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# The Equivalence of Hypercontractivity and Logarithmic Sobolev Inequality for $q$ $(-1 \le q \le 1)$ -Ornstein-Uhlenbeck Semigroup

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**Abstract** In this paper the author proves the equivalence of hypercontractivity and logarithmic Sobolev inequality for $q$-Ornstein-Uhlenbeck semigroup $U_t^{(q)} = \Gamma_q(\mathrm{e}^{-t}I)$ $(-1 \le q \le 1)$, where $\Gamma_q$ is a $q$-Gaussian functor.

**Keywords** $q$-Ornstein-Uhlenbeck semigroup, Hypercontractivity, Logarithmic Sobolev inequality

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## 1 Introduction

The study of hypercontractivity in quantum mechanics dates back to the work of Nelson [1] who showed that semiboundedness of certain Hamiltonians $H$ associated to a bosonic system can be obtained from the hypercontractivity of the semigroup $\mathrm{e}^{-tL} : L_2(\mathbb{R}^d, \mu) \to L_2(\mathbb{R}^d, \mu)$, where $L$ is the Dirichlet form operator for the Gaussian measure $\mu$ on $\mathbb{R}^d$. After some contributions (see [2–4]), Nelson finally proved in [5] that the previous semigroup is contractive from $L_p(\mathbb{R}^d, \mu)$ to $L_r(\mathbb{R}^d, \mu)$ if and only if $\mathrm{e}^{-2t} \le \frac{p-1}{r-1}$. By that time a new deep connection was shown by Gross in [6], who established the equivalence between the hypercontractivity of the semigroup $\mathrm{e}^{-tL}$, where $L$ is the Dirichlet form operator associated to the measure $\mu$, and the logarithmic Sobolev inequality verified by $\mu$. The extension of Nelson's theorem to the fermonic case started with Gross' papers (see [7–8]). Namely, he adapted the argument in the bosonic case by considering a suitable Clifford algebra $\mathcal{C}(\mathbb{R}^d)$ on the fermion Fock space and noncommutative $L_p$ space on this algebra after Segal [9]. In particular, hypercontractivity makes perfectly sense in this context by considering the corresponding Ornstein-Uhlenbeck semigroup

$$U_t^{(-1)} = \mathrm{e}^{-tN^{-1}} : L_2(\mathcal{C}(\mathbb{R}^d), \tau) \to L_2(\mathcal{C}(\mathbb{R}^d), \tau).$$

Here $N^{-1}$ denotes the fermion number operator. After some partial results (see [7, 10–11]), the optimal time hypercontractivity bound in the fermionic case was finally obtained by Carlen and Lieb in [12]:

$$\|U_t^{(-1)}\|_{L^p \to L^r} = 1 \text{ if and only if } \mathrm{e}^{-2t} \le \frac{p-1}{r-1}.$$

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As a by-product, Carlen and Lieb [12] proved that the above hypercontractivity in the fermion case is equivalent to the following logarithmic Sobolev inequality:

$$\tau(|a|^2)\log|a|^2 - \|a\|_2^2\log\|a\|_2^2 \le 2\,\langle a,\ N^{-1}a\rangle$$

for all $a \in \mathcal{C}(\mathbb{R}^d)$.

Biane [13] who extended Carlen and Lieb's work and obtained optimal time estimates for the $q$-Gaussian von Neumann algebras $\Gamma_q(-1 \le q \le 1)$ introduced by Bozèjko and Speicher [14] (see Section 2 for the construction of the von Neumann algebra $\Gamma_q$ and a precise definition of the $q$- Ornstein-Uhlenbeck semigroup on $\Gamma_q$). These algebras interpolate between the bosonic and fermonic frameworks, corresponding to $q = \pm 1$. The semigroup for $q = 0$ acts diagonally on free semi-circular variables in the context of Voiculescu's free probability theory (see [15]). As a consequence, Biane [13] derived from the optimal hypercontractivity inequality for $q$-Ornstein-Uhlenbeck semigroup $U_t^{(q)} = \Gamma_q(\mathrm{e}^{-t}I)$ a logarithmic Sobolev inequality (see [13, Corollary 1]):

$$\tau_q(|a|^2\log|a|^2) - \|a\|_2^2\log\|a\|_2^2 \le 2\,\varepsilon_q[a]$$

for all $a$ in $D(\varepsilon)$, where $D(\varepsilon)$ is the domain of Dirichlet form $\varepsilon_q[a] = \tau_q(a^*N^q a)$, $\tau_q(a) = \langle \Omega, a\Omega\rangle_q$ for all $a \in \Gamma_q(\mathcal{H})$, is the tracial state on $\Gamma_q(\mathcal{H})$, $\Omega$ is the vacuum vector.

In addition, under the framework of inductive limit $C^*$-algebra, Olkiewicz and Zegarlinski [16] established the relations between hypercontractivity and logarithmic Sobolev inequality on the basis of suitable regularity condition of corresponding noncommutative Dirichlet form.

We observe that Biane [13] did not characterize the relation between hypercontractivity and logarithmic Sobolev inequality for $q$-Ornstein-Uhlenbeck semigroup. Based on the above mentioned materials, the main work of this paper is to prove the equivalence of hypercontractivity and logarithmic Sobolev inequality for $q$-Ornstein-Uhlenbeck semigroup. This result refines the work of Biane [13].

This paper is organized as follows. In the first part we describe the $q$-Ornstein-Uhlenbeck semigroup which was introduced in [17], and related concepts and conclusions. In the second part, we state our main result and give the proof.

## 2 $q$-Ornstein-Uhlenbeck Semigroup

We begin by briefly reviewing the $q$-Fock spaces $\mathcal{F}_q$ and the von Neumann algebras $\Gamma_q$ (which are related to the creation and annihilation operators on $\mathcal{F}_q$), and the basic concepts and facts of Markov semigroups and associated noncommutative Dirichlet forms. It is emphasized that the Hilbert spaces appearing in this paper are all separable.

