Chin. Ann. Math. Ser. B
41(4), 2020, 627–644
DOI: 10.1007/s11401-020-0222-9

Chinese Annals of Mathematics, Series B

Long-Time Dynamics for a Nonlinear Viscoelastic Kirchhoff Plate Equation*

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Abstract This paper is devoted to study the long-time dynamics for a nonlinear viscoelastic Kirchhoff plate equation. Under some growth conditions of g and f, the existence of a global attractor is granted. Furthermore, in the subcritical case, this global attractor has finite Hausdorff and fractal dimensions.

Keywords Global attractor, Kirchhoff plate, Memory kernel, Finite Hausdorff dimensions
2000 MR Subject Classification 35B41, 37L30, 35L75

1 Introduction

In the past ten years many attentions were attracted by the following models of Kirchhoff plates subject to a viscoelastic damping

$$u_{tt} - \sigma\Delta u_{tt} + \alpha\Delta^2 u - \int_{-\infty}^{t} \mu(t-s)\Delta^2 u(s)\mathrm{d}s = \mathcal{F}, \tag{1.1}$$

where $\sigma > 0$ is the uniform plate thickness, the kernel $\mu > 0$ corresponds to the viscoelastic flexural rigidity, and $\mathcal{F} = \mathcal{F}(x, t, u, u_t, \cdots)$ represents additional damping and forcing terms. The unknown function $u = u(x, t)$ represents the transverse displacement of a plate filament with prescribed history $u_0(x, t), t \leq 0$. The derivation of the linear mathematical model (1.1) with $\mathcal{F} = 0$ is given in [1–2], by assuming viscoelastic stress-strain laws on an isotropic material occupying a region of $\mathbb{R}^3$ and constant Poissons ratio.

When $\sigma = \mu = 0$, i.e., neglecting the influence of viscoelastic term and rotational inertia term, the model (1.1) was extensively studied by Yang [3–6] and Yang and Jin [7]. More precisely they proved the global solvability and existence of finite-dimensional global attractors to a strongly damped system of the form

$$u_{tt} + \Delta^2 u - \mathrm{div}(|\nabla u|^{p-2}\nabla u) - \Delta u_t = h(x, u, u_t), \tag{1.2}$$

in bounded domains of $\mathbb{R}^N$ with critical and subcritical exponents p. In particular, in a two-dimensional setting with $p = 4$ and weak damping, (1.2) corresponds to the so called Kirchhoff-

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*This work was supported by the Natural Science Foundation of Guangdong Province (No. 2016A03 0310262).

Boussinesq model for nonlinear plates

$$u_{tt} + \Delta^2 u - \mathrm{div}(|\nabla u|^{p-2}\nabla u) + ku_t = \alpha\Delta(u^2) - f(u). \tag{1.3}$$

Chueshov and Lasiecka [8–9] established the existence of finite-dimensional global attractors to (1.3) under a weak damping ku_t instead of $-\Delta u_t$.

If $\sigma = 0$ and $\mathcal{F} = \mathrm{div}(|\nabla u|^{p-2}\nabla u) - g(u) + \Delta u_t + h(x), p \geq 2$, the model (1.1) becomes

$$u_{tt} + \alpha\Delta^2 u - \mathrm{div}(|\nabla u|^{p-2}\nabla u) - \int_0^\infty \mu(s)\Delta^2 u(t-s)\mathrm{d}s - \Delta u_t + g(u) = h(x). \tag{1.4}$$

In [10–11], Jorge Silva and Ma established the well-posedness, exponential stability and long-time dynamics of (1.4) Narciso [12] discussed the long-time behavior of the following evolution equation:

$$u_{tt} + \alpha\Delta^2 u - \int_{-\infty}^t \mu(t-s)\Delta^2 u(s)\mathrm{d}s + f(u) + g(u_t) = h(x).$$

By considering the nonlinear damping and source terms

$$g(u_t) \approx |u_t|^{p-1}u_t \quad \text{and} \quad f(u) \approx |u|^\alpha u - |u|^\beta u$$

with $1 < p \leq 3$ and $0 \leq \beta < \alpha \leq 2$, the author showed the existence of the global attractor. This work was extended by Conti and Geredeli [13] to a situation where g has any arbitray polynomial growth instead of cubic at most and f has no growth restriction.

