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Asymptotic Properties for a Semilinear Edge-Degenerate Parabolic Equation

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Abstract The author deals with a semi-linear edge-degenerate parabolic equation, and proves that the solution increases exponentially under the initial energy $J(u_0) \leq d$, where d is the mountain-pass level. Moreover, the author estimates the blow-up time and the blow-up rate for the solution under $J(u_0) < 0$.

Keywords Semilinear edge-degenerate parabolic equation, Exponential increase, Potential well method
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1 Introduction

Let X be a closed compact C^∞-smooth sub-manifold of dimension n embedded in the unit sphere of $\mathbb{R}^{n+1}$, and let Y be a bounded subset in $\mathbb{R}^q$ containing the origin. Assume $\mathbb{E} = [0,1) \times X \times Y$, which can be regarded as the local model near the boundary of stretched edge-manifolds, i.e., manifolds with edge singularities. Denote $\mathbb{E}_0$ for the interior of $\mathbb{E}$, and the boundary of $\mathbb{E}$ by $\partial\mathbb{E} = \{0\} \times X \times Y$ (see [2–3]).

In this paper, we consider initial boundary value problem with the following semi-linear edge-degenerate parabolic equation

$$\begin{cases} \partial_t u - \Delta_{\mathbb{E}} u - \varepsilon V u = |u|^{p-1}u, & 0 < t < T, \ (w,x,y) \in \mathbb{E}_0, \\ u(t,w,x,y) = 0, & 0 < t < T, \ (w,x,y) \in \partial\mathbb{E}, \\ u(0,w,x,y) = u_0(w,x,y), & (w,x,y) \in \mathbb{E}_0, \end{cases} \tag{1.1}$$

where $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$, and $\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ is called an edge Sobolev space (see [2, 6]), $T \in (0,+\infty]$ is the maximal existence time of the solution. $1 < p < \frac{N+2}{N-2}, N = 1+n+q \geq 3$ is the dimension of $\mathbb{E}$ and the coordinates $(w,x,y) = (w, x_1, \cdots, x_n, y_1, \cdots, y_q) \in \mathbb{E}$. The edge-Laplacian operator is defined as

$$\Delta_{\mathbb{E}} = \nabla_{\mathbb{E}}^2 = (w\partial_w)^2 + \partial_{x_1}^2 + \cdots + \partial_{x_n}^2 + (w\partial_{y_1})^2 + \cdots + (w\partial_{y_q})^2,$$

the corresponding gradient operator is $\nabla_{\mathbb{E}} = (w\partial_w, \partial_{x_1}, \cdots, \partial_{x_n}, w\partial_{y_1}, \cdots, w\partial_{y_q})$. Assume that V is a positive potential function which can be unbounded on the edge manifold $\mathbb{E}$ and is

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controlled by the following edge type Hardy's inequality

$$\int_{\mathbb{E}} w^q V|u|^2 \mathrm{d}\sigma \le C\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}$$

for all $u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$, where

$$\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E}) = \mathcal{H}_{2,0}^{0,\frac{n+1}{2}}(\mathbb{E}), \quad \mathrm{d}\sigma = (w^{-1}\mathrm{d}w)\mathrm{d}x_1\cdots\mathrm{d}x_n(w^{-1}\mathrm{d}y_1)\cdots(w^{-1}\mathrm{d}y_q).$$

For all $p \in (1, +\infty)$, the norm in the space $\mathcal{L}_p^{\frac{n+1}{p}}(\mathbb{E}) = \mathcal{H}_{p,0}^{0,\frac{n+1}{p}}(\mathbb{E})$ is defined as follow

$$\|u\|_{\mathcal{L}_p^{\frac{n+1}{p}}(\mathbb{E})} \triangleq \Big(\int_{\mathbb{E}} w^q|u|^p \mathrm{d}\sigma\Big)^{\frac{1}{p}}.$$

As in [2], we let

$$0 < \varepsilon < \frac{1}{C^{*2}}, \tag{1.2}$$

where

$$C^* \triangleq \sup\Big\{\frac{\|\sqrt{V}u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}}{\|\nabla_{\mathbb{E}} u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}} : u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}), \|\nabla_{\mathbb{E}} u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0\Big\}. \tag{1.3}$$

