1}F_m'(\mu)P_\mu x\mathrm{d}u\mathrm{d}v\Big|^2. \tag{4.8}$$

By Schur's result (see [21]) again, we have

$$\|\mathrm{d}_\mu \widetilde{G}(\nu)\| = \sup_{x\in S_1(l^2)} \Big|\frac{1}{\pi}\iint_{f_\mu(\Delta)} \Big(\frac{\nu}{1-|\mu|^2}\frac{\partial f_\mu}{\overline{\partial f_\mu}}\Big)\circ f_\mu^{-1}(P_\mu x)^2 \mathrm{d}u\mathrm{d}v\Big|. \tag{4.9}$$

Lemma 4.2 *For any $\mu\in M(\Delta)$, $\nu\in L^\infty(\Delta)$, and $x\in l^2$, we have*

$$\|\mathrm{d}_\mu \widetilde{G}(\nu)x\|^2 \le \frac{1}{\pi}\iint_{f_\mu(\Delta)} \Big|\frac{\nu}{1-|\mu|^2}\Big|^2 \circ f_\mu^{-1}|P_\mu x|^2 \mathrm{d}u\mathrm{d}v. \tag{4.10}$$

Proof We first recall the Hilbert transformation $\mathcal{H}$ defined by the Cauchy principle value integral

$$\mathcal{H}(\lambda)(\zeta) = \frac{1}{\pi}\iint_{\mathbb{C}} \frac{\overline{\lambda}(w)}{(w-\zeta)^2}\mathrm{d}u\mathrm{d}v. \tag{4.11}$$

It is well known that $\mathcal{H}$ is an isometric isomorphism on the space of square integral functions on $\mathbb{C}$.

It follows from (2.11) that

$$\frac{f_\mu'(z)}{(f_\mu(z)-w)^2} = \sum_{m=1}^\infty F_m'(\mu)(w)z^{-m-1}, \quad w\in f_\mu(\Delta),\ z\in f_\mu(\Delta^*). \tag{4.12}$$

Combining this with (4.8) we obtain

$$\begin{aligned}
&\|\mathrm{d}_\mu \widetilde{G}(\nu)x\|^2\\
&= \frac{1}{\pi^3}\iint_{\Delta^*}\Big|\sum_{m=1}^\infty \Big(\iint_{f_\mu(\Delta)}\Big(\frac{\nu}{1-|\mu|^2}\frac{\partial f_\mu}{\overline{\partial f_\mu}}\Big)\circ f_\mu^{-1}F_m'(\mu)P_\mu x\mathrm{d}u\mathrm{d}v\Big)z^{-m-1}\Big|^2\mathrm{d}x\mathrm{d}y\\
&= \frac{1}{\pi^3}\iint_{\Delta^*}\Big|\iint_{f_\mu(\Delta)}\Big(\frac{\nu}{1-|\mu|^2}\frac{\partial f_\mu}{\overline{\partial f_\mu}}\Big)\circ f_\mu^{-1}(w)\frac{f_\mu'(z)P_\mu x(w)}{(f_\mu(z)-w)^2}\mathrm{d}u\mathrm{d}v\Big|^2\mathrm{d}x\mathrm{d}y\\
&= \frac{1}{\pi^3}\iint_{f_\mu(\Delta^*)}\Big|\iint_{f_\mu(\Delta)}\Big(\frac{\nu}{1-|\mu|^2}\frac{\partial f_\mu}{\overline{\partial f_\mu}}\Big)\circ f_\mu^{-1}(w)\frac{P_\mu x(w)}{(\zeta-w)^2}\mathrm{d}u\mathrm{d}v\Big|^2\mathrm{d}\xi\mathrm{d}\eta\\
&= \frac{1}{\pi}\iint_{f_\mu(\Delta^*)}\Big|\mathcal{H}\Big(\overline{\Big(\frac{\nu}{1-|\mu|^2}\frac{\partial f_\mu}{\overline{\partial f_\mu}}\Big)\circ f_\mu^{-1}P_\mu x\chi(f_\mu(\Delta))}\Big)(\zeta)\Big|^2\mathrm{d}\xi\mathrm{d}\eta\\
&\le \frac{1}{\pi}\iint_{f_\mu(\Delta)}\Big|\frac{\nu}{1-|\mu|^2}\Big|^2\circ f_\mu^{-1}(w)|P_\mu x(w)|^2\mathrm{d}u\mathrm{d}v,
\end{aligned}$$

by the norm-preserving property of the Hilbert transformation $\mathcal{H}$, where $\chi(D)$ denotes the characteristic function of a set D.

