Chin. Ann. Math. Ser. B
41(4), 2020, 601–614
DOI: 10.1007/s11401-020-0220-y

Chinese Annals of Mathematics, Series B

A Unified Boundary Behavior of Large Solutions to Hessian Equations*

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Abstract This paper is concerned with strictly k-convex large solutions to Hessian equations $S_k(D^2u(x)) = b(x)f(u(x)),\ x \in \Omega$, where Ω is a strictly $(k-1)$-convex and bounded smooth domain in $\mathbb{R}^n$, $b \in C^\infty(\overline{\Omega})$ is positive in Ω, but may be vanishing on the boundary. Under a new structure condition on f at infinity, the author studies the refined boundary behavior of such solutions. The results are obtained in a more general setting than those in [Huang, Y., Boundary asymptotical behavior of large solutions to Hessian equations, *Pacific J. Math.,* **244**, 2010, 85–98], where f is regularly varying at infinity with index $p > k$.

Keywords Hessian equations, Strictly k-convex large solutions, Boundary behavior
2000 MR Subject Classification 35J60, 35B40

1 Introduction and the Main Results

For any $n \times n$ real symmetric matrix A, we let $\lambda(A)$ denote the eigenvalues of A and

$$S_k(A) = S_k(\lambda_1, \cdots, \lambda_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k} \tag{1.1}$$

denote the k-th elementary symmetric function, $k = 1, 2, \cdots, n$.

Define the set Γ_k which is the connected component of $\{\lambda \in \mathbb{R}^n : S_k(\lambda) > 0\}$ containing the positive cone

$$\Gamma^+ := \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n : \lambda_i > 0,\ i = 1, 2, \cdots, n\}.$$

It follows from [4] that

$$\Gamma^+ = \Gamma_n \subset \cdots \subset \Gamma_{k+1} \subset \Gamma_k \subset \cdots \subset \Gamma_1.$$

Let $D^2u(x) = \big(\frac{\partial^2 u(x)}{\partial x_i \partial x_j}\big)$ denote the Hessian of $u \in C^2(\Omega)$. We say that a function $u \in C^2(\Omega)$ is k-convex (or strictly k-convex) in Ω if $S_k(D^2u) \in \overline{\Gamma}_k$ (or $S_k(D^2u) \in \Gamma_k$) for all $x \in \Omega$ and $S_k(D^2u)$ turns to be elliptic in the class of k-convex functions.

While, for an open bounded subset Ω of $\mathbb{R}^n$ with boundary of class C^2 and every $\overline{x} \in \partial\Omega$, we denote $\kappa_1(\overline{x}), \cdots, \kappa_{n-1}(\overline{x})$ the principal curvatures of $\partial\Omega$ at $\overline{x}$. For $k \in \{1, 2, \cdots, n-1\}$, we say that Ω is k-convex (or strictly k-convex) if $(\kappa_1(\overline{x}), \cdots, \kappa_{n-1}(\overline{x})) \in \overline{\Gamma}_k$ (or $(\kappa_1(\overline{x}), \cdots, \kappa_{n-1}(\overline{x})) \in$

Manuscript received April 3, 2017. Revised November 29, 2017.
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*This work was supported by the National Natural Science Foundation of China (No. 11571295) and RP of Shandong Higher Education Institutions (No. J17KA173).

Γ_k). In this paper, we analyze the boundary behavior of strictly k-convex solutions to the following boundary blow-up problem

$$S_k(D^2u(x)) = b(x)f(u(x)), \quad x \in \Omega, \ u|_{\partial\Omega} = +\infty, \tag{1.2}$$

where the boundary condition means that $u(x) \to +\infty$ as $d(x) = \text{dist}(x, \partial\Omega) \to 0$, Ω is a strictly $(k-1)$-convex and bounded smooth domain in $\mathbb{R}^n$ with $n \geq 2$, f satisfies:

(f_1) $f \in C[0,\infty)$, $f(0) = 0$ and $f(s)$ is increasing on $[0,\infty)$(or (f_{01}) $f \in C(\mathbb{R})$, $f(s) > 0$, $\forall s \in \mathbb{R}$ and f is increasing on $\mathbb{R}$);

(f_2) the Keller-Osserman type condition (see [16, 29])

$$\Psi_k(r) := \int_r^\infty ((k+1)F(\tau))^{-\frac{1}{k+1}} \mathrm{d}\tau < \infty, \quad \forall r > 0, \quad F(\tau) := \int_0^\tau f(\upsilon)\mathrm{d}\upsilon.$$

b satisfies:

(b_1) $b \in C^\infty(\overline{\Omega})$ is positive in Ω.

And the solution is called 'a large solution' or 'an explosive solution'.

For convenience, we denote ψ_k the inverse of Ψ_k, i.e., ψ_k satisfies

$$\int_{\psi_k(t)}^\infty ((k+1)F(\tau))^{-\frac{1}{k+1}} \mathrm{d}\tau = t, \quad \forall\, t > 0. \tag{1.3}$$

While, we introduce two kinds of functions.

Firstly, the regularly varying function is introduced as follows.

Definition 1.1 *A positive continuous function f defined on $[a,\infty)$, for some $a > 0$, is called regularly varying at infinity with index ρ, denoted $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,*

$$\lim_{s\to\infty} \frac{f(\xi s)}{f(s)} = \xi^\rho. \tag{1.4}$$

In particular, when $\rho = 0$, f is called slowly varying at infinity.

Clearly, if $f \in RV_\rho$, then $L(s) := f(s)/s^\rho$ is slowly varying at infinity.

Some basic examples of slowly varying functions at infinity are

(i) every continuous function on $[a,\infty)$ which has a positive limit at infinity;

(ii) $(\ln s)^q$ and $(\ln(\ln s))^q$, $q \in \mathbb{R}$;

(iii) $\exp((\ln s)^q)$, $0 < q < 1$.

Secondly, let Λ denote the set of all positive nondecreasing functions θ in $C^1(0,\delta_0)$ ($\delta_0 > 0$) which satisfy

$$\lim_{t\to 0^+} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\Theta(t)}{\theta(t)}\Big) := D_\theta \in [0,\infty), \quad \Theta(t) := \int_0^t \theta(\upsilon)\mathrm{d}\upsilon. \tag{1.5}$$

Now let us return to problem (1.2).