Since $1 < p < \frac{N+2}{N-2}$, by [2, Propositions 2], we know that $\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \hookrightarrow \mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})$ compactly, and there exists a optimal positive constant C_* such that

$$\|u\|_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})} \le C_*\|\nabla_{\mathbb{E}} u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}. \tag{1.4}$$

Problem (1.1) was studied in [2, 6, 13], in order to introduce those results, we introduce some necessary notations, definitions and sets. We define the energy functional

$$J(u) \triangleq \frac{1}{2}\int_{\mathbb{E}} w^q|\nabla_{\mathbb{E}} u|^2 \mathrm{d}\sigma - \frac{1}{2}\int_{\mathbb{E}} w^q \varepsilon V|u|^2 \mathrm{d}\sigma - \frac{1}{p+1}\int_{\mathbb{E}} w^q|u|^{p+1}\mathrm{d}\sigma, \tag{1.5}$$

then the corresponding Nehari functional and Nehari manifold can be defined as follow:

$$I(u) \triangleq \int_{\mathbb{E}} w^q|\nabla_{\mathbb{E}} u|^2 \mathrm{d}\sigma - \int_{\mathbb{E}} w^q \varepsilon V|u|^2 \mathrm{d}\sigma - \int_{\mathbb{E}} w^q|u|^{p+1}\mathrm{d}\sigma, \tag{1.6}$$

$$\mathcal{N} \triangleq \Big\{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) : I(u) = 0, \int_{\mathbb{E}} w^q|\nabla_{\mathbb{E}} u|^2 \mathrm{d}\sigma \neq 0\Big\}. \tag{1.7}$$

Further, we denote the mountain-pass level by

$$d = \inf_{u\in\mathcal{N}} J(u). \tag{1.8}$$

For $\delta > 0$, we define

$$\begin{aligned} &I_\delta(u) \triangleq \delta\int_{\mathbb{E}} w^q|\nabla_{\mathbb{E}} u|^2 \mathrm{d}\sigma - \delta\int_{\mathbb{E}} w^q \varepsilon V|u|^2 \mathrm{d}\sigma - \int_{\mathbb{E}} w^q|u|^{p+1}\mathrm{d}\sigma,\\ &\mathcal{N}_\delta \triangleq \{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) : I_\delta(u) = 0\} \setminus \{0\},\\ &d_\delta \triangleq \inf_{u\in\mathcal{N}_\delta} J(u). \end{aligned} \tag{1.9}$$

For $0 < \delta < \frac{p+1}{2}$, we let

$$\begin{aligned} \mathcal{W}_\delta &= \{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) : J(u) < d(\delta), I_\delta(u) > 0\} \cup \{0\}, \\ \mathcal{V}_\delta &= \{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) : J(u) < d(\delta), I_\delta(u) < 0\}. \end{aligned} \tag{1.10}$$

Finally, we give the definition of the weak solution to problem (1.1).

Definition 1.1 (see [2, Definition 1.1]) *Function $u = u(t, w, x, y)$ is called a weak solution of problem* (1.1) *on* $[0, T) \times \mathbb{E}$ *in which T is either infinity or the limit of the existence interval of solution, if it satisfies*

(1) $u \in L^\infty(0, T; \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}))$ *and* $\partial_t u \in L^2(0, T; \mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E}))$;

(2) $u(0, w, x, y) = u_0(w, x, y)$ *in* $\mathbb{E}_0$;

(3) *for any* $v \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ *and* $t \in (0, T), u$ *satisfies:*

$$\int_{\mathbb{E}} w^q \partial_t u \cdot v \mathrm{d}\sigma + \int_{\mathbb{E}} w^q \nabla_{\mathbb{E}} u \cdot \nabla_{\mathbb{E}} v \mathrm{d}\sigma - \int_{\mathbb{E}} w^q \varepsilon V u \cdot v \mathrm{d}\sigma = \int_{\mathbb{E}} w^q |u|^{p-1} u \cdot v \mathrm{d}\sigma. \tag{1.11}$$

In [2], by introducing a family of potential wells, the authors proved that the weak solution of problem (1.1) exists globally and blows up in finite time under some appropriate initial conditions with $J(u_0) \le d$ respectively. Moreover, they obtained the exponential asymptotic behavior for the global weak solutions. For the blow-up weak solution, the authors given a lower bound estimate for blow-up time with $J(u_0) \le d$. The upper bound estimate of blow-up time was given in [13]. In [13], the authors given a upper bound estimate of blow-up time and blow-up rate for the blow-up solution of problem (1.1) with $0 \le J(u_0) < d$. In [6], using a new functional, the authors derived the possibility of finite time blow-up for the weak solution with high initial energy.