In the presence of the rotational inertia term ($\sigma > 0$), the model (1.1) with $\alpha = 1$ and $u(x,t) = 0, t \leq 0$ was considered by several authors. In [14], Barreto et al. studied the following viscoelastic equation:

$$u_{tt} - \sigma\Delta u_{tt} + \Delta^2 u - \int_0^t \mu(t-s)\Delta^2 u(s)\mathrm{d}s = 0. \tag{1.5}$$

They established that the energy decays to zero with the same rate of the kernel μ such as exponential and polynomial decay.

More recently, Jorge Silva et al. [15] investigated the well-posedness and the asymptotic behavior of energy to the following nonlinear viscoelastic Kirchhoff plate equation:

$$u_{tt} - \sigma\Delta u_{tt} + \Delta^2 u - \mathrm{div} f(\nabla u) - \int_0^t \mu(t-s)\Delta^2 u(s)\mathrm{d}s = 0. \tag{1.6}$$

Motivated by the works above, our goal of this paper is to discuss the long-time behavior of the following nonlinear viscoelastic Kirchhoff plate equation:

$$u_{tt} - \sigma\Delta u_{tt} + \alpha\Delta^2 u - \mathrm{div} f(\nabla u) - \int_0^\infty \mu(s)\Delta^2 u(t-s)\mathrm{d}s - \Delta u_t + g(u) = h(x), \tag{1.7}$$

with simply supported boundary condition

$$u = \Delta u = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R} \tag{1.8}$$

and initial conditions

$$u(x,\tau) = u_0(x,\tau) \quad \text{and} \quad u_t(x,\tau) = \partial_t u_0(x,\tau), \quad (x,\tau) \in \Omega \times (-\infty, 0], \tag{1.9}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $u_0 : \Omega \times (-\infty, 0] \to \mathbb{R}$ is the prescribed past history of u. Here, $\alpha > 0$ is a constant, μ is the memory kernel, $f : \mathbb{R}^N \to \mathbb{R}^N$ is a vector field, and g, h are forcing terms. Without loss of generality we can take $\sigma = 1$.

Following the framework proposed in [16–17], which uses an argument of [18], we shall add a new variable η^t to the system which corresponds to the relative displacement history. Let us define

$$\eta^t(x, s) = u(x, t) - u(x, t - s), \quad (x, s) \in \Omega \times \mathbb{R}^+,\ t \geq 0. \tag{1.10}$$

Differentiating (1.10) with respect to t we have

$$\eta_t^t(x, s) = -\eta_s^t(x, s) + u_t(x, t), \quad (x, s) \in \Omega \times \mathbb{R}^+,\ t \geq 0$$

and we can take as initial condition ($t = 0$)

$$\eta^0(x, s) = u_0(x, 0) - u_0(x, -s), \quad (x, s) \in \Omega \times \mathbb{R}^+.$$

Thus, the original memory term can be rewritten as

$$\int_0^\infty \mu(s)\Delta^2 u(t - s)\mathrm{d}s = \Big(\int_0^\infty \mu(s)\mathrm{d}s\Big)\Delta^2 u(t) - \int_0^\infty \mu(s)\Delta^2\eta^t(s)\mathrm{d}s.$$