Problem (1.2) was widely and deeply researched by many authors and in many contexts, see for instance [1, 18–19, 23, 27, 33] for $k = 1$, i.e., Laplace operator; and [6–8, 11, 20, 24–25, 32, 34] for $k = n$, i.e., Monge-Ampère operator. In particular, Matero [24] established the existence, uniqueness and asymptotic behavior such as $\lim\limits_{d(x)\to 0} \frac{\Psi_n(u(x))}{d(x)}$ of strictly convex solutions under the condition that $b \in C^\infty(\overline{\Omega})$ is positive on $\overline{\Omega}$. Then, by using a perturbation method and Karamata regularly varying theory, and constructing comparison functions, Cîrstea and Trombetti [8] established the existence, uniqueness and refined boundary behavior of solutions for $f \in RV_p$ with $p > n$. Their results were extended to problem (1.2) and k-curvature equation by Colesanti, Salani and Francini [9], Huang [13], Jian [16], Ji and Bao [15], Jin, Li and Xu [17], Nakamori and Takimoto [26], Salani [29], Takimoto [30], Zhang and Zhou [35], respectively. In particular, Huang [13] extended the results of [8] to problem (1.2) for $f \in RV_p$ with $p > k$.

For the existence, regularity theory and other properties of solutions for the Hessian equations, see for instance [2, 4–5, 10, 12, 14, 21–22, 31] and the references therein.

Inspired by the above works, in this paper we investigate the unified boundary behavior of strictly convex solutions to problem (1.2) under the following structure condition on f:

(f_3) there exists $C_{kf} \in (0, \infty]$ such that

$$\lim_{s\to+\infty} H_k'(s)\int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)} = C_{kf}, \quad H_k(s) := ((k+1)F(s))^{\frac{1}{k+1}}, \quad \forall s > 0.$$

A complete characterization of f in (f_3) is provided in Lemma 2.2.

Our main results are summarized as follows.

Theorem 1.1 *Let f satisfy* (f_1)–(f_3), *and b satisfy*

(b_1) *with the additional condition*;

(b_2) *there exist $\theta \in \Lambda$ and positive constants b_{ki} $(i = 1, 2)$ such that*

$$b_{k1} := \liminf_{d(x)\to 0} \frac{b(x)}{\theta^{k+1}(d(x))} \le b_{k2} := \limsup_{d(x)\to 0} \frac{b(x)}{\theta^{k+1}(d(x))}.$$

If

$$C_{kf} > 1, \tag{1.6}$$

or

$$C_{kf} = 1, \quad D_\theta > 0, \tag{1.7}$$

then for any strictly k-convex solution u to problem (1.2), *there hold*

$$1 \le \liminf_{d(x)\to 0} \frac{u(x)}{\psi_k(\xi_{k2}\Theta(d(x)))}, \quad \limsup_{d(x)\to 0} \frac{u(x)}{\psi_k(\xi_{k1}\Theta(d(x)))} \le 1, \tag{1.8}$$

where ψ_k is the solution to (1.3),

$$\xi_{k1} = \Big(\frac{b_{k1}}{M_k(1 - C_{kf}^{-1}(1 - D_\theta))}\Big)^{\frac{1}{k+1}}, \quad \xi_{k2} = \Big(\frac{b_{k2}}{m_k(1 - C_{kf}^{-1}(1 - D_\theta))}\Big)^{\frac{1}{k+1}},$$

and

$$M_k = \max_{\overline{x}\in\partial\Omega} S_k(\kappa_1(\overline{x}), \cdots, \kappa_{n-1}(\overline{x})); \quad m_k = \min_{\overline{x}\in\partial\Omega} S_k(\kappa_1(\overline{x}), \cdots, \kappa_{n-1}(\overline{x})). \tag{1.9}$$

In particular,

(i) *when* $C_{kf}=1$, u *verifies*

$$\lim_{d(x)\to 0}\frac{u(x)}{\psi_k(\Theta(d(x)))}=1; \tag{1.10}$$

(ii) *when* $\Omega=B_R$ *which is a ball of radius* R *centered at the origin,* $r=|x|$, $C_{kf}\in(1,\infty)$ *and* $b_{k1}=b_{k2}=b_{k0}$ *in* (b$_2$), u *verifies*

$$\lim_{r\to R}\frac{u(x)}{\psi_k(\Theta(R-r))}=\Big(\frac{b_{k0}R^{k-1}}{1-C_{kf}^{-1}(1-D_\theta)}\Big)^{\frac{1-C_{kf}}{k+1}}; \tag{1.11}$$

(iii) *when* $\Omega=B_R$, $C_{kf}=\infty$ *and* $b_{k1}=b_{k2}=b_{k0}$ *in* (b$_2$), u *verifies*

$$\lim_{r\to R}\frac{u(x)}{\psi_k(\xi_{k0}\Theta(R-r))}=1, \quad \textit{where } \xi_{k0}=(b_{k0}R^{k-1})^{\frac{1}{k+1}}. \tag{1.12}$$

If f *further satisfies the condition that*

$$\frac{f(s)}{s^k}\ \textit{increasing on } (0,\infty), \tag{1.13}$$

then problem (1.2) *has a unique strictly* k*-convex solution.*

Remark 1.1 Lemmas 2.1 and 2.2 ensure that $D_\theta\in[0,1]$ and that $C_{kf}\in[1,\infty]$. Hence (1.6) or (1.7) implies $1-C_{kf}^{-1}(1-D_\theta)>0$.

Remark 1.2 For the existence of the minimal (strictly) k-convex solution to problem (1.2), see Theorems 2.1 and 4.1 in [29].