### 2.1 The $q$-Fock spaces $\mathcal{F}_q$ and the von Neumann algebras $\Gamma_q$

Let $\mathcal{H}$ be a real Hilbert space with complexification $\mathcal{H}_{\mathcal{C}}$. Let $\Omega$ be a unit vector in a 1-dimensional complex Hilbert space (disjoint from $\mathcal{H}_{\mathcal{C}}$). We refer to $\Omega$ as the vacuum, and by convention define ${\mathcal{H}_{\mathcal{C}}}^{\otimes 0} \equiv \mathbb{C}\Omega$. The algebraic Fock space $\mathcal{F}(\mathcal{H})$ is defined as

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}^{\otimes n},$$

where the direct sum and tensor product are algebraic. For any $q \in [-1,1]$, we then define a Hermitian form $\langle \cdot,\cdot \rangle_q$ to be the conjugate-linear extension of

$$\begin{aligned}
&\langle \Omega, \Omega \rangle_q = 1;\\
&\langle f_1 \otimes f_2 \otimes \cdots \otimes f_j, g_1 \otimes g_2 \otimes \cdots \otimes g_k \rangle_q = \delta_{jk} \sum_{\pi \in S_k} q^{i(\pi)} \langle f_1, g_{\pi(1)} \rangle \cdots \langle f_k, g_{\pi(k)} \rangle
\end{aligned}$$

for $f_i, g_i \in \mathcal{H}$, where $S_k$ is the symmetric group on $k$ symbols, and $i(\pi)$ counts the number of inversions in $\pi$, that is

$$i(\pi) = \sharp\{(i,j) : 1 \le i < j \le k, \pi(i) > \pi(j)\}.$$

The form $\langle \cdot,\cdot \rangle_{-1}$ reduces to the standard Hermitian form associated to the Fermion Fock space. Similarly, the form $\langle \cdot,\cdot \rangle_{+1}$ yields the standard Hermitian form on the Boson Fock space. In each of these cases the form is degenerate, thus requiring that we take a quotient of $\mathcal{F}(\mathcal{H})$ before completing to form the Fermion or Boson Fock spaces. It is remarkable that, for $-1 < q < 1$, the form $\langle \cdot,\cdot \rangle_q$ is already non-degenerate on $\mathcal{F}(\mathcal{H})$ as below.

**Proposition 2.1** (see [18]) *The Hermitian form $\langle \cdot,\cdot \rangle_q$ is positive semi-definite on $\mathcal{F}(\mathcal{H})$. Moreover, it is an inner product on $\mathcal{F}(\mathcal{H})$ for $-1 < q < 1$.*

For $-1 < q < 1$, the $q$-Fock space $\mathcal{F}_q(\mathcal{H})$ is defined as the completion of $\mathcal{F}(\mathcal{H})$ with respect to the inner product $\langle \cdot,\cdot \rangle_q$. These spaces interpolate between the classical Boson and Fermion Fock spaces $\mathcal{F}_\pm(\mathcal{H})$ which are constructed by first taking the quotient of $\mathcal{F}(\mathcal{H})$ by the kernel of $\langle \cdot,\cdot \rangle_\pm$ and then completing.

As in the clasical theory, the spaces $\mathcal{F}_q$ come equipped with creation and annihilation operators. For any vector $f \in \mathcal{H} \subset \mathcal{H}_\mathbb{C}$, define the creation operator $c_q(f)$ on $\mathcal{F}_q(\mathcal{H})$ to extend

$$\begin{aligned}
&c_q(f)\Omega = f,\\
&c_q(f) f_1 \otimes \cdots \otimes f_k = f \otimes f_1 \otimes \cdots \otimes f_k.
\end{aligned}$$

The annihilation operator $c_q^*(f)$ is its adjoint, which the reader may compute satisfies

$$\begin{aligned}
&c_q^*(f)\Omega = 0,\\
&c_q^*(f) f_1 \otimes \cdots \otimes f_k = \sum_{j=1}^{k} q^{j-1} \langle f_j, f \rangle f_1 \otimes \cdots \otimes f_{j-1} \otimes f_{j+1} \otimes \cdots \otimes f_k.
\end{aligned}$$

**Remark 2.1** Since the cases $q = \pm 1$ are well known, in the sequel we shall only give proof of statements for $-1 < q < 1$.

It can be verified that these operators fulfill the relation

$$c_q^*(g) c_q(f) - q c_q(f) c_q^*(g) = \langle f, g \rangle I_{\mathcal{F}_q(\mathcal{H})}$$

for all $f, g \in \mathcal{H}_\mathbb{C}$, where $I_{\mathcal{F}_q(\mathcal{H})}$ is the identity on $\mathcal{F}_q(\mathcal{H})$.

The operators $c_q, c_q^*$ satisfy the above $q$-commutation relations, which interpolate between the canonical commutation relations (CCR for short) and canonical anti-commutation relations (CAR for short) usually associated to the Boson and Fermion Fock spaces. For both Bosons and Fermions, the operators $c_q, c_q^*$ also satisfy additional (anti) commutation relation. In the Boson case, for example, $c(f)$ and $c(g)$ commute for any choices of $f$ and $g$. However, that if $q \neq \pm 1$ there are no relations between $c_q(f)$ and $c_q(g)$ if $\langle f, g\rangle = 0$.

When $-1 < q < 1$, the operators $c_q(f)$ and $c_q^*(f)$ are bounded on $\mathcal{F}_q(\mathcal{H})$ with

$$\|c_q(f)\| = \|c_q^*(f)\| = \begin{cases} \|f\|(1-q)^{-\frac{1}{2}}, & \text{if } 0 \leq q < 1, \\ \|f\|, & \text{if } -1 < q < 0, \end{cases}$$

and they are adjoints of each other with respect to our scalar product $\langle \cdot, \cdot \rangle_q$.

Now we can define $\Gamma_q(\mathcal{H})$ for a real Hilbert space $\mathcal{H}$, as the von Neumann algebra of operators on $\mathcal{F}_q(\mathcal{H})$ generated by the selfadjoint $q$-Gaussian operators

$$\omega(f) = c_q(f) + c_q^*(f), \quad f \in \mathcal{H}.$$

It was proved in [14] that $\Omega$ is a cyclic and separating trace vector for $\Gamma_q(\mathcal{H})$ and the state $\tau_q(a) = \langle \Omega, a\Omega\rangle$ for a $\in \Gamma_{\mathrm{q}}(\mathcal{H})$, is a faithful normal trace on $\Gamma_q(\mathcal{H})$.