Lemma 4.3 *For any $\mu\in M(\Delta)$ and $\nu\in L^\infty(\Delta)$, we have*

$$\|\mathrm{d}_\mu \widetilde{G}(\nu)\|_e \lesssim C(\|\mu\|_\infty)\inf\{\|\nu|_{\Delta-E}\|_\infty : E\subset \Delta \mathit{compact}\}. \tag{4.13}$$

Proof For any compact subset E of Δ and any degenerating sequence $(x^{(j)})$ in l^2, we obtain from (4.10) that

$$\begin{aligned}
&\|\mathrm{d}_\mu \widetilde{G}(\nu)x^{(j)}\|^2\\
&\le \frac{1}{\pi}\iint_{f_\mu(\Delta)}\Big|\frac{\nu}{1-|\mu|^2}\Big|^2\circ f_\mu^{-1}|P_\mu x^{(j)}|^2\mathrm{d}u\mathrm{d}v\\
&\lesssim C(\|\mu\|_\infty)\Big(\|\nu\|_\infty^2\iint_{f_\mu(E)}|P_\mu x^{(j)}|^2\mathrm{d}u\mathrm{d}v + \|\nu|_{\Delta-E}\|_\infty^2\iint_{f_\mu(\Delta-E)}|P_\mu x^{(j)}|^2\mathrm{d}u\mathrm{d}v\Big).
\end{aligned}$$

Recalling that $\{P_\mu x^{(j)}\}$ converges to zero locally uniformly in $f_\mu(\Delta)$ when $j\to\infty$, we obtain $\limsup\limits_{j\to\infty}\|\mathrm{d}_\mu\widetilde{G}(\nu)x^{(j)}\|\le C(\|\mu\|_\infty)\|\nu|_{\Delta-E}\|_\infty$. By the arbitrariness of E and $(x^{(j)})$, we obtain (4.13) as desired.

Proof of Theorem 2.2 It follows from Lemmas 3.1, 4.1 and 4.3.

5 Proof of Theorem 2.3

In this section, we will prove Theorem 2.3. Recall that, for $p\ge 2$, a compact operator L from a Hilbert space H into itself is a p-Schatten class operator if and only if $\sum\limits_{j=1}^{\infty}\|Lx^{(j)}\|^p<\infty$ for any orthonormal basis $(x^{(j)})$ of H (see [35]). A 2-Schatten class operator is also called a Hilbert-Schmidt operator. Now Lemma 3.1 implies that (1) $\Rightarrow$ (2). We need to prove (2) $\Rightarrow$ (3) $\Rightarrow$ (1). For simplicity, set as before that $f=f_\mu$, $g=f^{-1}$, $D=f(\Delta)$, $D^*=f(\Delta^*)$, and $F_m(\mu)=F_m$, $P_\mu=P$.

(2) $\Rightarrow$ (3) Suppose that there exists $\widetilde{\nu}\in L^\infty(\Delta)$ with $\mathrm{d}_\mu\mathcal{S}(\widetilde{\nu})=\mathrm{d}_\mu\mathcal{S}(\nu)$ such that $\widetilde{\nu}\circ g\in\mathcal{L}^p(D)$. Since $\mathrm{d}_\mu\widetilde{G}(\widetilde{\nu})=\mathrm{d}_\mu\widetilde{G}(\nu)$, we may assume without loss of generality that $\nu\circ g\in\mathcal{L}^p(D)$. Let $(x^{(j)})$ be any orthonormal basic of l^2. By (4.10) we have

$$\|\mathrm{d}_\mu\widetilde{G}(\nu)x^{(j)}\|^2\lesssim C_1(\|\mu\|_\infty)\iint_D|\nu|^2\circ g|Px^{(j)}|^2\mathrm{d}u\mathrm{d}v.$$