Remark 1.3 Some basic examples of functions which satisfy (f$_3$) are the following:

(I) For $C_{kf}=1$,

(1) $f(s)=c_0\exp((\ln s)^q)$, $q>1$, $c_0>0$, $s\ge S_0$, where $S_0>0$ is a large constant;

(2) $f(s)=c_0\exp(s^q)$, $c_0>0$, $q>0$, $s\ge S_0$. In particular, when $F(s)=c_0\exp(s)$, $s>S_0$, $\psi_k(t)=k\ln(k+1)-(k+1)\ln c_0-(k+1)\ln t$ for sufficiently small $t>0$;

(3) $f(s)=c_0\exp(s\ln s)$, $c_0>0$, $s\ge S_0$;

(4) $f\in C^1(S_0,\infty)$ and $\lim\limits_{s\to\infty}\frac{f'(s)F(s)}{f^2(s)}=1$ (see Lemma 2.2);

(5) $F(s)=c_0\exp\big(\int_{S_0}^s\frac{\mathrm{d}\tau}{\zeta(\tau)}\big)$, $c_0>0$, $s>S_0$, where ζ is a positive C^1-function on $[S_0,\infty)$ and $\lim\limits_{\tau\to\infty}\zeta'(\tau)=0$ (see Lemma 2.2).

(II) For $C_{kf}=\infty$,

(1) when $F(s)=(k+1)^{-1}c_0^{k+1}s^{k+1}(\ln s)^{q(k+1)}$, $c_0>0$, $s\ge S_0$ with $q>1$, $\psi_k(t)=\exp((c_0(q-1)t)^{-\frac{1}{q-1}})$ for sufficiently small $t>0$;

(2) $F(s)=(k+1)^{-1}c_0^{k+1}s^{k+1}\exp((k+1)s^q)$, $c_0>0$, $q\in(0,1)$, $s\ge S_0$;

(3) when $F(s)=(k+1)^{-1}c_0^{k+1}s^{k+1}(\ln s)^{k+1}(\ln(\ln s))^{q(k+1)}$, $c_0>0$, $q>1$, $s\ge S_0$, $\psi_k(t)=\exp(\exp((c_0(q-1)t)^{-\frac{1}{q-1}}))$ for sufficiently small $t>0$.

(III) For $C_{kf}\in(1,\infty)$,

when $F(s)=c_0s^{p+1}L(s)$, $c_0>0$, $s\ge S_0$, where $p>k$ and L is slowly varying at infinity, $C_{kf}=\frac{p+1}{p-k}$. In particular, when $f(s)=c_0s^p$, $s\ge 0$,

$$\psi_k(t)=\Big(\frac{p+1}{k+1}\Big(\frac{k+1}{c_0(p-k)}\Big)^{k+1}\Big)^{\frac{1}{p-k}}t^{-\frac{k+1}{p-k}}, \quad \forall t>0.$$

The outline of this paper is as follows. In Section 2 we give some preliminary. The proof of Theorem 1.1 is provided in Section 3.

2 Preliminaries

Our approach relies on Karamata regular variation theory, which was first introduced and established by Karamata in 1930 and is a basic tool in stochastic process (see [3, 28]).

In this section, we present some basic facts from the theory and some preparations.

Corresponding to Definition 1.1, we also say that a positive continuous function θ defined on $(0, a)$ for some $a > 0$, is regularly varying at zero with index ρ (and denoted by $\theta \in RVZ_\rho$) if $t \to \theta\big(\frac{1}{t}\big)$ belongs to $RV_{-\rho}$.

Definition 2.1 *For some $a > 0$, a positive continuous function f defined on $[a, \infty)$ is called rapidly varying at infinity if for each $\rho > 1$,*

$$\lim_{s \to \infty} \frac{f(s)}{s^\rho} = \infty. \tag{2.1}$$

Definition 2.2 *For some $a > 0$, a positive continuous function f defined on $[a, \infty)$, is called rapidly varying at infinity if for each $\xi > 1$,*

$$\lim_{s \to \infty} \frac{f(\xi s)}{f(s)} = \infty. \tag{2.2}$$

Proposition 2.1 (Uniform Convergence Theorem) *If $f \in RV_\rho$, then* (1.4) *in Definition* 1.1 *holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$.*

Proposition 2.2 (Representation Theorem) *A function L is slowly varying at infinity if and only if it may be written as the form*

$$L(s) = z(s) \exp\Big(\int_{a_1}^{s} \frac{y(\tau)}{\tau} \mathrm{d}\tau\Big), \quad s \geq a_1, \tag{2.3}$$

for some $a_1 \geq a$, where the functions z and y are continuous and for $s \to \infty$, $y(s) \to 0$ and $z(s) \to c_0$, with $c_0 > 0$.

We say that a function is normalized slowly varying at infinity if it can be written as

$$\widehat{L}(s) = c_0 \exp\Big(\int_{a_1}^{s} \frac{y(\tau)}{\tau} \mathrm{d}\tau\Big), \quad s \geq a_1, \tag{2.4}$$

where y, c_0, a_1 are defined as above and a function is normalized regularly varying at infinity with index ρ if it can be written as

$$f(s) = s^\rho \widehat{L}(s), \quad s \geq a_1, \tag{2.5}$$

which is denoted by $f \in NRV_\rho$.

Equivalently, a function $f \in NRV_\rho$ if and only if

$$f \in C^1[a_1, \infty) \text{ for some } a_1 > 0 \quad \text{and} \quad \lim_{s \to \infty} \frac{sf'(s)}{f(s)} = \rho. \tag{2.6}$$

Similarly, θ is called normalized regularly varying at zero with index ρ, denoted by $\theta \in NRVZ_\rho$ if $t \to \theta\big(\frac{1}{t}\big)$ belongs to $NRV_{-\rho}$.