Let $L^p(\Gamma_q(\mathcal{H}), \tau_q)$ be the non-commutative $L^p$-spaces with the trace $\tau_q$, for $1 \leq p \leq \infty$; i.e., $L^p(\Gamma_q(\mathcal{H}), \tau_q)$ is the completion of $\Gamma_q(\mathcal{H}) = L^\infty(\Gamma_q(\mathcal{H}), \tau_q)$ with respect to the norm

$$\|a\|_{L^p}^p = \tau_q[(a^*a)^{\frac{p}{2}}], \quad 1 \leq p < \infty.$$

These spaces share all the functional analytic features of the classical $L^p$-spaces, such as the uniform convexity for $p \in (0, \infty)$, duality between $L^p(\Gamma_q(\mathcal{H}), \tau_q)$ and $L^{p'}(\Gamma_q(\mathcal{H}), \tau_q)$ with $p^{-1} + p'^{-1} = 1$, and Riesz-Thorin interpolation, Hölder's and Clarkson's inequalities.

Among the properties of these spaces, in the sequel we will use in particular the following one.

**Proposition 2.2** (see [19, Lemma 3.1]) *If $t \in \mathbb{R} \to \varphi(t) \in L_+^p, 1 < p < \infty$ is differentiable (with respect to the $L^p$-norm) at $t_0$, and $\varphi(t_0) \neq 0$. Then $t \in \mathbb{R} \to \tau_q(\varphi(t))^p \in \mathbb{R}_+$ is differentiable at $t_0$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau_q(\varphi(t)^p)|_{t=t_0} = p\tau_q\Big(\varphi(t_0)^{p-1}\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)\Big|_{t=t_0}\Big).$$

**Remark 2.2** Since $\tau_q$ is a faithful trace, the map

$$\Phi : \Gamma_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{H})$$

defined as $\Phi(a) = a(\Omega)$ is a continuous imbedding of $\Gamma_q(\mathcal{H})$ into $\mathcal{F}_q(\mathcal{H})$ which extends to an unitary isomorphism of $L^2(\Gamma_q(\mathcal{H}), \tau_q)$ with $\mathcal{F}_q(\mathcal{H})$.

### 2.2 Markov semigroups and Dirichlet form on the above noncommutative space $L^2(\Gamma_q(\mathcal{H}), \tau_q)$

In this subsection we briefly recall the concepts and Deny-Beurling

correspondence of Markov semigroup and the associated Dirichlet form on $L^2(\Gamma_q(\mathcal{H}), \tau_q)$. The details can refer to [20–26].

When $a \in L_h^2(\Gamma_q(\mathcal{H}), \tau_q)$, the symbol $a \wedge 1$ will denote the projection of $a$ onto the closed convex set $\{x \in L_+^2(\Gamma_q(\mathcal{H}), \tau_q) : x \leq 1\}$, where 1 is the unit of $\Gamma_q(\mathcal{H})$.

**Definition 2.1** *A closed, densely defined, nonnegative quadratic form $(\varepsilon, D(\varepsilon))$ on $L^2(\Gamma_q(\mathcal{H}), \tau_q)$ is said to be*

(1) *real if for $a \in D(\varepsilon)$ then $J(a) \in D(\varepsilon)$ and $\varepsilon[J(a)] = \varepsilon[a]$;*

(2) *a Dirichlet form if it is real and $\varepsilon[a \wedge 1] \leq \varepsilon[a]$, for $a \in D(\varepsilon) \cap L^2_h(\Gamma_q(\mathcal{H}), \tau_q)$;*

(3) *a completely Dirichlet form if the canonical extension $(\varepsilon^n, D(\varepsilon^n))$ to $L^2(\Gamma_q(\mathcal{H}) \otimes M_n(\mathbb{C}), \tau_q^n)$ : $\varepsilon^n[[a_{ij}]_{i,j=1}^n] := \sum_{i,j=1}^n \varepsilon[a_{ij}]$, is a Dirichlet form for all $n \geq 1$, where $[a_{ij}]_{i,j=1}^n \in D(\varepsilon^n) := D(\varepsilon) \otimes M_n(\mathbb{C})$, and $\tau_q^n = \tau_q \otimes tr_n$ is the faithful, normal trace on the von Neumann algebra $M_n(\Gamma_q(\mathcal{H})) = \Gamma_q(\mathcal{H}) \otimes M_n(\mathbb{C})$, here $tr_n$ is a normalized trace on $M_n(\mathbb{C})$.*

**Proposition 2.3** (see [20, Proposition 4.5 and Proposition 4.10, 25, Lemma 2.3, 26, Proposition 2.12]) *Let $(\varepsilon, D(\varepsilon))$ be a closed, densely defined, nonnegative real quadratic form, and $1 \in D(\varepsilon)$, where 1 is the unit of $\Gamma_q(\mathcal{H})$. Then the following statements are equivalent*:

(1) *$(\varepsilon, D(\varepsilon))$ is a Dirichlet form with respect to* 1.

(2) *For every real-valued Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$, which satifies $|\varphi(t) - \varphi(s)| \leq c_\varphi |t - s|, \forall t, s \in \mathbb{R}$ and $\varphi(0) = 0$, where $c_\varphi$ is a positive constant, we have $\varepsilon[\varphi(x)] \leq \varepsilon[x]$ whenever $x \in D(\varepsilon) \cap L^2_h(\Gamma_q(\mathcal{H}), \tau_q)$.*

(3) *$\varepsilon(1, x) \geq 0$ for all $x \in D(\varepsilon) \cap L^2_+(\Gamma_q(\mathcal{H}))$, and $\varepsilon[|x|] \leq \varepsilon[x]$ for all $x \in D(\varepsilon) \cap L^2_h(\Gamma_q(\mathcal{H}))$.*

Recall that a sub-Markov semigroup $\{T_t\}_{t\geq 0}$ on $\Gamma_q(\mathcal{H}) = L^\infty(\Gamma_q(\mathcal{H}), \tau_q)$ is a semigroup consisting of positive normal linear operators on $\Gamma_q(\mathcal{H})$, such that $T_t 1 \leq 1$ and the map $t \to T_t(a)$ from $[0, \infty)$ to $\Gamma_q(\mathcal{H})$ is continuous with respect to the $\sigma$-weak topology on $\Gamma_q(\mathcal{H})$ for each $a \in \Gamma_q(\mathcal{H})$; if $\{T_t \otimes I_n\}$ is sub-Markovian on the von Neumann algebra $\Gamma_q(\mathcal{H}) \otimes M_n(\mathbb{C})$ for all $n \geq 1$, then $\{T_t\}_{t\geq 0}$ is called to be completely sub-Markov semigroup, where $I_n$ is the unit of matrix algebra $M_n(\mathbb{C})$.