Then assuming for simplicity that $\alpha = 1 + \int_0^\infty \mu(s)\mathrm{d}s$, (1.7) becomes

$$u_{tt} - \Delta u_{tt} - \mathrm{div} f(\nabla u) + \Delta^2 u + \int_0^\infty \mu(s)\Delta^2\eta^t(s)\mathrm{d}s - \Delta u_t + g(u) = h(x), \tag{1.11}$$

$$\eta_t^t(x, s) = -\eta_s^t(x, s) + u_t(x, s), \tag{1.12}$$

with boundary condition

$$u = \Delta u = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \quad \eta^t = \Delta\eta^t = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+ \tag{1.13}$$

and initial conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \eta^t(x, 0) = 0, \quad \eta^0(x, s) = \eta_0(x, s), \tag{1.14}$$

where

$$\begin{cases} u_0(x) = u_0(x, 0), & x \in \Omega, \\ u_1(x) = \partial_t u_0(x, t)|_{t=0}, & x \in \Omega, \\ \eta_0(x, s) = u_0(x, 0) - u_0(x, -s), & (x, s) \in \Omega \times \mathbb{R}^+. \end{cases}$$

2 Preliminaries

In this section we recall some fundamentals of the theory of infinite-dimensional dynamical systems which can be founded in classic references such as [19–22]. Below we follow more closely the book by Chueshov and Lasiecka [23–24].

Theorem 2.1 *A dissipative dynamical system $(\mathcal{H}, S(t))$ has a compact global attractor if and only if it is asymptotically smooth.*

The proof of asymptotic smoothness property can be very delicate. Here we use the following "compensated compactness"result in [23–24] and [25–26] for other applications.

Theorem 2.2 *Suppose that for any bounded positively invariant set $B \subset \mathcal{H}$ and for any $\varepsilon > 0$, there exists $T = T(\varepsilon, B)$ such that*

$$\|S(T)x - S(T)y\|_{\mathcal{H}} \le \varepsilon + \phi_T(x, y), \quad \forall\, x, y \in B,$$

where $\phi_T : B \times B \to \mathbb{R}$ satisfies

$$\liminf_{n\to\infty} \liminf_{m\to\infty} \phi_T(z_n, z_m) = 0, \tag{2.1}$$

for any sequence $\{z_n\}_{n\in\mathbb{N}}$ in B. Then $S(t)$ is asymptotic smooth in $\mathcal{H}$.

Let X, Y, Z be three reflexive Banach spaces with X compactly embedded in Y and put $\mathcal{H} = X \times Y \times Z$. Consider the dynamical system $(\mathcal{H}, S(t))$ given by an evolution operator

$$S(t)w = (u(t), u_t(t), \eta^t), \quad w = (u_0, u_1, \eta_0) \in \mathcal{H}, \tag{2.2}$$

where the functions u and η^t have regularity

$$u \in C(\mathbb{R}^+; X) \cap C^1(\mathbb{R}^+; Y), \quad \eta^t \in C(\mathbb{R}^+; Z). \tag{2.3}$$

Then one says that $(\mathcal{H}, S(t))$ is quasi-stable on a set $B \subset \mathcal{H}$ if there exists a compact seminorm n_X on X and nonnegative scalar functions $a(t)$ and $c(t)$, locally bounded in $[0, \infty)$, and $b(t) \in L^1(\mathbb{R}^+)$ with $\lim\limits_{t\to+\infty} b(t) = 0$, such that,

$$\|S(t)w_1 - S(t)w_2\|^2_{\mathcal{H}} \le a(t)\|w_1(t) - w_2(t)\|^2_{\mathcal{H}} \tag{2.4}$$

and

$$\|S(t)w_1 - S(t)w_2\|^2_{\mathcal{H}} \le b(t)\|w_1(t) - w_2(t)\|^2_{\mathcal{H}} + c(t) \sup_{0<s<t} [n_X(u(s) - v(s))]^2, \tag{2.5}$$

for any $w_1 = (u, u_t, \eta^t), w_2 = (v, v_t, \xi^t) \in B$. The inequality (2.5) is often called stabilizability inequality.