Proposition 2.3 *If functions L, L_1 are slowly varying at infinity, then*

(i) L^ρ *(for every $\rho \in \mathbb{R}$), $c_1 L + c_2 L_1$ ($c_1 \geq 0,\ c_2 \geq 0$ with $c_1 + c_2 > 0$), $L \cdot L_1$, $L \circ L_1$ (if $L_1(s) \to \infty$ as $s \to \infty$) are also slowly varying at infinity*;

(ii) *for every $\varepsilon > 0$ and $s \to \infty$, $s^\varepsilon L(s) \to \infty$ and $s^{-\varepsilon} L(s) \to 0$;*

(iii) *for $\rho \in \mathbb{R}$ and $s \to \infty$, $\frac{\ln(L(s))}{\ln s} \to 0$ and $\frac{\ln(s^\rho L(s))}{\ln s} \to \rho$.*

Proposition 2.4 *Let $f \in RV_\rho$ with $\rho \neq 0$. If f is strictly monotonous, then the inverse of f belongs to $RV_{\frac{1}{\rho}}$.*

Proposition 2.5 (Asymptotic Behaviour) *If a function L is slowly varying at infinity, then for $a \geq 0$ and $t \to \infty$,*

(i) $\int_a^t s^\rho L(s)\mathrm{d}s \cong (\rho+1)^{-1} t^{1+\rho} L(t)$ *for $\rho > -1$;*

(ii) $\int_t^\infty s^\rho L(s)\mathrm{d}s \cong (-\rho-1)^{-1} t^{1+\rho} L(t)$ *for $\rho < -1$.*

Proposition 2.6 (Asymptotic Behaviour) (see [3, Karamata's Theorem]) *If a function $z \in RV_{-1}$ and $\int_s^\infty z(\tau)\mathrm{d}\tau < \infty$, $s > 0$, then $\int_s^\infty z(\tau)\mathrm{d}\tau$ is slowly varying at infinity and* $\lim\limits_{s\to\infty} \frac{sz(s)}{\int_s^\infty z(\tau)\mathrm{d}\tau} = 0.$

Lemma 2.1 (see [33, Lemma 2.1]) *Let $\theta \in \Lambda$. We have*

(i) $\lim\limits_{t\to 0^+} \frac{\Theta(t)}{\theta(t)} = 0$;

(ii) $\lim\limits_{t\to 0^+} \frac{\Theta(t)\theta'(t)}{\theta^2(t)} = 1 - \lim\limits_{t\to 0^+} \frac{\mathrm{d}}{\mathrm{d}t}\big(\frac{\Theta(t)}{\theta(t)}\big) = 1 - D_\theta$ *and $D_\theta \in [0,1]$;*

(iii) *when $D_\theta \in (0,1]$, $\theta \in NRVZ_{\frac{1-D_\theta}{D_\theta}}$ and $\Theta \in NRVZ_{D_\theta^{-1}}$;*

(iv) *when $D_\theta = 0$, θ is rapidly varying to zero at zero.*

Recall that

$$\Psi_k(s) = \int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)}, \quad \forall s > 0, \text{ where } H_k(\tau) = ((k+1)F(\tau))^{\frac{1}{k+1}}. \tag{2.7}$$

Our results in the section are summarized as follows.

Lemma 2.2 *Let f satisfy* (f_1)–(f_2). *We have*

(i_1) *if f satisfies* (f_3), *then $C_{kf} \geq 1$;*

(i_2) *f satisfies* (f_3) *with $C_{kf} \in (1,\infty)$ if and only if $F \in NRV_{p(k+1)}$ with $p = \frac{C_{kf}}{C_{kf}-1} > 1$;*

(i_3) *if f satisfies* (f_3) *with $C_{kf} = 1$, then F is rapidly varying at infinity*;

(i_4) *if*

$$\lim_{s\to\infty} \frac{f'(s)F(s)}{f^2(s)} = 1, \tag{2.8}$$

then f satisfies (f_3) *with $C_{kf} = 1$;*

(i_5) *if $F \in NRV_{k+1}$, then f satisfies* (f_3) *with $C_{kf} = \infty$;*

(i_6) $\lim\limits_{s\to\infty} \frac{((k+1)F(s))^{\frac{k}{k+1}}}{f(s)\Psi_k(s)} = C_{kf}^{-1}$.

Proof (i_1) Let $C_{kf} \in (0,\infty]$ and

$$I_k(s) = H_k'(s) \int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)}, \quad \forall s > 0.$$

Integrating $I_k(\upsilon)$ from a ($a > 0$) to s and integration by parts, we obtain

$$\int_a^s I_k(\upsilon)\mathrm{d}\upsilon = H_k(s)\int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)} - H_k(a)\int_a^\infty \frac{\mathrm{d}\tau}{H_k(\tau)} + s - a, \quad \forall s > a.$$

It follows from the l'Hospital's rule that

$$0 \le \lim_{s\to\infty} \frac{H_k(s)\int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)}}{s} = \lim_{s\to\infty} \frac{\int_a^s I_k(\upsilon)\mathrm{d}\upsilon}{s} - 1 = \lim_{s\to\infty} I_k(s) - 1 = C_{kf} - 1, \tag{2.9}$$

i.e., $C_{kf} \ge 1$.

(i_2) The necessity is as follows. For $C_{kf} \in (1, \infty)$, we see that

$$\lim_{s\to\infty} \frac{H_k(s)}{sH_k'(s)} = \lim_{s\to\infty} \frac{H_k(s)\displaystyle\int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)}}{sH_k'(s)\displaystyle\int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)}} = \frac{1}{C_{kf}} \lim_{s\to\infty} \frac{H_k(s)\displaystyle\int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)}}{s} = \frac{C_{kf} - 1}{C_{kf}},$$

i.e., $H_k \in NRV_{\frac{C_{kf}}{C_{kf}-1}}$ and $F \in NRV_{\frac{C_{kf}(k+1)}{C_{kf}-1}}$.

For the sufficiency, if $F \in NRV_{p(k+1)}$ with $p > 1$, we have $H_k \in NRV_p$ and

$$\lim_{s\to\infty} \frac{sH_k'(s)}{H_k(s)} = p.$$

By the representation theorem, $H_k(s)$ can be written as

$$H_k(s) = s^p \widehat{L}(s), \quad \forall s \ge S_0 \text{ for sufficiently large } S_0,$$

where $\widehat{L}$ is normalized slowly varying at infinity.