However, up to our knowledge, there is no paper consider asymptotic behavior of blow-up solution for problem (1.1), further, we note that the upper bound estimate of blow-up time with $J(u_0) < 0$ is still unsolved. The purpose of this paper is to study those questions.

First, by utilizing the method in [7, 9, 10, 12, 14], we get a upper bound estimate of blow-up time and blow-up rate for the blow-up solution to problem (1.1) with $J(u_0) < 0$.

Theorem 1.1 *Let* $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}), J(u_0) < 0$. *Assume $u(t) = u(x, t)$ is a weak solution of problem* (1.1), *then $u(t)$ blows up at a finite time T. Moreover, the upper bound estimate of T be given by*

$$T \le -\frac{1}{p^2 - 1} \frac{\|u_0\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}}{J(u_0)}.$$

Moreover, the upper bound for the blow-up rate can be given by

$$\|u(\cdot, t)\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \le 2^{\frac{2}{1-p}} \Big[-\frac{(p+1)J(u_0)}{(p-1)\|u_0\|^{p+1}_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}} \Big]^{\frac{1}{1-p}} (T - t)^{-\frac{1}{p-1}}.$$

Secondly, we prove that the solution of problem (1.1) increase exponentially under the initial energy $J(u_0) \le d$.

Theorem 1.2 *Let $J(u_0) \le d, I(u_0) < 0$. Assume $u(t) = u(x,t)$ is a weak solution of problem* (1.1)*, then $u(t)$ increase exponentially in $\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})$-norm. In detail, we have*

(1) *if $J(u_0) < d, I(u_0) < 0$, then for all $t \in [0, T)$ we get*

$$\|u(t)\|^{p+1}_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})} \ge C_5 \mathrm{e}^{C_3 t} - \frac{d}{C_4};$$

(2) *if $J(u_0) = d, I(u_0) < 0$, then there exists a sufficiently small t_1, for all $t \in [t_1, T)$ we get*

$$\|u(t)\|^{p+1}_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})} \ge C_6 \mathrm{e}^{C_3 t} - \frac{d}{C_4},$$

where C_i $(i = 3, 4, 5, 6)$ are positive, time-independent constants and will be given later.

The rest of this paper is organized as follows. In Section 2, we give some important lemmas, which will be used in the proof of the main results. In Section 3, we give the proof of the above theorems.

2 Preliminaries

We begin this section with following lemma, which will be used later.

Lemma 2.1 (see [2, Remark 1]) *Let u be a solution of problem* (1.1)*, then the functional $J(u(t))$ defined in* (1.5) *is non-increasing with respect to t and*

$$\int_0^t \|\partial_\tau u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \mathrm{d}\tau + J(u(t)) = J(u_0), \quad 0 \le t < T. \tag{2.1}$$

Next, we introduce the edge type Hölder inequality and Poincaré inequality.

Lemma 2.2 (see [2]) *If $u \in \mathcal{L}_p^{\frac{n+1}{p}}(\mathbb{E}), v \in \mathcal{L}_{p'}^{\frac{n+1}{p'}}(\mathbb{E})$ with $p, p' \in (1, +\infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then we have the following edge type Hölder inequality*

$$\int_{\mathbb{E}} w^q |uv| \mathrm{d}\sigma \le \Big(\int_{\mathbb{E}} w^q |u|^p \mathrm{d}\sigma\Big)^{\frac{1}{p}} \Big(\int_{\mathbb{E}} w^q |v|^{p'} \mathrm{d}\sigma\Big)^{\frac{1}{p'}}.$$

Moreover, if $u(w,x,y) \in \mathcal{H}_{p,0}^{1,\gamma}(\mathbb{R}_+^N)$ for $1 < p < +\infty, \gamma \in \mathbb{R}_+$, then we have the following edge type Poincaré inequality

$$\|u(w,x,y)\|_{\mathcal{L}_p^\gamma(\mathbb{E})} \le d_{\mathbb{E}} \|\nabla_{\mathbb{E}} u(w,x,y)\|_{\mathcal{L}_p^\gamma(\mathbb{E})}, \tag{2.2}$$

where $d_{\mathbb{E}}$ is the diameter of $\mathbb{E}$.