For $1 \leq p < \infty$, we have the following concepts of sub-Markov semigroup on $L^p(\Gamma_q(\mathcal{H}), \tau_q)$.

**Definition 2.2** *A strongly continuous contractive semigroup $\{T_t = \mathrm{e}^{-tL}\}_{t\geq 0}$ consisting of bounded linear operators on $L^p(\Gamma_q(\mathcal{H}), \tau_q)$, $p \in [1, \infty)$, is said to be sub-Markov semigroup, if $0 \leq x \leq 1$ then $0 \leq T_t x \leq 1$, for all $t \geq 0$, where $L$ is the infinitesimal generator of the semigroup $T_t$, 1 is the unit of $\Gamma_q(\mathcal{H})$; furthermore, if $\{T_t \otimes I_n\}_{t\geq 0}$ on $L^p(\Gamma_q(\mathcal{H}), \tau_q) \otimes M_n(\mathbb{C})$ is sub-Markovian for all $n \geq 1$, then $\{T_t\}_{t\geq 0}$ is called to be completely sub-Markovian.*

**Theorem 2.1** (Beurling-Deny Correspondence, see [25, Theorems 2.7–2.8, 26, Theorem 3.3]) *Given a strongly continuous symmetric semigroup $T_t = \mathrm{e}^{-tL}$ with infinitesimal generator $L$, and the associated quadratic form $\varepsilon[x] = \langle \sqrt{L}x, \sqrt{L}x \rangle$, for $x \in D(\varepsilon) = D(\sqrt{L})$. Then the following are equivalent*:

(1) *The form $\varepsilon$ is a* (*completely*) *Dirichlet form.*

(2) *The semigroup $T_t = \mathrm{e}^{-tL}$ is* (*completely*) *sub-Markovian.*

**Remark 2.3** (1) From [26, Proposition 3.1] and [27, Theorem 3.3] the (completely) sub-Markov semigroup on $\Gamma_q(\mathcal{H})$ can be extended to (completely) sub-Markov semigroup on $L^p(\Gamma_q(\mathcal{H}), \tau_q)$.

(2) Given a Dirichlet form $\varepsilon(,)$, if the unit $1 \in D(\varepsilon)$ then it is called a conservative Dirichlet form; sub-Markov semigroup $T_t$ is Markovian in case $T_t(1) = 1$. From the above Theorem 2.1 it is easy to check that $\{T_t\}$ is (completely) Markovian if and only if the associated quadratic

form is (completely) conservative Dirichlet form.

### 2.3 $q$-Ornstein-Uhlenbeck semigroup

Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a contraction between real Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ with complexification $T_{\mathbb{C}}$, then the linear map defined on elementary tensors by

$$F_q(T)(f_1 \otimes \cdots \otimes f_n) = T_{\mathbb{C}} f_1 \otimes \cdots \otimes T_{\mathbb{C}} f_n$$

extends to a contraction

$$F_q(T) : \mathcal{F}_q(\mathcal{H}_1) \to \mathcal{F}_q(\mathcal{H}_2).$$

Now we define $q$-Gaussian functor $\Gamma_q$ as a map

$$\Gamma_q(T) : \Gamma_q(\mathcal{H}_1) \to \Gamma_q(\mathcal{H}_2)$$

as follows:

(1) $\Gamma_q(T)\omega(f) = \omega(Tf)$ for $f \in \mathcal{H}$;

(2) $(\Gamma_q(T)(X))\Omega = F_q(T)(X\Omega)$.

**Proposition 2.4** (see [17, Theorem 2.11]) *Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a contraction between real Hilbert spaces, then there exists a unique map $\Gamma_q(T) : \Gamma_q(\mathcal{H}_1) \to \Gamma_q(\mathcal{H}_2)$ such that*

$$\Gamma_q(T)(X)\Omega = F_q(T)(X\Omega)$$

*for every $X \in \Gamma_q(\mathcal{H}_1)$. Moreover, the map $\Gamma_q(T)$ is bounded, normal, unital, completely positive and the trace preserving.*

$\Gamma_q(T)$ is a functor, that is, if $T_1 : \mathcal{H}_1 \to \mathcal{H}_2,\ T_2 : \mathcal{H}_2 \to \mathcal{H}_3$ are contractions, then

$$\Gamma_q(T_2 T_1) = \Gamma_q(T_2)\Gamma_q(T_3).$$

Thus, by the above Remark 2.3, $\Gamma_q(T)$ has a completely positive and unital extension on all $L^p(\Gamma_q(\mathcal{H}), \tau_q)$-spaces.

Now we can introduce $q$-Ornstein-Uhlenbeck semigroup as follows (see [17]).

**Definition 2.3** *Let $\mathcal{H}$ be a real Hilbert space, and $T_t = \mathrm{e}^{-t} I_{\mathcal{H}}, t \geq 0$; the completely positive maps $U_t^{(q)} = \Gamma_q(T_t)$ on $\Gamma_q(\mathcal{H}) = L^\infty(\Gamma_q(\mathcal{H}), \tau_q)$ form a semigroup, called the q-Ornstein-Uhlenbeck semigroup.*

By the meaning of $U_t^{(q)}$ we see that it is a completely Markov semigroup on $\Gamma_q(\mathcal{H})$. From the above Remark 2.3 it implies that $U_t^{(q)}$ can be extended to completely Markov semigroup on all noncommutative $L^p(\Gamma_q(\mathcal{H}), \tau_q)$-spaces and $U_t^{(q)} = \mathrm{e}^{-tN^q}$, where its generator on $L^2(\Gamma_q(\mathcal{H}), \tau_q)$ is the number operator given by

$$N^q \Omega = 0;$$

and

$$N^q(f_1 \otimes \cdots \otimes f_n) = n(f_1 \otimes \cdots \otimes f_n), \quad f_j \in \mathcal{H}_{\mathbb{C}}(j = 1, 2, \cdots, n).$$

Then by the above Theorem 2.1 the corresponding quadratic form $\varepsilon[x] = \langle \sqrt{N^q}x, \sqrt{N^q}x \rangle$ for all $x \in D(\sqrt{N^q})$, is a completely conservative Dirichlet form.