It follows from Propositions 2.3 (i) and 2.5 (ii) that

$$\begin{aligned}
&\lim_{s\to\infty} H_k'(s)\int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)}\\
&= \lim_{s\to\infty} \frac{sH_k'(s)}{H_k(s)} \lim_{s\to\infty} \frac{H_k(s)}{s}\int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)}\\
&= p\lim_{s\to\infty} s^{p-1}\widehat{L}(s)\int_s^\infty \tau^{-p}(\widehat{L}(\tau))^{-1}\mathrm{d}\tau = \frac{p}{p-1} = C_{kf}.
\end{aligned}$$

(i_3) When $C_{kf} = 1$, we see by the proof of (i_2) that

$$\lim_{s\to\infty} \frac{sH_k'(s)}{H_k(s)} = +\infty. \tag{2.10}$$

Consequently, for an arbitrary $\rho > 1$, there exists $S_0 > 0$ such that

$$\frac{H_k'(s)}{H_k(s)} > \frac{\rho+1}{s}, \quad \forall s \ge S_0.$$

Integrating the above inequality from S_0 to s, we obtain

$$\ln(H_k(s)) - \ln(H_k(S_0)) > (\rho+1)(\ln s - \ln S_0), \quad \forall s > S_0,$$

i.e.,

$$\frac{H_k(s)}{s^\rho} > \frac{H_k(S_0)s}{S_0^{\rho+1}}, \quad \forall s > S_0.$$

Letting $s \to \infty$, we see by Definition 2.1 that H_k is rapidly varying at infinity. So does F.
There is another proof. Let

$$\frac{H_k'(s)}{H_k(s)} = \frac{y_k(s)}{s}, \quad \forall s > 0.$$

Integrating the above inequality from $s_0 > 0$ to s, we obtain

$$H_k(s) = c_k \exp\Big(\int_{s_0}^{s} \frac{y_k(\tau)}{\tau} \mathrm{d}\tau\Big), \quad s > S_0,$$

where $c_k = H_k(S_0)$.

Since $\lim_{s\to+\infty} y_k(s) = \infty$, we see that for each $\xi > 1$,

$$\frac{H_k(\xi s)}{H_k(s)} = \exp\Big(\int_{s}^{\xi s} \frac{y_k(\tau)}{\tau} \mathrm{d}\tau\Big) = \exp\Big(\int_{1}^{\xi} \frac{y_k(s\upsilon)}{\upsilon} \mathrm{d}\upsilon\Big) \to +\infty \quad as\ s \to \infty.$$

So H_k is rapidly varying at infinity by Definition 2.2. So does F.

(i_4) By (2.8) and the generalized l'Hospital's rule, we see that

$$\lim_{s\to\infty} \frac{F(s)}{sf(s)} = \lim_{s\to\infty} \frac{\frac{F(s)}{f(s)}}{s} = \lim_{s\to\infty} \frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{F(s)}{f(s)}\Big) = 1 - \lim_{s\to\infty} \frac{F(s)f'(s)}{f^2(s)} = 0. \tag{2.11}$$

Consequently, for an arbitrary $p > k$, there exists $S_p > 0$ such that

$$\frac{f(s)}{F(s)} > \frac{p+2}{s}, \quad s \geq S_p.$$

Integrating from S_p to s, we obtain

$$F(s) \geq \frac{F(S_p)}{S_p^{p+2}} s^{p+2}, \quad s > S_p.$$

So

$$f(s) \geq \frac{(p+2)F(S_p)}{S_p^{p+2}} s^{p+1}, \quad s > S_p,$$

which implies

$$\lim_{s\to\infty} \frac{f(s)}{s^p} = \infty.$$

Next, since

$$F(s) = \int_0^s f(\tau)\mathrm{d}\tau \leq sf(s), \quad s > 0,$$

we have that

$$0 < \frac{(F(s))^{\frac{k}{k+1}}}{f(s)} \leq \Big(\frac{s^k}{f(s)}\Big)^{\frac{1}{k+1}}, \quad s > 0.$$

So

$$\lim_{s\to\infty} \frac{(F(s))^{\frac{k}{k+1}}}{f(s)} = 0.$$

By $H_k(s) = ((k+1)F(s))^{\frac{1}{k+1}}$ and ($\mathbf{f}_2$), we see that

$$H_k'(s) = \frac{f(s)}{((k+1)F(s))^{\frac{k}{k+1}}},$$
$$\frac{f(s)\Psi_k(s)}{((k+1)F(s))^{\frac{k}{k+1}}} = H_k'(s)\int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)}, \quad \forall s > 0. \tag{2.12}$$

It follows by the l'Hospital's rule and (2.8) that

$$\lim_{s\to\infty} \frac{\frac{\Psi_k(s)}{((k+1)F(s))^{\frac{k}{k+1}}}}{f(s)}$$
$$= \lim_{s\to\infty} \frac{f^2(s)}{(k+1)F(s)f'(s) - kf^2(s)}$$
$$= \lim_{s\to\infty} \frac{1}{(k+1)\frac{F(s)f'(s)}{f^2(s)} - k} = 1.$$

(i_5) When $F \in NRV_{k+1}$, we see that $H_k \in NRV_1$, $(H_k)^{-1} \in NRV_{-1}$ and

$$\lim_{s\to\infty} \frac{sH_k'(s)}{H_k(s)} = 1.$$

It follows from (f_2) and Proposition 2.6 that

$$\lim_{s\to\infty} \frac{s}{H_k(s)\int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)}} = 0.$$

Thus

$$\lim_{s\to\infty} H_k'(s)\int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)} = \lim_{s\to\infty} \frac{sH_k'(s)}{H_k(s)} \lim_{s\to\infty} \frac{H_k(s)\int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)}}{s} = \infty.$$

(i_6) Follows from (f_3) and (2.12).