Lemma 2.3 (see [2, Lemma 2.4]) *Assume that $u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}), \|\nabla_{\mathbb{E}} u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \ne 0$, we have*

(1) $\lim\limits_{\lambda \to 0} J(\lambda u) = 0, \quad \lim\limits_{\lambda \to +\infty} J(\lambda u) = -\infty;$

(2) *there exists a unique* $\lambda^* = \lambda^*(u)$, *that is,*

$$\lambda^* = \Big(\frac{\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} - \varepsilon\|\sqrt{V}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}}{\|u\|^{p+1}_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}}\Big)^{\frac{1}{p-1}} \tag{2.3}$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t}J(\lambda u)\Big|_{\lambda=\lambda^*} = 0,$$

and $J(\lambda u)$ *is strictly increasing on* $0 \le \lambda \le \lambda^*$, *strictly decreasing on* $\lambda^* \le \lambda < +\infty$ *and takes the maximum at* $\lambda = \lambda^*$;

(3) $I(\lambda u) > 0$ *for* $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ *for* $\lambda^* < \lambda < +\infty$, *and* $I(\lambda^* u) = 0$.

By the conclusion of Lemma 2.3, we now give the concrete value of d defined in (1.8) as follows.

Lemma 2.4 *For the mountain-pass level* d *defined in* (1.8), *we have*

$$d = \frac{p-1}{2(p+1)}\Big(\frac{1-\varepsilon C^{*2}}{C_*^2}\Big)^{\frac{p+1}{p-1}}. \tag{2.4}$$

Proof In fact, by [1, 5], the mountain pass level d may also be characterized as

$$d = \inf_{u\in\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\setminus\{0\}} \sup_{\lambda\ge 0} J(\lambda u).$$

By Lemma 2.3 we know there exists a unique positive constant λ^* defined in (2.3) such that $\lambda^* u \in \mathcal{N}$, and $J(\lambda u)$ attains its maximum at $\lambda = \lambda^*$. Hence,

$$d = \inf_{u\in\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\setminus\{0\}} \sup_{\lambda\ge 0} J(\lambda u) = \inf_{u\in\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\setminus\{0\}} J(\lambda^* u). \tag{2.5}$$

By the definition of $J(u)$ in (1.5) and (2.3), we have

$$\begin{aligned} J(\lambda^* u) &= \frac{{\lambda^*}^2}{2}\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} - \frac{{\lambda^*}^2}{2}\varepsilon\|\sqrt{V}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{{\lambda^*}^{p+1}}{p+1}\|u\|^{p+1}_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})} \\ &= \frac{p-1}{2(p+1)}\Big(\frac{\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} - \varepsilon\|\sqrt{V}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}}{\|u\|^2_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}}\Big)^{\frac{p+1}{p-1}} \\ &= \frac{p-1}{2(p+1)}\Big[\Big(1 - \frac{\varepsilon\|\sqrt{V}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}}{\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}}\Big) \times \frac{\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}}{\|u\|^2_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}}\Big]^{\frac{p+1}{p-1}}. \end{aligned}$$

Then by the definition of C^*, C_* in (1.3) and (1.4) respectively, we get

$$
\begin{aligned}
&\inf_{u\in\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\setminus\{0\}} J(\lambda^* u)\\
&= \inf_{u\in\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\setminus\{0\}} \frac{p-1}{2(p+1)}\Big[\Big(1-\frac{\varepsilon\|\sqrt{V}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}}{\|\nabla_{\mathbb{E}}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}}\Big)\times\frac{\|\nabla_{\mathbb{E}}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}}{\|u\|^2_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}}\Big]^{\frac{p+1}{p-1}}\\
&= \frac{p-1}{2(p+1)}\Big(\frac{1-\varepsilon C^{*2}}{C_*^2}\Big)^{\frac{p+1}{p-1}}.
\end{aligned}
$$

Lemma 2.5 (see [2, Lemma 2.7]) *Assume $d(\delta)$ is defined in* (1.9), *then it satisfies the following properties*:

(1) $\lim\limits_{\delta\to 0} d(\delta)=0, d\big(\frac{p+1}{2}\big)=0$ *and* $d(\delta)<0$ *for all* $\delta>\frac{p+1}{2}$;

(2) $d(\delta)$ *is increasing on* $0<\delta\le 1$, *decreasing on* $1\le\delta\le\frac{p+1}{2}$ *and takes the maximum* $d=d(1)$ *at* $\delta=1$.