Theorem 2.3 *Let $(\mathcal{H}, S(t))$ be given by (2.2) and satisfying (2.3). If $(\mathcal{H}, S(t))$ possesses a compact global attractor $\mathcal{A}$ and is quasi-stable on $\mathcal{A}$, then the attractor $\mathcal{A}$ has finite fractal dimension.*

3 Assumptions and the Main Result

We start this section introducing the following Hilbert spaces

$$V_0 = L^2(\Omega), \quad V_1 = H^1_0(\Omega), \quad V_2 = H^2(\Omega) \cap H^1_0(\Omega)$$

and

$$V_3 = \{u \in H^3(\Omega) | u = \Delta u = 0 \text{ on } \partial\Omega\},$$

equipped with respective inner products and norms,

$$\begin{aligned}
(u, v)_{V_1} &= (\nabla u, \nabla v) \quad \text{and} \quad \|u\|_{V_1} = \|\nabla u\|_2,\\
(u, v)_{V_2} &= (\Delta u, \Delta v) \quad \text{and} \quad \|u\|_{V_2} = \|\Delta u\|_2,\\
(u, v)_{V_3} &= (\nabla\Delta u, \nabla\Delta v) \quad \text{and} \quad \|u\|_{V_3} = \|\nabla\Delta u\|_2.
\end{aligned}$$

As usual, $\|\cdot\|_p$ denotes the L^p-norms as well as $(\cdot,\cdot)$ denotes either the L^2-inner product or else a duality pairing between a Banach space V and its dual V'. The constants $\lambda_0, \lambda_1, \lambda_2 > 0$ represent the embedding constants

$$\lambda_0\|u\|_2^2 \le \|\nabla u\|_2^2, \quad \lambda_1\|u\|_2^2 \le \|\Delta u\|_2^2, \quad \lambda_2\|\nabla u\|_2^2 \le \|\Delta u\|_2^2 \quad \text{for } u \in V_2. \tag{3.1}$$

In order to consider the relative displacement η^t as a new variable, one introduces the weighted L^2-spaces

$$\mathcal{M}_i := L^2_\mu(\mathbb{R}^+; V_i) = \Big\{\xi : \mathbb{R}^+ \to V_i \;\Big|\; \int_0^\infty \mu(s)\|\xi(s)\|^2_{V_i}\mathrm{d}s < \infty\Big\}, \quad i = 0, 1, 2, 3,$$

which are Hilbert spaces endowed with inner products and norms

$$(\xi_1, \xi_2)_{\mu,i} = \int_0^\infty \mu(s)(\xi_1(s), \xi_2(s))_{V_i}\mathrm{d}s$$

and

$$\|\xi\|^2_{\mu,i} = \int_0^\infty \mu(s)\|\xi(s)\|^2_{V_i}\mathrm{d}s, \quad i = 0, 1, 2, 3$$

respectively.

Now let us introduce the phase spaces

$$\mathcal{H} = V_2 \times V_1 \times \mathcal{M}_2 \quad \text{and} \quad \mathcal{H}_1 = V_3 \times V_2 \times \mathcal{M}_3, \tag{3.2}$$

equipped with norms

$$\|(u, v, \xi)\|_{\mathcal{H}} = \|\Delta u\|_2^2 + \|\nabla v\|_2^2 + \|\xi\|^2_{\mu,2}$$

and

$$\|(u, v, \xi)\|_{\mathcal{H}_1} = \|\nabla\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|\xi\|^2_{\mu,3}$$

respectively.

Next we impose some hypotheses on f, g and μ.

Assumption A.1 Concerning the forcing term $f : \mathbb{R}^N \to \mathbb{R}^N$ is a C^1-vector field given by $f = (f_1, \cdots, f_N)$ such that

$$|\nabla f_j(z)| \le k_j(1 + |z|^{\frac{p_j - 1}{2}}), \quad \forall z \in \mathbb{R}^N, \tag{3.3}$$

where, for every $j = 1, \cdots, N$, we consider $k_j > 0$ and p_j satisfying

$$p_j \ge 1 \quad \text{if} \quad N = 1, 2 \quad \text{and} \quad 1 \le p_j \le \frac{N+2}{N-2} \quad \text{if} \quad N \ge 3. \tag{3.4}$$

Moreover, f is a conservative vector field with $f = \nabla F$, where $F : \mathbb{R}^N \to \mathbb{R}$ is a real valued function satisfying

$$-\beta_0 - \frac{\beta}{2}|z|^2 \le F(z) \le f(z)\cdot z + \frac{\beta}{2}|z|^2, \quad \forall z \in \mathbb{R}^N, \tag{3.5}$$

where $\beta_0 \ge 0$ and $\beta \in \big[0, \frac{\lambda_2}{2}\big)$.