Recall that ψ_k satisfies

$$\int_{\psi_k(t)}^\infty \frac{\mathrm{d}\tau}{H_k(\tau)} = t, \quad \forall t > 0, \text{ where } H_k(\tau) = ((k+1)F(\tau))^{\frac{1}{k+1}}.$$

Lemma 2.3 *Let f satisfy* (f_1)–(f_3). *We have*

(i_1) $-\psi_k'(t) = H_k(\psi_k(t)) = ((k+1)F(\psi_k(t)))^{\frac{1}{k+1}}$, $t > 0$, $\psi_k''(t) = \frac{f(\psi_k(t))}{((k+1)F(\psi_k(t)))^{\frac{k-1}{k+1}}}$ *and* $\psi_k(t) > 0$, $t > 0$, $\psi_k(0) := \lim_{t\to 0^+} \psi_k(t) = +\infty$;

(i_2) $\lim_{t\to 0} \frac{\psi_k'(t)}{t\psi_k''(t)} = -C_{kf}^{-1}$;

(i_3) *when $C_{kf} \in [1,\infty)$, $\psi \in NRVZ_{1-C_{kf}}$ and $-\psi_k' \in NRVZ_{-C_{kf}}$;*

(i_4) *when $C_{kf} = \infty$, ψ_k is rapidly varying to infinity at zero.*

Proof (i_1) It is easy to be obtained.

(i_2) and (i_3) For $C_{kf} \in [1, \infty)$, let $s = \psi_k(t)$. It follows from (2.9) and (f_3) that

$$\lim_{t\to 0} \frac{t\psi_k'(t)}{\psi_k(t)} = -\lim_{t\to 0} \frac{tH_k(\psi_k(t))}{\psi_k(t)} = -\lim_{s\to\infty} \frac{H_k(s)}{s} \int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)} = -(C_{kf} - 1), \tag{2.13}$$

$$\lim_{t\to 0} \frac{t\psi_k''(t)}{\psi_k'(t)} = -\lim_{t\to 0} tH_k'(\psi_k(t)) = -\lim_{s\to\infty} H_k'(s) \int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)} = -C_{kf}, \tag{2.14}$$

i.e., $\psi_k \in NRVZ_{1-C_{kf}}$ and $-\psi_k' \in NRVZ_{-C_{kf}}$.

(i_4) When $C_{kf} = \infty$, we see by using the l'Hospital's rule that

$$\lim_{s\to\infty} \frac{H_k(s) \displaystyle\int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)}}{s} = \lim_{s\to\infty} H_k'(s) \int_s^\infty \frac{\mathrm{d}\tau}{H_k(\tau)} - 1 = +\infty,$$

i.e., $\lim\limits_{t\to 0^+} \frac{t\psi_k'(t)}{\psi_k(t)} = -\infty$.

Similarly as the proof to (i_3) in Lemma 2.2, we can show that ψ_k is rapidly varying to infinity at zero.

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

For any $\delta > 0$, we let

$$\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\}.$$

When Ω is C^m-smooth for $m \geq 2$, choose $\delta_1 > 0$ such that (see [10, Lemmas 14.16 and 14.17])

$$d \in C^m(\Omega_{\delta_1}) \quad \text{and} \quad |\nabla d(x)| = 1, \quad \forall x \in \Omega_{\delta_1}. \tag{3.1}$$

Let $\overline{x}$ be the projection of the point $x \in \Omega_{\delta_1}$ to $\partial\Omega$, and $\kappa_1(\overline{x}), \kappa_2(\overline{x}), \cdots, \kappa_{n-1}(\overline{x})$ are the principal curvatures of $\partial\Omega$ at $\overline{x}$. Then

$$D^2(d(x)) = \operatorname{diag}\Big[\frac{-\kappa_1(\overline{x})}{1 - d(x)\kappa_1(\overline{x})}, \ \cdots, \ \frac{-\kappa_{n-1}(\overline{x})}{1 - d(x)\kappa_{n-1}(\overline{x})}, \ 0\Big].$$

Lemma 3.1 (see [13, Corollary 2.3]) *Let h be a C^2-function on $(0, \delta_1)$. Then, we have*

$$\begin{aligned} &S_k(D^2 h(d(x))) \\ &= (-h'(d(x)))^k S_k\Big(\frac{\kappa_1(\overline{x})}{1 - d(x)\kappa_1(\overline{x})}, \cdots, \frac{\kappa_{n-1}(\overline{x})}{1 - d(x)\kappa_{n-1}(\overline{x})}\Big) \\ &\quad + (-h'(d(x)))^{k-1} h''(d(x)) S_{k-1}\Big(\frac{\kappa_1(\overline{x})}{1 - d(x)\kappa_1(\overline{x})}, \cdots, \frac{\kappa_{n-1}(\overline{x})}{1 - d(x)\kappa_{n-1}(\overline{x})}\Big). \end{aligned}$$

Proof of Theorem 1.1 For an arbitrary $\varepsilon \in (0, \min\{\frac{1}{2}, \frac{b_{k1}}{2}\})$, let

$$\xi_{+\varepsilon} = \Big(\frac{(b_{k2} + \varepsilon)(1 + \varepsilon) + \varepsilon}{m_k(1 - C_{kf}^{-1}(1 - D_\theta))}\Big)^{\frac{1}{k+1}} \quad \text{and} \quad \xi_{-\varepsilon} = \Big(\frac{(b_{k1} - \varepsilon)(1 - \varepsilon) - \varepsilon}{M_k(1 - C_{kf}^{-1}(1 - D_\theta))}\Big)^{\frac{1}{k+1}},$$

where m_k, M_k are given as in (1.9), b_{k1} and b_{k2} are given as in (b_2).

Using Lemmas 2.1–2.3 and

$$\begin{aligned}&\Theta \in C^2(0,\delta_0)\cap C([0,\delta_0)), \quad \Theta(0)=0,\\ &\Theta(d(x))=\int_{\psi_k(\Theta(d(x)))}^{\infty}((k+1)F(\tau))^{-\frac{1}{k+1}}\mathrm{d}\tau,\end{aligned} \tag{3.2}$$

we see that

$$\lim_{d(x)\to 0}\frac{\Theta(d(x))}{\theta(d(x))}\cdot\frac{\kappa_k(\overline{x})}{1-d(x)\kappa_k(\overline{x})}=0; \tag{3.3}$$

$$\lim_{d(x)\to 0}\frac{\Theta(d(x))\theta'(d(x))}{\theta^2(d(x))}=1-D_\theta; \tag{3.4}$$

$$\begin{aligned}&\lim_{d(x)\to 0}\frac{((k+1)F(\psi_k(\Theta(d(x)))))^{\frac{k}{k+1}}}{\Theta(d(x))f(\psi_k(\Theta(d(x))))}=C_{kf}^{-1};\\ &\lim_{d(x)\to 0}\prod_{i=1}^{k-1}(1-d(x)\kappa_i(\overline{x}))=1;\\ &m_k\xi_{+\varepsilon}^{k+1}((1-C_{kf}^{-1}(1-D_\theta))-(b_{k2}+\varepsilon)(1+\varepsilon)=\varepsilon;\\ &M_k\xi_{-\varepsilon}^{k+1}(1-C_{kf}^{-1}(1-D_\theta))-(b_{k1}-\varepsilon)(1-\varepsilon)=-\varepsilon.\end{aligned}$$