For a constant $e\in(0,d)$, by Lemma 2.5 we know that the equation $d(\delta)=e$ admits two roots. Then, we next show that both sets defined in (1.10) are invariant sets of the solution to problem (1.1).

Lemma 2.6 (see [2, Lemma 2.9]) *Let* $u_0\in\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}), e\in(0,d)$. *Suppose that* $\delta_1<1<\delta_2$ *are two roots of the equation of* $d(\delta)=e$, *then*

(1) *all weak solutions of problem* (1.1) *with* $J(u_0)=e$ *belong to* $\mathcal{W}_\delta$ *for* $\delta\in(\delta_1,\delta_2)$ *provided that* $I(u_0)>0$;

(2) *all weak solutions of problem* (1.1) *with* $J(u_0)=e$ *belong to* $\mathcal{V}_\delta$ *for* $\delta\in(\delta_1,\delta_2)$ *provided that* $I(u_0)<0$.

Lemma 2.7 (see [13, Lemma 2.2]) *Let* $u(t)=u(x,t)$ *be the weak solution of problem* (1.1). *If* $J(u_0)<d, I(u_0)<0$, *then*

$$\|\nabla_{\mathbb{E}}u_0\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}>\alpha_1=\Big(\frac{1-\varepsilon C^{*2}}{C_*^{p+1}}\Big)^{\frac{1}{p-1}}. \tag{2.6}$$

Moreover, there exists a positive constant $\alpha_2>\alpha_1$ *such that*

$$\|\nabla_{\mathbb{E}}u(t)\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\ge\alpha_2,\quad \forall t\ge 0 \tag{2.7}$$

and

$$\|u(t)\|_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}\ge C_*\alpha_2,\quad \forall t\ge 0. \tag{2.8}$$

Lemma 2.8 (see [13, Lemma 2.3]) *For all* $t\in[0,T)$, *let*

$$H(t)=d-J(u(t)). \tag{2.9}$$

If $J(u_0)<d, I(u_0)<0$, *then we have*

$$0<H(0)\le H(t)<\frac{1}{p+1}\|u\|^{p+1}_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}.$$

3 The Proof of Main Results

On the basis of the above lemmas, we are now in a position to prove our main results. We first prove Theorem 1.1.

Proof of Theorem 1.1 Let $u(t) = u(x,t)$ be a weak solution of problem (1.1). We define

$$f(t) = \frac{1}{2}\int_{\mathbb{E}} w^q |u(t)|^2 \mathrm{d}\sigma \tag{3.1}$$

and

$$\begin{aligned} g(t) &= -(p+1)J(u(t)) \\ &= -\frac{p+1}{2}\int_{\mathbb{E}} w^q |\nabla_{\mathbb{E}} u(t)|^2 \mathrm{d}\sigma + \frac{p+1}{2}\int_{\mathbb{E}} w^q \varepsilon V |u(t)|^2 \mathrm{d}\sigma + \int_{\mathbb{E}} w^q |u(t)|^{p+1} \mathrm{d}\sigma. \end{aligned} \tag{3.2}$$

By the definition of $f(t)$, taking $v = u$ in (1.11) we get

$$\begin{aligned} f'(t) &= \int_{\mathbb{E}} w^q u(t) \partial_\tau u(t) \mathrm{d}\sigma \\ &= -\int_{\mathbb{E}} w^q |\nabla_{\mathbb{E}} u(t)|^2 \mathrm{d}\sigma + \int_{\mathbb{E}} w^q \varepsilon V |u(t)|^2 \mathrm{d}\sigma + \int_{\mathbb{E}} w^q |u(t)|^{p+1} \mathrm{d}\sigma. \end{aligned} \tag{3.3}$$

Combining the definition of $g(t)$ and (2.1) we have

$$g'(t) = -(p+1)\frac{\mathrm{d}}{\mathrm{d}t} J(u(t)) = (p+1)\|\partial_\tau u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \geq 0, \tag{3.4}$$

then by $J(u_0) < 0$ and the definition of $g(t)$ in (3.2) we have $g(0) > 0$. So it follows from above inequality that $g(t) > 0$ for all $t \in [0,T)$. Due to the assumption in (1.2) we know that $1 - \varepsilon C^{*2} > 0$, then combine (1.3) we get

$$\begin{aligned} &-\int_{\mathbb{E}} w^q |\nabla_{\mathbb{E}} u|^2 \mathrm{d}\sigma + \int_{\mathbb{E}} w^q \varepsilon V |u|^2 \mathrm{d}\sigma \\ &\leq -\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + \varepsilon C^{*2} \|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &= (\varepsilon C^{*2} - 1)\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &\leq 0. \end{aligned} \tag{3.5}$$