Remark 3.1 Observe that the vector field satisfying conditions (3.3) and (3.5) includes not only usual p-Laplacian operator but also other forms. Then we give examples of vector fields.

Let us consider

$$F(z) = \frac{1}{p}|z|^p, \quad z = (z_1, \cdots, z_N) \in \mathbb{R}^N, \quad p \geq 2.$$

We note that $f = \nabla F$. Then we have

$$f(z) = |z|^{p-2}z.$$

It is also easy to verify that (3.3) and (3.5) hold true. Therefore, this vector field generates the following p-Laplacian operator

$$\text{div} f(\nabla u) = \text{div}(|\nabla u|^{p-2}\nabla u).$$

Another case of p-Laplacian operator arises when we consider the vector field $f = (f_1, \cdots, f_N)$ whose components $f_j, j = 1, \cdots, N$ are given by

$$f_j(z) = |z_j|^{p-2}z_j, \quad \forall\ z = (z_1, \cdots, z_N) \in \mathbb{R}^N,$$

where $p \geq 2$. In this case

$$\text{div} f(\nabla u) = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\Big(\Big|\frac{\partial u}{\partial x_j}\Big|^{p-2}\frac{\partial u}{\partial x_j}\Big).$$

To illustrate another vector field, different one of p-Laplacian type, we consider $f = \nabla F$, where the potential function is given by

$$F(z) = \ln(\sqrt{|z|^2 + 1}), \quad z = (z_1, \cdots, z_N) \in \mathbb{R}^N.$$

In such case we have

$$f(z) = \frac{z}{|z|^2 + 1}, \quad \forall\ z \in \mathbb{R}^N,$$

which vanishes when $z \to \infty$. It is easy to check that F and f satisfy (3.3) and (3.5). Thus

$$\text{div}(f(\nabla u)) = \text{div}\Big(\frac{\nabla u}{|\nabla u|^2 + 1}\Big).$$

Assumption A.2 With respect to $g : \mathbb{R} \to \mathbb{R}$ we assume that

$$g(0) = 0, \quad |g(u) - g(v)| \leq \sigma_0(1 + |u|^q + |v|^q)|u - v|, \quad \forall u, v \in \mathbb{R}, \tag{3.6}$$

where $\sigma_0 > 0$ and

$$0 < q \leq \frac{4}{N-4} \quad \text{if} \quad N \geq 5 \quad \text{and} \quad q > 0 \quad \text{if} \quad 1 \leq N \leq 4. \tag{3.7}$$

In addition, we assume that for some $\sigma_1 \geq 0$,

$$-\sigma_1 \leq G(u) \leq g(u)u, \quad \forall u \in \mathbb{R}, \tag{3.8}$$

where $G(z) = \int_0^z g(s)\text{d}s$.

Assumption A.3 The memory kernel is required to satisfy the following hypotheses

$$\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \mu'(s) \le 0, \quad \mu(s) \ge 0, \tag{3.9}$$

and there exist $\mu_0, \delta > 0$ such that

$$\int_0^\infty \mu(s)\mathrm{d}s = \mu_0 \tag{3.10}$$

and

$$\mu'(s) + \delta\mu(s) \le 0, \quad \forall s \in \mathbb{R}^+. \tag{3.11}$$

Remark 3.2 Applying (3.4) it follows from Sobolev embedding that

$$V_2 \hookrightarrow W_0^{1,p_j+1}(\Omega), \quad \forall j = 1, \cdots, N.$$

Thereby, the constants $\mu_{p_1}, \cdots, \mu_{p_N} > 0$ represent the embedding constants for

$$\|\nabla u\|_{p_j+1} \le \mu_{p_j} \|\triangle u\|_2^2, \quad j = 1, \cdots, N. \tag{3.12}$$