It follows from (b_2) that there is a sufficiently small $\delta_\varepsilon\in(0,\min\{1,\frac{\delta_1}{2}\})$ corresponding to ε, such that for $\sigma\in(0,\delta_\varepsilon)$,

$$(b_{k1}-\varepsilon)\theta^{k+1}(d(x)-\sigma)\le(b_{k1}-\varepsilon)\theta^{k+1}(d(x))<b(x), \quad x\in D_\sigma^-=\Omega_{2\delta_\varepsilon}/\overline{\Omega}_\sigma; \tag{3.5}$$

$$b(x)<(b_{k2}+\varepsilon)\theta^{k+1}(d(x))\le(b_{k2}+\varepsilon)\theta^{k+1}(d(x)+\sigma), \quad x\in D_\sigma^+=\Omega_{2\delta_\varepsilon-\sigma}; \tag{3.6}$$

$$1-\varepsilon<\prod_{i=1}^{k-1}(1-d(x)\kappa_i(\overline{x}))<1+\varepsilon, \quad x\in\Omega_{2\delta_\varepsilon}. \tag{3.7}$$

Furthermore, there hold for $x\in\Omega_{2\delta_\varepsilon}$,

$$\begin{aligned}&M_k\Big(\xi_{-\varepsilon}^{k+1}\frac{\Theta(d(x))}{\theta(d(x))}\cdot\frac{\kappa_k(\overline{x})}{1-d(x)\kappa_k(\overline{x})}\cdot\frac{((k+1)F(\psi_k(\xi_{-\varepsilon}\Theta(d(x)))))^{\frac{k}{k+1}}}{\xi_{-\varepsilon}\Theta(d(x))f(\psi_k(\xi_{-\varepsilon}\Theta(d(x))))}\\ &+\xi_{-\varepsilon}^{k+1}-\xi_{-\varepsilon}^{k+1}\frac{\Theta(d(x))\theta'(d(x))}{\theta^2(d(x))}\cdot\frac{((k+1)F(\psi_k(\xi_{-\varepsilon}\Theta(d(x)))))^{\frac{k}{k+1}}}{\xi_{-\varepsilon}\Theta(d(x))f(\psi_k(\xi_{-\varepsilon}\Theta(d(x))))}\Big)\\ &-(b_{k1}-\varepsilon)(1-\varepsilon)<0;\\ &m_k\Big(\xi_{+\varepsilon}^{k+1}\cdot\frac{\Theta(d(x))}{\theta(d(x))}\frac{\kappa_k(\overline{x})}{1-d(x)\kappa_k(\overline{x})}\frac{((k+1)F(\psi_k(\xi_{+\varepsilon}\Theta(d(x)))))^{\frac{k}{k+1}}}{\xi_{+\varepsilon}\Theta(d(x))f(\psi_k(\xi_{+\varepsilon}\Theta(d(x))))}\\ &+\xi_{+\varepsilon}^{k+1}-\xi_{+\varepsilon}^{k+1}\frac{\Theta(d(x))\theta'(d(x))}{\theta^2(d(x))}\cdot\frac{((k+1)F(\psi_k(\xi_{+\varepsilon}\Theta(d(x)))))^{\frac{k}{k+1}}}{\xi_{+\varepsilon}\Theta(d(x))f(\psi_k(\xi_{+\varepsilon}\Theta(d(x))))}\Big)\\ &-(b_{k2}+\varepsilon)(1+\varepsilon)>0.\end{aligned}$$

Let

$$d_1(x)=d(x)-\sigma; \quad d_2(x)=d(x)+\sigma; \tag{3.8}$$

$$\overline{u}_\varepsilon=\psi_k(\xi_{-\varepsilon}\Theta(d_1(x))), \quad x\in D_\sigma^-; \quad \underline{u}_\varepsilon=\psi_k(\xi_{+\varepsilon}\Theta(d_2(x))), \quad x\in D_\sigma^+. \tag{3.9}$$

By Lemma 3.1, where we let $h=\psi_k(\xi_{-\varepsilon}\Theta(t))$ and a direct computation, we see that for $x\in D_\sigma^-$,