Combining (3.3), (3.2), $p > 1$ and above inequality we have

$$f'(t) \geq g(t) > 0, \quad \forall t \in [0,T). \tag{3.6}$$

Then $f(t) > 0$ for all $t \in (0,T)$.

By (3.1), (3.4), Schwartzs inequality and above inequality, we obtain

$$\begin{aligned} f(t)g'(t) &= \frac{p+1}{2}\|u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \|\partial_\tau u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &\geq \frac{p+1}{2}\Big(\int_{\mathbb{E}} w^q u(t) \partial_t u(t) \mathrm{d}\sigma\Big)^2 \\ &= \frac{p+1}{2}[f'(t)]^2 \\ &\geq \frac{p+1}{2} f'(t) g(t), \end{aligned}$$

which can be rewritten as

$$\frac{g'(t)}{g(t)} \geq \frac{p+1}{2} \frac{f'(t)}{f(t)}. \tag{3.7}$$

Integrating above inequality from 0 to t we get

$$\frac{g(t)}{[f(t)]^{\frac{p+1}{2}}} \geq \frac{g(0)}{[f(0)]^{\frac{p+1}{2}}},$$

then by (3.6), we have

$$\frac{f'(t)}{[f(t)]^{\frac{p+1}{2}}} \geq \frac{g(0)}{[f(0)]^{\frac{p+1}{2}}}. \tag{3.8}$$

Integrating inequality (3.8) from 0 to t, we see

$$\frac{1}{[f(t)]^{\frac{p-1}{2}}} \leq \frac{1}{[f(0)]^{\frac{p-1}{2}}} - \frac{p-1}{2} \frac{g(0)}{[f(0)]^{\frac{p+1}{2}}} t. \tag{3.9}$$

Clearly, (3.9) cannot hold for all time, this means $f(t)$ blows up at some finite time T, i.e.,

$$\lim_{t \to T} f(t) = +\infty. \tag{3.10}$$

Next, we estimate the blow-up time and blow-up rate. Let $t \to T$ in (3.9), by (3.10) and the definition of $f(t), g(t)$ we get

$$T \leq \frac{2}{p-1} \frac{f(0)}{g(0)} = -\frac{1}{p^2-1} \frac{\|u_0\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}}{J(u_0)}.$$

Moreover, integrating inequality (3.8) from t to T, by (3.10) we have

$$f(t) \leq (T-t)^{-\frac{2}{p-1}} \left[\frac{2g(0)}{(p-1)[f(0)]^{\frac{p+1}{2}}} \right]^{\frac{2}{1-p}},$$

then by the definition of $f(t)$ and $g(t)$ we have

$$\|u(\cdot,t)\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \leq 2^{\frac{2}{1-p}} \left[-\frac{(p+1)J(u_0)}{(p-1)\|u_0\|^{p+1}_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}} \right]^{\frac{1}{1-p}} (T-t)^{-\frac{1}{p-1}}.$$

Next, we prove Theorem 1.2 and the idea of the proof comes form [4, 8, 11, 15].

Proof of Theorem 1.2 The proof is divided into two steps.

Step 1 the case $(J(u_0) < d)$ Assume $u(x,t)$ is a solution of problem (1.1), we define a functional

$$G(t) = H(t) + f(t), \quad \forall t \in [0,T), \tag{3.11}$$

where $H(t), f(t)$ defined in (2.9) and (3.1) respectively, then by (2.1), (3.3) and the definition of $I(u)$ we get

$$\begin{aligned} G'(t) &= -\frac{\mathrm{d}}{\mathrm{d}t}J(u(t)) + f'(t) \\ &= \|\partial_\tau u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} - \|\nabla_{\mathbb{E}} u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + \varepsilon\|\sqrt{V}u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + \|u(t)\|^{p+1}_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})} \\ &= \|\partial_\tau u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} - I(u(t)). \end{aligned} \tag{3.12}$$