Furthermore, $U_t^{(q)}$ has the Feller property as below.

**Proposition 2.5** *$U_t^{(q)}$ is a Feller semigroup, that is, $U_t^{(q)}\Gamma_q(\mathcal{H}) \subseteq \Gamma_q(\mathcal{H})$ and $U_t^{(q)}\Gamma_q^+(\mathcal{H}) \subseteq \Gamma_q^+(\mathcal{H})$.*

**Proof** Indeed, given a fixed orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ in $\mathcal{H}$, for any subset $I = \{i_1, i_2, \cdots, i_n\} \subseteq \mathbb{N}$, by [17, Proposition 2.7] we can construct the $q$-Wick product $\psi_I = \psi(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) \in \Gamma_q(\mathcal{H})$ by induction as below:

$$\begin{aligned}
&\psi(e_{i_k}) = \omega(e_{i_k}) = c_q(e_{i_k}) + c_q^*(e_{i_k});\\
&\psi(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) = \omega(e_{i_1})\psi(e_{i_2} \otimes \cdots \otimes e_{i_n})\\
&\qquad\qquad\qquad - \sum_{k=1}^{n} q^{k-1}\langle e_{i_1}, e_{i_k}\rangle \psi(e_{i_1} \otimes \cdots \otimes \check{e_{i_k}} \otimes \cdots \otimes e_{i_n}).
\end{aligned}$$

Then from the construction of the $q$-Ornstein-Uhlenbeck semigroup $U_t^{(q)}$, we have $U_t^{(q)}\psi_I = \mathrm{e}^{-tn}\psi_I$. Therefore, $U_t^{(q)}\Gamma_q(\mathcal{H}) \subseteq \Gamma_q(\mathcal{H})$ and $U_t^{(q)}\Gamma_q^+(\mathcal{H}) \subseteq \Gamma_q^+(\mathcal{H})$.

**Proposition 2.6** *The subset of invertible positive operators in $D(\varepsilon)$ is dense in every noncommutative space $L^p(\Gamma_q(\mathcal{H}), \tau_q)$ with respect to $L^p$-norm for all $p \geq 1$.*

**Proof** Given a fixed orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ in $\mathcal{H}$. For any subset $I = \{i_1, i_2, \cdots, i_n\} \subseteq \mathbb{N}$, we can construct the $q$-Wick product $\psi_I = \psi(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) \in \Gamma_q(\mathcal{H})$ (see the above Proposition 2.5). Then from the construction of the $q$-Ornstein-Uhlenbeck semigroup $U_t^{(q)} = \mathrm{e}^{-tN^q}$, we have $N^q\psi_I = n\psi_I$, from which follows that $\psi_I \in D(\varepsilon)$. Denote by $\widetilde{\Gamma_q(\mathcal{H})}$ the set of all finite linear combinations of such operators in $\psi_I$. It follows that $\widetilde{\Gamma_q(\mathcal{H})} \subset D(\varepsilon)$ since $D(\varepsilon)$ is a subspace of $L^2(\Gamma_q(\mathcal{H}), \tau_q)$. Since $\widetilde{\Gamma_q(\mathcal{H})}$ is dense in $\Gamma_q(\mathcal{H})$ with respect to the operator norm, and $\Gamma_q(\mathcal{H})$ is dense in every $L^p(\Gamma_q(\mathcal{H}), \tau_q)$ with respect to $L^p$-norm for all $p \geq 1$, notice that the operator norm is stronger than $L^p$-norm, then $\widetilde{\Gamma_q(\mathcal{H})}$ is dense in every $L^p(\Gamma_q(\mathcal{H}), \tau_q)$ with respect to $L^p$-norm for all $p \geq 1$ also. It follows that the subset of positive operators in $\widetilde{\Gamma_q(\mathcal{H})}$ is dense in every $L^p(\Gamma_q(\mathcal{H}), \tau_q)$ $(p \geq 1)$. Furthermore, since $N^q 1 = 0$, where 1 is the unit in $\Gamma_q(\mathcal{H})$, so that $1 \in D(\varepsilon)$. Thus, for any positive operator $x \in \widetilde{\Gamma_q(\mathcal{H})}$, then $x + \frac{1}{n}1$ is invertible for any natural number $n$, and $x + \frac{1}{n}1$ converges to $x$ in the operator norm. This shows that the subset of invertible positive operators in $\widetilde{\Gamma_q(\mathcal{H})}$, and then the subset of invertible positive operators in $D(\varepsilon)$ is dense in every noncommutative space $L^p(\Gamma_q(\mathcal{H}), \tau_q)$ with respect to $L^p$-norm for all $p \geq 1$.

## 3 The Main Results and Their Proofs

The main theme of this section is to prove the equivalence of hypercontractivity and logarithmic Sobolev inequality for $q$-Ornstein-Uhlenbeck semigroup $\{U_t^{(q)}\}(-1 \leq q \leq 1)$.

### 3.1 Hypercontractivity and logarithmic Sobolev inequality

First, recall the concept of hypercontractivity as follows.

**Definition 3.1** *A Markov semigroup $\{T_t\}_{t\geq 0}$ in the interpolating family $L^p(\Gamma_q(\mathcal{H}), \tau_q)$ is called hypercontractivity, if for every $1 < p < r < \infty$ there exist $a(p, r) \geq 0$ and $b(p, r) > 0$ such that $\forall t > a(p, r)$ we have $\|T_t x\|_r \leq b(p, r)\|x\|_p$, for each $x \in L^p(\Gamma_q(\mathcal{H})$; it is called strictly hypercontractivity if the constant $b(p, r) \leq 1$.*

**Remark 3.1** Given $a>0$ and $b\geq 0$, let $p(t)=1+(p-1)\mathrm{e}^{\frac{2t}{a}}$. Then the above hypercontractivity in Definition 3.1 is equivalent to the following statement.