$$\begin{aligned}
&S_k(D^2\overline{u}_\varepsilon(x))-(b_{k1}-\varepsilon)\theta^{k+1}(d_1(x))f(\psi_k(\xi_{-\varepsilon}\Theta(d_1(x))))\\
&=(-\xi_{-\varepsilon}\theta(d_1(x))\psi_k'(\xi_{-\varepsilon}\Theta(d_1(x))))^kS_k\Big(\frac{\kappa_1(\overline{x})}{1-d(x)\kappa_1(\overline{x})},\cdots,\frac{\kappa_{n-1}(\overline{x})}{1-d(x)\kappa_{n-1}(\overline{x})}\Big)\\
&\quad+(-\xi_{-\varepsilon}\theta(d_1(x))\psi_k'(\xi_{-\varepsilon}\Theta(d_1(x))))^{k-1}(\xi_{-\varepsilon}^2\theta^2(d_1(x))\psi_k''(\xi_{-\varepsilon}\Theta(d_1(x)))\\
&\quad+\xi_{-\varepsilon}\theta'(d_1(x))\psi_k'(\xi_{-\varepsilon}\Theta(d_1(x)))S_{k-1}\Big(\frac{\kappa_1(\overline{x})}{1-d(x)\kappa_1(\overline{x})},\cdots,\frac{\kappa_{n-1}(\overline{x})}{1-d(x)\kappa_{n-1}(\overline{x})}\Big)\\
&\quad-(b_{k1}-\varepsilon)\theta^{k+1}(d_1(x))f(\psi_k(\xi_{-\varepsilon}\Theta(d_1(x))))\\
&\le(1-\varepsilon)^{-1}\theta^{k+1}(d_1(x))f(\psi_k(\xi_{-\varepsilon}\Theta(d_1(x))))\Big(M_k\Big(\xi_{-\varepsilon}^{k+1}\frac{\Theta(d_1(x))}{\theta(d_1(x))}\cdot\frac{\kappa_k(\overline{x})}{1-d(x)\kappa_k(\overline{x})}\\
&\quad\frac{((k+1)F(\psi_k(\xi_{-\varepsilon}\Theta(d_1(x)))))^{\frac{k}{k+1}}}{\xi_{-\varepsilon}\Theta(d_1(x))f(\psi_k(\xi_{-\varepsilon}\Theta(d_1(x))))}+\xi_{-\varepsilon}^{k+1}-\xi_{-\varepsilon}^{k+1}\frac{\Theta(d_1(x))\theta'(d_1(x))}{\theta^2(d_1(x))}\cdot\\
&\quad\frac{((k+1)F(\psi_k(\xi_{-\varepsilon}\Theta(d_1(x)))))^{\frac{k}{k+1}}}{\xi_{-\varepsilon}\Theta(d_1(x))f(\psi_k(\xi_{-\varepsilon}\Theta(d_1(x))))}\Big)-(b_{k1}-\varepsilon)(1-\varepsilon)\Big)\\
&\le 0,
\end{aligned}$$

which means that $\overline{u}_\varepsilon$ is a supersolution to equation (1.2) in D_σ^-.

In a similar way, we can show that $\underline{u}_\varepsilon=\psi_k(\xi_{+\varepsilon}\Theta(d_2(x)))$ is a subsolution to equation (1.2) in D_σ^+.

Now let $u\in C^2(\Omega)$ be an arbitrary strictly k-convex solution to problem (1.2) and let $M>0$ be a sufficiently large constant such that

$$u\le\overline{u}_\varepsilon+M \text{ on } d(x)=2\delta_\varepsilon,\quad \underline{u}_\varepsilon\le u+M,\quad \text{on } d(x)=2\delta_\varepsilon-\sigma. \tag{3.10}$$

We observe that $\overline{u}_\varepsilon(x)\to\infty$ as $d(x)\to\sigma$, and $u|_{\partial\Omega}=+\infty>\underline{u}_\varepsilon|_{\partial\Omega}$. It follows from the comparison principle for k-Hessians (see [16, Lemma 2.1]) that

$$u\le\overline{u}_\varepsilon+M \text{ in } D_\sigma^-,\quad \underline{u}_\varepsilon\le u+M\quad \text{in } D_\sigma^+, \tag{3.11}$$

i.e.,

$$1-\frac{M}{\psi_k(\xi_{+\varepsilon}\Theta(d_2(x)))}\le\frac{u(x)}{\psi_k(\xi_{+\varepsilon}\Theta(d_2(x)))},\quad x\in D_\sigma^+,$$

and

$$\frac{u(x)}{\psi_k(\xi_{-\varepsilon}\Theta(d_1(x)))}\le1+\frac{M}{\psi_k(\xi_{-\varepsilon}\Theta(d_1(x)))},\quad x\in D_\sigma^-.$$

Hence, for $x\in D_\sigma^-\cap D_\sigma^+$, by letting $\sigma\to0$, we have

$$1-\frac{M}{\psi_k(\xi_{+\varepsilon}\Theta(d(x)))}\le\frac{u(x)}{\psi_k(\xi_{+\varepsilon}\Theta(d(x)))},\quad \frac{u(x)}{\psi_k(\xi_{-\varepsilon}\Theta(d(x)))}\le1+\frac{M}{\psi_k(\xi_{-\varepsilon}\Theta(d(x)))}.$$

Then

$$1\le\lim_{d(x)\to0}\inf\frac{u(x)}{\psi_k(\xi_{+\varepsilon}\Theta(d(x)))},\quad \lim_{d(x)\to0}\sup\frac{u(x)}{\psi_k(\xi_{-\varepsilon}\Theta(d(x)))}\le1. \tag{3.12}$$

Thus letting $\varepsilon\to0$ in (3.12), we obtain (1.8).

Moreover, one can see by Lemma 2.3 that for $C_{kf} \in [1, \infty)$,

$$\lim_{d(x)\to 0} \frac{\psi_k(\xi\Theta(d(x)))}{\psi_k(\Theta(d(x)))} = \xi^{1-C_{kf}},$$

holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$.

Finally we prove the uniqueness of solutions.

For arbitrary fixed $\varepsilon \in (0,1)$, let u_1, u_2 be two arbitrary strictly k-convex solutions satisfying problem (1.2). It suffices to show that $u_1 \leq u_2$ in Ω.

Since $\lim\limits_{d(x)\to 0} \frac{u_1(x)}{u_2(x)} = 1$, we see that there is a sufficiently small constant $\delta_\varepsilon > 0$ corresponding to ε, such that

$$u_1(x) < (1+\varepsilon)u_2(x), \quad x \in \Omega_{\delta_\varepsilon}.$$

By $\frac{f(s)}{s^k}$ is increasing on $(0, \infty)$, we deduce that

$$S_k(D^2((1+\varepsilon)u_2(x))) = (1+\varepsilon)^k b(x) f(u_2(x)) \leq b(x) f((1+\varepsilon)u_2(x)), \quad x \in \Omega.$$

It follows by the comparison principle for k-Hessian equations that $u_1 \leq (1+\varepsilon)u_2$ in Ω. Letting $\varepsilon \to 0$, we obtain $u_1 \leq u_2$ in Ω. This completes the proof of Theorem 1.1.

Acknowledgement The author is greatly indebted to the anonymous referees for the very valuable suggestions and comments which improved the quality of the presentation.

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