Also, condition (3.7) implies that

$$V_2 \hookrightarrow L^{2(q+1)}.$$

In addition, assumptions (3.6) and (3.8) include nonlinear terms of the form

$$g(u) \approx |u|^q u \pm |u|^\theta u, \quad 0 < \theta < q.$$

Given initial data $(u_0, u_1, \eta_0) \in \mathcal{H}$ and $h \in V_0$, a function $z = (u, u_t, \eta^t) \in C([0,T], \mathcal{H})$ is called a weak solution of the problem (1.11)–(1.14) if it satisfies the initial condition $z(0) = (u_0, u_1, \eta_0)$ and

$$\begin{aligned}
&(u_{tt}, \omega) + (\nabla u_{tt}, \nabla\omega) + (\Delta u, \Delta\omega) + (\nabla u_t, \nabla\omega) + (f(\nabla u), \nabla\omega) \\
&+ \int_0^\infty (\Delta\eta^t, \Delta\omega) + (g(u) - h, \omega) = 0, \\
&(\eta_t^t + \eta_s^t, \xi)_{\mu,2} = (u_t(t), \xi)_{\mu,2},
\end{aligned}$$

for all $\omega \in V_1, \xi \in \mathcal{M}_2$ and a.e. $t \in [0,T]$.

The energy corresponding to the system (1.11)–(1.14) is defined as

$$\begin{aligned}
E(t) = {}& \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{2}\|\eta^t\|_{\mu,2}^2 \\
&+ \int_\Omega F(\nabla u)\mathrm{d}x + \int_\Omega (G(u) - hu)\mathrm{d}x.
\end{aligned} \tag{3.13}$$

Applying Faedo-Galerkin method and combining the arguments of Jorge Silva [15] with those of Jorge Silva and Ma [10], we can obtain the following result.

Theorem 3.1 *Assume that assumptions* A.1–A.3 *hold and consider* $h \in V_0$*. Then we have*
(i) *If initial data* $(u_0, u_1, \eta_0) \in \mathcal{H}$*, then problem* (1.11)–(1.14) *has a weak solution*

$$(u, u_t, \eta^t) \in C([0,T], \mathcal{H}), \quad \forall\, T > 0,$$

satisfying

$$u \in L^\infty(0,T;V_2), \quad u_t \in L^\infty(0,T;V_1),$$
$$(\mathrm{I}-\Delta)u_{tt} \in L^\infty_{loc}(\mathbb{R}^+, V_2'), \quad \eta^t \in L^\infty(0,T;\mathcal{M}_2).$$

(ii) *If initial data* $(u_0, u_1, \eta_0) \in \mathcal{H}_1$, *then problem* (1.11)–(1.14) *has a stronger solution satisfying*

$$u \in L^\infty(0,T;V_3), \quad u_t \in L^\infty(0,T;V_2),$$
$$(\mathrm{I}-\Delta)u_{tt} \in L^\infty_{loc}(\mathbb{R}^+, V_1'), \quad \eta^t \in L^\infty(0,T;\mathcal{M}_3).$$

(iii) *Let* $z_1(t) = (u, u_t, \eta^t), z_2(t) = (v, v_t, \xi^t)$ *be weak solutions of problem* (1.11)–(1.14) *corresponding to initial data* $z_1(0) = (u_0, u_1, \eta_0), z_2(0) = (v_0, v_1, \xi_0)$. *Then one has*

$$\|z_1(t) - z_2(t)\|^2_{\mathcal{H}} \le C_T \|z_1(0) - z_2(0)\|^2_{\mathcal{H}}, \quad t \ge 0,$$

for some constant $C_T = C(\|z_1(0)\|_{\mathcal{H}}, \|z_2(0)\|_{\mathcal{H}}, T) > 0$. *In particular, problem* (1.11)–(1.14) *has a unique weak solution.*