For all $p\in(0,\infty)$ one has

$$\|T_t x\|_{p(t)}\leq \mathrm{e}^{b(\frac{1}{p}-\frac{1}{p(t)})}\|x\|_p$$

for $\forall t\geq 0$ and each $x\in L^p(\Gamma_q(\mathcal{H}),\tau_q)$.

In the commutative case, it is equivalent to $\|T_t x\|_4\leq c\|x\|_2, \forall t\geq 0$, here $c$ is a positive constant (see [28, Chapter 3, Theorem 3.2.2]).

Let $\mathrm{Ent}(x)=\tau_q(x\log x)-\|x\|_{L^2}\log\|x\|_{L^2}$ denote the relative entropy of a positive element $x$. Now we can state the main result of this paper.

**Theorem 3.1** *Given constants $a>0$ and $b\geq 0$. For $p>1$, let $p(t)=1+(p-1)\mathrm{e}^{\frac{2t}{a}}, b(t)=b\big(\frac{1}{p}-\frac{1}{p(t)}\big)$. Then the following statements are equivalent*:

(1) $\|U_t^{(q)}x\|_{p(t)}\leq \mathrm{e}^{b(t)}\|x\|_p$ *for all* $x\in\Gamma_q(\mathcal{H})$, $\forall p>1$, $\forall t\geq 0$;

(2) $\mathrm{Ent}(|x|^2)\leq 2a\varepsilon[x]+b\|x\|_2^2$ *for all* $x\in D(\varepsilon)$.

We need the following lemma which plays a crucial role for proving the above Theorem 3.1. It was proved by Biane in [13] (see [13, Lemma 3]), its special case in the Clifford algebra setting had been proved by Gross in [8] (see [8, Lemma 1.1]). Here we use spectral decomposition to give a new proof that is different from Biane's. This lemma is also very interesting itself.

**Lemma 3.1** *For all invertible positive $x\in D(\varepsilon)$, and $1<p<\infty$, one has*

$$\varepsilon[x^{\frac{p}{2}}]\leq \frac{p^2}{4(p-1)}\varepsilon(x,x^{p-1}).$$

*That is*

$$\langle x^{\frac{p}{2}},N^q x^{\frac{p}{2}}\rangle\leq \frac{p^2}{4(p-1)}\langle x,N^q x^{p-1}\rangle.$$

**Proof** First, notice that $x$ is invertible and positive, then there exists a constant $c>0$ such that $\mathrm{Spec}(x)\subseteq[c,\|x\|]$. Hence, by the above Proposition 2.3 item (2) combining the function calculus of $x$ it is easy to check that $x^{\frac{p}{2}}$ and $x^{p-1}$ are in $D(\varepsilon)$. By the definition and spectrum decomposition of $N^q$,

$$\langle x^{\frac{p}{2}},N^q x^{\frac{p}{2}}\rangle=\lim_{t\to 0}\frac{1}{t}\tau_q[(x^{\frac{p}{2}}-U_t^{(q)}x^{\frac{p}{2}})x^{\frac{p}{2}}];$$
$$\langle x,N^q x^{p-1}\rangle=\langle N^q x,x^{p-1}\rangle=\lim_{t\to 0}\frac{1}{t}\tau_q[(x-U_t^{(q)}x)x^{p-1}].$$

So, it suffices to prove

$$\tau_q[(x^{\frac{p}{2}}-U_t^{(q)}x^{\frac{p}{2}})x^{\frac{p}{2}}]\leq \frac{p^2}{4(p-1)}\tau_q[(x-U_t^{(q)}x)x^{p-1}]. \tag{3.1}$$

For any fixed $t>0$. Since $U_t^{(q)}$ is symmetric and Markovian, then for $\varphi$ and $\psi$ positive and continuous functions on $\mathbb{R}$, $\tau_q[\varphi(x)U_t^{(q)}(\psi(x))]$ is positive and linear in $\varphi$ and $\psi$. Moreover for $\varphi$ and $\psi$ such that $\varphi(\alpha)\leq c|\alpha|$ and $\psi(\alpha)\leq c'|\alpha|$, where $c,c'$ are constants. As a general property for normal traces on von Neumann algebras (see [25–26]), there exists a positive measure $\mu_x$ on $\mathbb{R}\backslash\{0\}\times\mathbb{R}\backslash\{0\}$ with support contained in $\mathrm{Spec}(x)\times\mathrm{Spec}(x)$ such that $\mu_x(\alpha,\beta)=\mu_x(\beta,\alpha)$ and

$$\tau_q[\varphi(x)U_t^{(q)}(\psi(x))]=\int\int\varphi(\alpha)\psi(\beta)\mathrm{d}\mu_x(\alpha,\beta).$$

Consider now the quadratic form $\tau_q[x(1-U_t^{(q)})x]$. Notice that $U_t^{(q)}(1)=1$, we then have

$$\begin{aligned}\tau_q[\varphi(x)(1-U_t^{(q)})\varphi(x)] &= \tau_q[\varphi(x)^2(1-U_t^{(q)}(1))]+\tau_q[\varphi(x)^2U_t^{(q)}(1)-\varphi(x)U_t^{(q)}(\varphi(x))]\\ &= \tau_q[\varphi(x)^2U_t^{(q)}(1)-\varphi(x)U_t^{(q)}(\varphi(x))]=\tau_q[\varphi(x)^2-\varphi(x)U_t^{(q)}(\varphi(x))].\end{aligned}$$

Therefore we have

$$\tau_q[\varphi(x)(1-U_t^{(q)})\varphi(x)]=\frac{1}{2}\int\int[\varphi(\alpha)-\varphi(\beta)]^2\mathrm{d}\mu_x(\alpha,\beta). \tag{3.2}$$

In the following let $\varphi(\alpha)=\alpha^{\frac{p}{2}},\alpha\in\mathrm{Spec}(x)$. Take $\varphi(x)=x^{\frac{p}{2}}$ in the above equation (3.2), we obtain

$$\tau_q[(x^{\frac{p}{2}}-U_t^{(q)}x^{\frac{p}{2}})x^{\frac{p}{2}}]=\frac{1}{2}\int\int(\alpha^{\frac{p}{2}}-\beta^{\frac{p}{2}})^2\mathrm{d}\mu_x(\alpha,\beta).$$