Using the definition of $J(u), I(u), H(t)$ and (1.3), we obtain

$$\begin{aligned} I(u) &= (p+1)J(u) - \frac{p-1}{2}\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{p-1}{2}\varepsilon\|\sqrt{V}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &\le (p+1)d - (p+1)H(t) - \frac{p-1}{2}(1-\varepsilon C^{*2})\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}. \end{aligned} \tag{3.13}$$

Since $1-\varepsilon C^{*2} > 0$, by (2.7) we have

$$\begin{aligned} &\frac{p-1}{2}(1-\varepsilon C^{*2})\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &= \frac{p-1}{2}(1-\varepsilon C^{*2})\Big(\frac{\alpha_2^2-\alpha_1^2}{\alpha_2^2}\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{\alpha_1^2}{\alpha_2^2}\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\Big) \\ &\ge \frac{p-1}{2}(1-\varepsilon C^{*2})\Big(\frac{\alpha_2^2-\alpha_1^2}{\alpha_2^2}\Big)\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{p-1}{2}(1-\varepsilon C^{*2})\alpha_1^2. \end{aligned} \tag{3.14}$$

It follows form the value of d, α_1 in (2.4) and (2.6) respectively that

$$(p+1)d = \frac{p-1}{2}(1-\varepsilon C^{*2})\alpha_1^2. \tag{3.15}$$

Then combine (3.12)–(3.15) we have

$$\begin{aligned} G'(t) &\ge \|\partial_\tau u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} - (p+1)d + (p+1)H(t) + \frac{p-1}{2}(1-\varepsilon C^{*2})\|\nabla_{\mathbb{E}} u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &\ge (p+1)H(t) + \frac{p-1}{2}(1-\varepsilon C^{*2})\Big(\frac{\alpha_2^2-\alpha_1^2}{\alpha_2^2}\Big)\|\nabla_{\mathbb{E}} u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}. \end{aligned}$$

By Lemma 2.8 that $H(t) > 0$, then take

$$C_1 = \min\Big\{p+1, \quad \frac{p-1}{2}(1-\varepsilon C^{*2})\Big\},$$

so we get

$$G'(t) \ge C_1(H(t) + \|\nabla_{\mathbb{E}} u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}). \tag{3.16}$$

On the other hand, by (2.2) we have

$$G(t) = H(t) + \frac{1}{2}\|u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \le H(t) + \frac{1}{2}(d_{\mathbb{E}})^2\|\nabla_{\mathbb{E}} u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})},$$

taking $C_2=\max\left\{1,\frac{1}{2}(d_{\mathbb{E}})^2\right\}$, then

$$G(t)\le C_2(H(t)+\|\nabla_{\mathbb{E}}u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}). \tag{3.17}$$

Combining (3.16) and (3.17) we get

$$G'(t)\ge C_3G(t),$$

where $C_3=\frac{C_1}{C_2}>0$. It follows form Gronwall inequality that

$$G(t)\ge G(0)\mathrm{e}^{C_3t},\quad \forall t\in[0,T). \tag{3.18}$$

Finally, we estimate the functional $G(t)$. By Lemma 2.6 we know that $\mathcal{V}_1$ is a invariant set, then it follows form $0<J(u_0)<d, I(u_0)<0$ that

$$I(u(t))<0,\quad \forall t\in[0,T). \tag{3.19}$$

If $J(u_0)\le 0$, by the fact that $J(u(t))$ is non-increasing respect to t, we have $J(u(t))\le 0$ for all $t\in[0,T)$. Then by the definition of $J(u(t))$ and $I(u(t))$ we get

$$\frac{1}{2}I(u(t))<J(u(t))\le 0,$$

so for $J(u_0)<d, I(u_0)<0$, we have (3.19) holds.