Remark 3.3 The well-posedness of problem (1.11)–(1.14) given by Theorem 3.1 implies that the one-parameter family of operators $S(t) : \mathcal{H} \to \mathcal{H}$ defined by

$$S(t)(u_0, u_1, \eta_0) = (u(t), u_t(t), \eta^t(t)), \quad t \ge 0, \tag{3.14}$$

where $(u(t), u_t(t), \eta^t(t))$ is the unique weak solution of the system (1.11)–(1.14), satisfies the semigroup properties

$$S(0) = \mathrm{I} \quad \text{and} \quad S(t+s) = S(t) \circ S(s), \quad t, s \ge 0,$$

and defines a nonlinear C_0-semigroup. Then problem (1.11)–(1.14) can be viewed as a nonlinear infinite dynamical system $(\mathcal{H}, S(t))$.

Our main result in this present paper is the following.

Theorem 3.2 *Assume that assumptions* A.1–A.3 *hold and consider* $h \in V_0$. *Then we have*

(i) *The dynamical system* $(\mathcal{H}, S(t))$ *corresponding to the system* (1.11)–(1.14) *has a compact global attractor* $\mathcal{A} \subset \mathcal{H}$.

(ii) *If in* (3.4) *and* (3.7) *we assume subcritical conditions*

$$1 \le p_j < \frac{N+2}{N-2} \quad \text{if} \quad N \ge 3 \quad \text{and} \quad 0 < q < \frac{4}{N-4} \quad \text{if} \quad N \ge 5, \tag{3.15}$$

then the corresponding global attractor $\mathcal{A}$ *has finite fractal dimension.*

4 Proof of the Main Result

In this section we will apply the abstract results presented in Section 2 to prove Theorem 3.2. The proof is divided three steps. The first step is to show that the dynamical system $(\mathcal{H}, S(t))$ is dissipative. The second step is to verify the asymptotic smoothness. Then the existence of a compact global attractor is guaranteed by Theorem 2.1. The final step is to prove the quasi-stability property which implies that the fractal dimension of the attractor is finite, as stated in Theorem 2.3.

4.1 Existence of an absorbing set

Lemma 4.1 (Absorbing Set) *Under assumptions of Theorem* 3.2*, the semigroup* $S(t)$ *defined by* (3.14) *has a bounded absorbing set* $\mathcal{B} \in \mathcal{H}$*.*

Proof Multiplying (1.11) by u_t and (1.12) by η^t and integrating over Ω, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\|\nabla u_t\|_2^2 - (\eta^t, \eta_s^t)_{\mu,2}. \tag{4.1}$$

From (3.9), we see that

$$\begin{aligned}(\eta^t, \eta_s^t)_{\mu,2} &= \frac{1}{2}\int_0^\infty \mu(s)\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\eta^t(s)\|_2^2\mathrm{d}s \\ &= \Big[\frac{1}{2}\mu(s)\|\Delta\eta^t(s)\|_2^2\Big]_0^\infty - \frac{1}{2}\int_0^\infty \mu'(s)\|\Delta\eta^t(s)\|_2^2\mathrm{d}s.\end{aligned}$$

Using (3.11) we arrive at

$$\mu(s) \le \mu(0)\mathrm{e}^{-\delta s}, \quad \forall s \in \mathbb{R}^+.$$

And this implies

$$\lim_{s\to\infty} \mu(s) = 0.$$

According to (1.10), namely, the definition of $\eta^t(x, s)$, one can easily see that

$$\eta^t(x, 0) = u(x, t) - u(x, t - 0) = 0,$$

which implies that

$$\|\Delta\eta^t(0)\|_2^2 = 0.$$

Thus

$$(\eta^t, \eta_s^t)_{\mu,2} = -\frac{1}{2}\int_0^\infty \mu'(s)\|\Delta\eta^t(s)\|_2^2\mathrm{d}s,$$

which together with (3.11) gives

$$(\eta^t, \eta_s^t)_{\mu,2} \ge \frac{\delta}{2}\|\eta^t\|_{\mu,2}^2. \tag{4.2}$$