Similarly,

$$\tau_q[x^{p-1}(x-U_t^{(q)}x)]=\frac{1}{2}\int\int(\alpha-\beta)(\alpha^{p-1}-\beta^{p-1})\mathrm{d}\mu_x(\alpha,\beta).$$

Then (3.1) holds true from the above two formulas combining the following fact

$$(a^{\frac{p}{2}}-b^{\frac{p}{2}})^2\le\frac{p^2}{4(p-1)}(a-b)(a^{p-1}-b^{p-1}),\quad a,b\ge 0,\ p>1.$$

**The Proof of Theorem 3.1** First, from [12, Theorem 3] and [13, Lemma 4] we see that the norm of $U_t^{(q)}$ from $L_h^p(\Gamma_q(\mathcal{H}),\tau_q)$ to $L_h^r(\Gamma_q(\mathcal{H}),\tau_q)$ $(p,r>1)$ is achieved on the positive cone $L_+^p(\Gamma_q(\mathcal{H}),\tau_q)$. So it is sufficient to consider hypercontractivity on positive cones. Given an invertible positive $x\in D(\varepsilon)$, put $\varphi(t)=\mathrm{e}^{-b(t)}\|U_t^{(q)}x\|_{p(t)}$. By [29, Lemma 2] and the above Proposition 2.2, a straightforward calculus shows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log\varphi(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(-b(t)+\log\|U_t^{(q)}x\|_{p(t)})\\ &= -b'(t)+\frac{1}{\|U_t^{(q)}x\|_{p(t)}}\frac{\mathrm{d}\|U_t^{(q)}x\|_{p(t)}}{\mathrm{d}t}\\ &= -b'(t)+\frac{1}{\|U_t^{(q)}x\|_{p(t)}}\frac{\mathrm{d}}{\mathrm{d}t}[\tau_q(U_t^{(q)}x)^{p(t)}]^{\frac{1}{p(t)}}.\end{aligned}\tag{I}$$

Since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[\tau_q(U_t^{(q)}x)^{p(t)}]^{\frac{1}{p(t)}} &= \|U_t^{(q)}x\|_{p(t)}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{1}{p(t)}\log\tau_q(U_t^{(q)}x)^{p(t)}\Big]\\ &= \|U_t^{(q)}x\|_{p(t)}\Big[-\frac{p'(t)}{p^2(t)}\log\|U_t^{(q)}x\|_{p(t)}^{p(t)}+\frac{1}{p(t)}\frac{1}{\|U_t^{(q)}x\|_{p(t)}^{p(t)}}\frac{\mathrm{d}\|U_t^{(q)}x\|_{p(t)}^{p(t)}}{\mathrm{d}t}\Big],\end{aligned}$$

and since

$$\begin{aligned}\frac{\mathrm{d}\|U_t^{(q)}x\|_{p(t)}^{p(t)}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\tau_q[(U_t^{(q)}x)^{p(t)}]\\ &= \tau_q\Big[(U_t^{(q)}x)^{p(t)}\Big(p'(t)\log U_t^{(q)}x+p(t)(U_t^{(q)}x)^{-1}\frac{\mathrm{d}U_t^{(q)}x}{\mathrm{d}t}\Big)\Big],\end{aligned}$$

then take the above equation to formula (I), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\log\varphi(t) &= -b'(t) - \frac{p'(t)}{p^2(t)}\log\|U_t^{(q)}x\|_{p(t)}^{p(t)} \\
&\quad + \frac{1}{p(t)\|U_t^{(q)}x\|_{p(t)}^{p(t)}}\tau_q\Big[(U_t^{(q)}x)^{p(t)}\Big(p'(t)\log U_t^{(q)}x + p(t)(U_t^{(q)}x)^{-1}\frac{\mathrm{d}U_t^{(q)}x}{\mathrm{d}t}\Big)\Big] \\
&= -b'(t) - \frac{p'(t)}{p^2(t)}\log\|U_t^{(q)}x\|_{p(t)}^{p(t)} + \frac{1}{p(t)\|U_t^{(q)}x\|_{p(t)}^{p(t)}}\tau_q[p'(t)(U_t^{(q)}x)^{p(t)}\log U_t^{(q)}x] \\
&\quad + \tau_q\Big[p(t)(U_t^{(q)}x)^{p(t)-1}\frac{\mathrm{d}U_t^{(q)}x}{\mathrm{d}t}\Big]. \qquad \text{(II)}
\end{aligned}
$$

Notice that $\frac{\mathrm{d}U_t^{(q)}x}{\mathrm{d}t} = -N^q(U_t^{(q)}x)$, so that

$$
\tau_q\Big[(U_t^{(q)}x)^{p(t)-1}\frac{\mathrm{d}U_t^{(q)}x}{\mathrm{d}t}\Big] = -\varepsilon((U_t^{(q)}x)^{p(t)-1}, U_t^{(q)}x).
$$

On the other hand,

$$
\begin{aligned}
\mathrm{Ent}((U_t^{(q)}x)^{p(t)}) &= \tau_q[(U_t^{(q)}x)^{p(t)}\log(U_t^{(q)}x)^{p(t)}] - \tau_q[(U_t^{(q)}x)^{p(t)}\log\tau_q[(U_t^{(q)}x)^{p(t)}]] \\
&= \tau_q[(U_t^{(q)}x)^{p(t)}\log(U_t^{(q)}x)^{p(t)}] - \|U_t^{(q)}x\|_{p(t)}^{p(t)}\log\|U_t^{(q)}x\|_{p(t)}^{p(t)}.
\end{aligned}
$$

Combing with the above formula (II) one can obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\log\varphi(t) &= -b'(t) + \frac{p'(t)}{p(t)^2}\frac{1}{\|U_t^{(q)}x\|_{p(t)}^{p(t)}}\mathrm{Ent}((U_t^{(q)}x)^{p(t)}) \\
&\quad - \frac{1}{\|U_t^{(q)}x\|_{p(t)}^{p(t)}}\varepsilon((U_t^{(q)}x)^{p(t)-1}, U_t^{(q)}x). \qquad \text{(III)}
\end{aligned}
$$