Together with (3.19) and the definitions of $I(u), C^*$ we obtain

$$\begin{aligned}\|\nabla_{\mathbb{E}}u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}&<\varepsilon\|\sqrt{V}u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}+\|u(t)\|^{p+1}_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}\\&\le\varepsilon C^{*2}\|\nabla_{\mathbb{E}}u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}+\|u(t)\|^{p+1}_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})},\end{aligned}$$

i.e.,

$$\|\nabla_{\mathbb{E}}u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}<\frac{1}{1-\varepsilon C^{*2}}\|u(t)\|^{p+1}_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}. \tag{3.20}$$

Then by the definitions of $G(t), H(t), f(t), J(u)$ and C^*, we have

$$\begin{aligned}G(t)&\le d+\frac{1}{2}\varepsilon\|\sqrt{V}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}+\frac{1}{p+1}\|u\|^{p+1}_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}+\frac{1}{2}\|u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\\&\le d+\frac{1}{2}\varepsilon C^{*2}\|\nabla_{\mathbb{E}}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}+\frac{1}{p+1}\|u\|^{p+1}_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}+\frac{1}{2}\|u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})},\end{aligned}$$

then combine edge type Poincaré inequality and (3.20), we get

$$\begin{aligned}G(t)&\le d+\frac{1}{2}\varepsilon C^{*2}\|\nabla_{\mathbb{E}}u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}+\frac{1}{p+1}\|u(t)\|^{p+1}_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}+\frac{1}{2}(d_{\mathbb{E}})^2\|\nabla_{\mathbb{E}}u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\\&\le d+\Big(\frac{\varepsilon C^{*2}+(d_{\mathbb{E}})^2}{2(1-\varepsilon C^{*2})}+\frac{1}{p+1}\Big)\|u(t)\|^{p+1}_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}.\end{aligned} \tag{3.21}$$

Taking

$$C_4 = \frac{\varepsilon {C^*}^2 + (d_{\mathbb{E}})^2}{2(1 - \varepsilon {C^*}^2)} + \frac{1}{p+1} > 0,$$

then combine (3.18) and (3.21) we have

$$\|u(t)\|^{p+1}_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})} \geq C_5 \mathrm{e}^{C_3 t} - \frac{d}{C_4}, \quad \forall t \in [0, T), \tag{3.22}$$

where

$$C_5 = \frac{1}{C_4}\Big(d - J(u_0) + \frac{1}{2}\|u_0\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\Big) > 0.$$

Then we know that the solution of problem (1.1) grows as an exponential function in $\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})$-norm.

Step 2 the case $(J(u_0) = d)$ By $J(u_0) = d > 0, I(u_0) < 0$ and the continuities of $J(u(t))$ and $I(u(t))$ with respect to t, we know that there exists a sufficiently small $t_1 > 0$ such that $J(u(t_1)) > 0$ and $I(u(t)) < 0$ for $0 \leq t \leq t_1$. Then for $0 \leq t \leq t_1$ it follows from (3.3) that

$$\begin{aligned}
&\int_{\mathbb{E}} w^q u(t) \partial_\tau u(t) \mathrm{d}\sigma \\
&= -\int_{\mathbb{E}} w^q |\nabla_{\mathbb{E}} u(t)|^2 \mathrm{d}\sigma + \int_{\mathbb{E}} w^q \varepsilon V |u(t)|^2 \mathrm{d}\sigma + \int_{\mathbb{E}} w^q |u(t)|^{p+1} \mathrm{d}\sigma \\
&= -I(u(t)) > 0.
\end{aligned} \tag{3.23}$$

On the other hand, by edge type Hölder inequality it is easy to see that

$$\int_{\mathbb{E}} w^q |u \partial_\tau u| \mathrm{d}\sigma \leq \|\partial_\tau u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \|u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}.$$

Then by (3.23), for $0 \leq t \leq t_1$, we have $\|\partial_\tau u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} > 0$.

It follows from the continuity of $\int_0^t \|\partial_\tau u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \mathrm{d}\tau$ that

$$0 < d - \int_0^{t_1} \|\partial_\tau u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \mathrm{d}\tau < d.$$

Then from (2.1) we can get

$$J(u(t_1)) = d - \int_0^{t_1} \|\partial_\tau u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \mathrm{d}\tau < d.$$

Thus if we take $t = t_1$ as the initial time, we can see $J(u(t_1)) < d, I(u(t_1)) < 0$, the remainder of the proof is similar as that in Step 1 and we can obtain

$$\|u(t)\|^{p+1}_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})} \geq C_6 \mathrm{e}^{C_3 t} - \frac{d}{C_4}, \quad \forall t \in [t_1, T),$$

where

$$C_6 = \frac{1}{C_4}\Big(d - J(u(t_1)) + \frac{1}{2}\|u(t_1)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\Big) > 0.$$

Namely, the solution of problem (1.1) grows exponentially in $\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})$-norm.

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