The Hölder inequality yields

$$\begin{aligned}\|\mathrm{d}_\mu\widetilde{G}(\nu)x^{(j)}\|^p&\lesssim C_2(\|\mu\|_\infty)\iint_D|\nu|^p\circ g|Px^{(j)}|^2\mathrm{d}u\mathrm{d}v\Big(\iint_D|Px^{(j)}|^2\mathrm{d}u\mathrm{d}v\Big)^{\frac{p}{2}-1}\\&\lesssim C_2(\|\mu\|_\infty)\iint_D|\nu|^p\circ g|Px^{(j)}|^2\mathrm{d}u\mathrm{d}v,\end{aligned}$$

which implies that

$$\sum_{j=1}^{\infty}\|\mathrm{d}_\mu\widetilde{G}(\nu)x^{(j)}\|^p\lesssim C_2(\|\mu\|_\infty)\iint_D|\nu(g(w))|^p\Big(\sum_{j=1}^{\infty}|Px^{(j)}(w)|^2\Big)\mathrm{d}u\mathrm{d}v.\tag{5.1}$$

Noting that

$$Px^{(j)}(w)=\sum_{m=1}^{\infty}\frac{x_m^{(j)}}{\sqrt{m}}F'_m(w)=\Big\langle\Big(\frac{F'_m(w)}{\sqrt{m}}\Big),\overline{x^{(j)}}\Big\rangle,$$

and $(x^{(j)})$ is an orthonormal basic of l^2, we conclude that

$$\sum_{j=1}^{\infty}|Px^{(j)}(w)|^2=\sum_{j=1}^{\infty}\Big|\Big\langle\Big(\frac{F'_m(w)}{\sqrt{m}}\Big),\overline{x^{(j)}}\Big\rangle\Big|^2=\sum_{m=1}^{\infty}\frac{|F'_m(w)|^2}{m}.\tag{5.2}$$

It follows from (4.3), (5.1) and (5.2) that

$$\sum_{j=1}^{\infty}\|\mathrm{d}_\mu\widetilde{G}(\nu)x^{(j)}\|^p\lesssim C_3(\mu)\iint_D|\nu(g(w))|^p\lambda_D^2(w)\mathrm{d}u\mathrm{d}v<\infty.$$

Thus, $\mathrm{d}_\mu\widetilde{G}(\nu)$ is a p-Schatten class operator from l^2 into itself.

(3) $\Rightarrow$ (1) We will use the atomic decomposition of Bergman functions (see [34]) following an idea of Jones [13]. Let $\mathcal{A}^2$ denote the Bergman space in the usual sense, which is the Hilbert space of all holomorphic functions ϕ in Δ^* with the inner product and norm

$$\langle \phi, \psi \rangle = \frac{1}{\pi} \iint_{\Delta^*} \phi(w)\overline{\psi(w)}\mathrm{d}u\mathrm{d}v, \quad \|\phi\| = \Big(\frac{1}{\pi} \iint_{\Delta^*} |\phi(w)|^2 \mathrm{d}u\mathrm{d}v\Big)^{\frac{1}{2}} < \infty. \tag{5.3}$$

For each $x \in l^2$, set

$$P_0 x(w) = \sum_{m=1}^{\infty} \sqrt{m} x_m w^{-m-1}, \quad w \in \Delta^*. \tag{5.4}$$

It is easy to see that P_0 is an isometric linear isomorphism from l^2 onto $\mathcal{A}^2$. In particular, for $z \in \Delta^*$, we consider x_z as in the proof of Lemma 4.1. Then a direct computation shows that

$$\phi_z(w) \doteq P_0 x_z(w) = \frac{|z|^2 - 1}{(1 - zw)^2}, \quad w \in \Delta^*. \tag{5.5}$$

Let ρ denote the hyperbolic distance in Δ^*. Precisely,

$$\rho(z, \zeta) = \frac{1}{2} \log \frac{1 + \left|\dfrac{\zeta - z}{1 - \overline{z}\zeta}\right|}{1 - \left|\dfrac{\zeta - z}{1 - \overline{z}\zeta}\right|}, \quad z, \zeta \in \Delta^*.$$