Assume (1). Since $b(0) = 0$ and $p(0) = p$, it follows that $\varphi(0) = \|x\|_p$, then by the hypercontractivity of $(U_t^{(q)})$ implies that $\varphi'(0) \le 0$, which gives, via the above formula (III),

$$
\mathrm{Ent}((U_t^{(q)}x)^{p(t)}) \le \frac{p^2(t)\|U_t^{(q)}x\|_{p(t)}^{p(t)}}{p'(t)}\Big[b'(t) + \frac{1}{\|U_t^{(q)}x\|_{p(t)}^{p(t)}}\varepsilon((U_t^{(q)}x)^{p(t)-1}, U_t^{(q)}x)\Big].
$$

Let $t = 0$ and $p = 2$, from which follows that $p(0) = 2$, $p'(0) = \frac{2}{a}$, $b'(0) = \frac{1}{2a}$. Therefore, from the above inequality it implies that

$$
\mathrm{Ent}(x^2) \le 2a\varepsilon[x] + b\|x\|_2^2.
$$

Now, given any positive element $y \in D(\varepsilon)$. For any fixed natural number $n$, then $y + \frac{1}{n}x$ is invertible positive in $D(\varepsilon)$. By the above proof we have

$$
\mathrm{Ent}\Big(\Big(y + \frac{1}{n}x\Big)^2\Big) \le 2a\varepsilon\Big[y + \frac{1}{n}x\Big] + b\Big\|y + \frac{1}{n}x\Big\|_2^2.
$$

Notice that

$$
\varepsilon\Big[y + \frac{1}{n}x\Big] = \Big\langle y + \frac{1}{n}x, N^q\Big(y + \frac{1}{n}x\Big)\Big\rangle = \varepsilon[y] + \frac{1}{n^2}\varepsilon[x] + \frac{2}{n}\varepsilon(x, y).
$$

Let $n \to \infty$, and combining the continuity of norm we obtain

$$\mathrm{Ent}(y^2) \le 2a\varepsilon[y] + b\|y\|_2^2.$$

Finally, for general $z \in D(\varepsilon)$, from the above proof we have

$$\mathrm{Ent}(|z|^2) \le 2a\varepsilon[|z|] + b\|z\|_2^2.$$

Since $\varepsilon[|z|] \le \varepsilon[z]$ (see the above Proposition 2.3), then combining the above inequality implies that

$$\mathrm{Ent}(|z|^2) \le 2a\varepsilon[z] + b\|z\|_{L^2}^2.$$

Conversely, assume (2). For a given invertible positive $x \in D(\varepsilon)$, we have

$$\mathrm{Ent}(x^2) \le 2a\varepsilon[x] + b\|x\|_2^2.$$

Replace $x$ with $x^{\frac{p}{2}}$, one can get

$$\mathrm{Ent}(x^p) \le 2a\varepsilon[x^{\frac{p}{2}}] + b\|x^{\frac{p}{2}}\|_2^2.$$

By Lemma 3.1 it implies that

$$\mathrm{Ent}(x^p) \le \frac{ap^2}{2(p-1)}\varepsilon(x^{p-1}, x) + b\|x\|_p^p.$$

Set $x$ be $U_t^{(q)}x$ and $p$ be $p(t)$ in the above formula, we have

$$\mathrm{Ent}((U_t^{(q)}x)^{p(t)}) \le \frac{ap(t)^2}{2(p(t)-1)}\varepsilon((U_t^{(q)}x)^{p-1}, U_t^{(q)}x) + b\|U_t^{(q)}x\|_{p(t)}^{p(t)}. \tag{IV}$$

Notice that the above formula (III),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log\varphi(t) = {}& \frac{p'(t)}{p^2(t)\|U_t^{(q)}x\|_{p(t)}^{p(t)}}\Big[\mathrm{Ent}((U_t^{(q)}x)^{p(t)}) \\ & - \frac{p(t)^2}{p'(t)}\varepsilon((U_t^{(q)}x)^{p(t)-1}, U_t^{(q)}x) - \frac{b'(t)p(t)^2}{p'(t)}\|U_t^{(q)}x\|_{p(t)}^{p(t)}\Big].\end{aligned} \tag{V}$$

Since $b(t) = b\big(\frac{1}{p} - \frac{1}{p(t)}\big)$, $p(t) = 1 + (p-1)\mathrm{e}^{\frac{2t}{a}}$, then $b'(t) = \frac{p'(t)}{p(t)^2}$, $p'(t) = \frac{2}{a}(p-1)\mathrm{e}^{\frac{2t}{a}}$. From which follows that $\frac{p(t)^2}{p'(t)} = \frac{ap(t)^2}{2(p(t)-1)}$, and $\frac{b'(t)p(t)^2}{p'(t)} = b$. Compare (IV) and (V), then $\frac{\mathrm{d}}{\mathrm{d}t}\log\varphi(t) \le 0$, thus, $\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) \le 0$. Therefore, $\varphi(t) \le \varphi(0) = \|x\|_p$. Since the subset of invertible positive elements in $D(\varepsilon)$ is dense in all $L^p(\Gamma_q(\mathcal{H}), \tau_q)$ for all $p > 1$ (see the above Proposition 2.6), and since $\varphi(t)$ is continuous in the operator norm, which in turn yields the hypercontractivity of $\{U_t^{(q)}\}$.

Biane (see [13, Corollary 1]) derived a logarithmic Sobolev inequality from strictly hypercontractivity. In fact, if $a = 1$ and $b(t) = 0$ in the above Theorem 3.1, the following direct consequence shows that strictly hypercontractivity and the logarithmic Sobolev inequality in [13, Corollary 1] is equivalent.

**Corollary 3.1** *The conditions are the same as in the above Theorem* 3.1. *Then the following statements are equivalent*:

(1) $\|U_t^{(q)}x\|_{p(t)} \le \|x\|_p$ *for all* $x \in \Gamma_q(\mathcal{H})$, $\forall p > 1$, $\forall t \ge 0$;

(2) $\mathrm{Ent}(|x|^2) \le 2\,\varepsilon[x]$ *for all* $x \in D(\varepsilon)$.

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