A sequence (z_j) in Δ^* is called an r-lattice if $\Delta^* = \bigcup_{j=1}^{\infty} D(z_j, r)$ and $\rho(z_i, z_j) \geq \frac{r}{2}$ whenever $i \neq j$, where $D(z, r) = \{\zeta : \rho(z, \zeta) < r\}$ is the hyperbolic disk. For each $x = (x_m) \in l^2$, set

$$Lx(w) = \sum_{j=1}^{\infty} x_j \phi_{z_j}(w) = \sum_{j=1}^{\infty} x_j P_0 x_{z_j}(w) = \sum_{j=1}^{\infty} x_j \frac{|z_j|^2 - 1}{(1 - z_j w)^2}. \tag{5.6}$$

The atomic decomposition of Bergman functions says that, when r is small, L is a bounded operator from l^2 onto $\mathcal{A}^2$ (see [35]).

Now we suppose that $\mathrm{d}_\mu \widetilde{G}(\nu)$ is a p-Schatten class operator from l^2 into itself. Then $\mathrm{d}_\mu \widetilde{G}(\nu) \circ P_0^{-1} \circ L$ is also a p-Schatten class operator from l^2 into itself. Set $e_j = (x_m) \in l^2$ by $x_j = 1$ and $x_m = 0$ whenever $m \neq j$. Then $e_1, e_2, \cdots, e_j, \cdots$ is an orthonormal basic for l^2. So we have

$$\sum_{j=1}^{\infty} \|\mathrm{d}_\mu \widetilde{G}(\nu) \circ P_0^{-1} \circ L(e_j)\|^p < \infty.$$

Noting that $P_0^{-1} \circ L(e_j) = P_0^{-1}(\phi_{z_j}) = x_{z_j}$, we obtain

$$\sum_{j=1}^{\infty} |\langle \mathrm{d}_\mu \widetilde{G}(\nu) x_{z_j}, \overline{x_{z_j}} \rangle|^p \leq \sum_{j=1}^{\infty} \|\mathrm{d}_\mu \widetilde{G}(\nu) x_{z_j}\|^p < \infty,$$

which is by (4.2) equivalent to

$$\sum_{j=1}^{\infty} |\mathrm{d}_\mu \mathcal{S}(\nu)(z_j)|^p (|z_j|^2 - 1)^{2p} < \infty. \tag{5.7}$$

On the other hand, from [17] we known that for any analytic function ϕ in Δ^*, it holds that

$$\iint_{\Delta^*} |\phi(z)|^p(|z|^2-1)^{2p-2}\mathrm{d}x\mathrm{d}y \asymp \sum_{j=1}^{\infty}|\phi(z_j)|^p(|z_j|^2-1)^{2p},$$

which implies by (5.7) that

$$\iint_{\Delta^*} |\mathrm{d}_\mu S(\nu)(z)|^p(|z|^2-1)^{2p-2}\mathrm{d}x\mathrm{d}y < \infty$$

as desired.

It should be pointed out that (1) $\Leftrightarrow$ (2) in Theorem 2.3 was first proved by Zhai [33], even for $1 \le p < 2$. Now we show that (2) $\Rightarrow$ (3) in Theorem 2.3 also holds when $1 \le p < 2$. We state this as following proposition.

Proposition 5.1 *Let $p \ge 1$ be a fixed number. If $\mu \in M$ and $\nu \in L^\infty$ satisfy $\nu \circ f_\mu^{-1} \in \mathcal{L}^p(f_\mu(\Delta))$, then $\mathrm{d}_\mu\widetilde{G}(\nu)$ is a p-Schatten class operator.*

Proof Recall that, for $p \ge 1$, a compact operator L from a Hilbert space H into itself is a p-Schatten class operator if and only if $\sum_{j=1}^{\infty}|\langle Lx^{(j)}, x^{(j)}\rangle|^p < \infty$ for any orthonormal basis $(x^{(j)})$ of H (see [35]). Set as above that $f = f_\mu$, $g = f^{-1}$, $D = f(\Delta)$, $D^* = f(\Delta^*)$, $F_m(\mu) = F_m$, $P_\mu = P$. It follows from (4.6) that for $x, y \in l^2$

$$\langle \mathrm{d}_\mu\widetilde{G}(\nu)x, y\rangle = \frac{1}{\pi}\iint_D \Big(\frac{\nu}{1-|\mu|^2}\frac{\partial f}{\overline{\partial f}}\Big)\circ gPxP\overline{y}\mathrm{d}u\mathrm{d}v. \tag{5.8}$$

Let $(x^{(j)})$ be any orthonormal basic of l^2. Then we have from (5.8) that

$$|\langle \mathrm{d}_\mu\widetilde{G}(\nu)x^{(j)}, x^{(j)}\rangle| \lesssim C(\|\mu\|_\infty)\iint_D |\nu|\circ g|Px^{(j)}P\overline{x^{(j)}}|\mathrm{d}u\mathrm{d}v. \tag{5.9}$$

Then

$$\sum_{j=1}^{\infty}|\langle \mathrm{d}_\mu\widetilde{G}(\nu)x^{(j)}, x^{(j)}\rangle| \lesssim C(\|\mu\|_\infty)\iint_D |\nu|\circ g\Big(\sum_{j=1}^{\infty}|Px^{(j)}P\overline{x^{(j)}}|\Big)\mathrm{d}u\mathrm{d}v.$$

Noting that

$$\Big(\sum_{j=1}^{\infty}|Px^{(j)}(w)P\overline{x^{(j)}}(w)|\Big)^2 \le \sum_{j=1}^{\infty}|Px^{(j)}(w)|^2\sum_{j=1}^{\infty}|P\overline{x^{(j)}}(w)|^2 = \Big(\sum_{m=1}^{\infty}\frac{|F_m'(w)|^2}{m}\Big)^2$$

by (5.2), we conclude by (4.3) again that

$$\sum_{j=1}^{\infty}|Px^{(j)}(w)P\overline{x^{(j)}}(w)| \lesssim C_1(\mu)\lambda_D^2(w). \tag{5.10}$$

Consequently,

$$\sum_{j=1}^{\infty}|\langle \mathrm{d}_\mu\widetilde{G}(\nu)x^{(j)}, x^{(j)}\rangle| \lesssim C_2(\mu)\iint_D |\nu|\circ g\lambda_D^2\mathrm{d}u\mathrm{d}v < \infty.$$

This proves the proposition in the case $p = 1$.

When $p > 1$, we obtain from (5.9) that

$$
\begin{aligned}
&|\langle \mathrm{d}_\mu \widetilde{G}(\nu)x^{(j)}, x^{(j)}\rangle|^p \\
&\lesssim C_3(\|\mu\|_\infty) \iint_D |\nu|^p \circ g |Px^{(j)} P\overline{x^{(j)}}| \mathrm{d}u\mathrm{d}v \Big(\iint_D |Px^{(j)}|^2 \mathrm{d}u\mathrm{d}v \iint_D |P\overline{x^{(j)}}|^2 \mathrm{d}u\mathrm{d}v \Big)^{\frac{p-1}{2}} \\
&\lesssim C_3(\|\mu\|_\infty) \iint_D |\nu|^p \circ g |Px^{(j)} P\overline{x^{(j)}}| \mathrm{d}u\mathrm{d}v.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{j=1}^{\infty} |\langle \mathrm{d}_\mu \widetilde{G}(\nu)x^{(j)}, x^{(j)}\rangle|^p &\lesssim C_3(\|\mu\|_\infty) \iint_D |\nu|^p \circ g \Big(\sum_{j=1}^{\infty} |Px^{(j)} P\overline{x^{(j)}}| \Big) \mathrm{d}u\mathrm{d}v \\
&\lesssim C_4(\mu) \iint_D |\nu|^p \circ g \lambda_D^2 \mathrm{d}u\mathrm{d}v < \infty,
\end{aligned}
$$

by (5.10). This completes the proof.

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