This proves that (4.1) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le -\|\nabla u_t\|_2^2 - \frac{\delta}{2}\|\eta^t\|_{\mu,2}^2. \tag{4.3}$$

Next let us consider the perturbed energy

$$E_\varepsilon(t) = E(t) + \varepsilon\Psi(t), \quad \varepsilon > 0,$$

with

$$\Psi(t) = \int_\Omega u_t(t)u(t)\mathrm{d}x - \int_\Omega \triangle u_t(t)u(t)\mathrm{d}x. \tag{4.4}$$

Let us show that there exists a constant $C_1 > 0$ such that

$$|E_\varepsilon(t) - E(t)| \leq \varepsilon C_1(E(t) + \|h\|_2^2 + |\Omega|), \quad \forall t \geq 0, \quad \forall \varepsilon > 0. \tag{4.5}$$

Indeed, from (3.8), (3.1) and Young inequality, we get

$$\int_\Omega (G(u) - hu)\mathrm{d}x \geq -\frac{1}{4}\|\Delta u\|_2^2 - \sigma_1|\Omega| - \frac{1}{\lambda_1}\|h\|_2^2. \tag{4.6}$$

Combining (3.5) with (3.1), we can see that

$$\int_\Omega F(\nabla u)\mathrm{d}x \geq -\beta_0|\Omega| - \frac{\beta}{2}\|\nabla u\|_2^2 \geq -\beta_0|\Omega| - \frac{\beta}{2\lambda_2}\|\Delta u\|_2^2. \tag{4.7}$$

Then using (3.13)–(4.7) we obtain

$$\begin{aligned} E(t) + \frac{1}{\lambda_1}\|h\|_2^2 + (\beta_0 + \sigma_1)|\Omega| &\geq \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\eta^t\|_2^2 + \Big(\frac{1}{4} - \frac{\beta}{2\lambda_2}\Big)\|\Delta u\|_2^2 \\ &\geq \frac{1}{2}C_2\|(u(t), u_t(t), \eta^t)\|_{\mathcal{H}}^2, \end{aligned} \tag{4.8}$$

where $C_2 = \min\{1, \frac{1}{2} - \frac{\beta}{\lambda_1}\}$.

Using Young inequality, (3.1) and (4.8), we have

$$\begin{aligned} |\Psi(t)| &\leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 \\ &\leq \frac{1}{2}\Big(\frac{1}{\lambda_0} + 1\Big)\|\nabla u_t\|_2^2 + \frac{1}{2}\Big(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\Big)\|\Delta u\|_2^2 \\ &\leq \frac{C_3C_4}{C_2}(E(t) + \|h\|_2^2 + |\Omega|), \end{aligned} \tag{4.9}$$

where

$$C_3 = \max\Big\{\lambda_0 + 1, \frac{1}{\lambda_1} + \frac{1}{\lambda_2}\Big\}, \quad C_4 = \max\Big\{1, \frac{1}{\lambda_1}, \beta_0 + \sigma_1\Big\}.$$

Then taking $C_1 = \frac{C_3C_4}{C_2}$ the inequality (4.5) follows.

Next let us prove that there exist constants $C_5, C_6 > 0$ such that

$$\Psi'(t) \leq -E(t) + C_5\|\nabla u_t(t)\|_2^2 + C_6\|\eta^t\|_{\mu,2}^2. \tag{4.10}$$

From definition of $\Psi(t)$, we see that

$$\Psi'(t) = \int_\Omega (u_{tt} - \Delta u_{tt})u\mathrm{d}x + \|u_t\|_2^2 + \|\nabla u_t\|_2^2. \tag{4.11}$$

Using (1.11) we obtain

$$\begin{aligned} \int_\Omega (u_{tt} - \Delta u_{tt})u\mathrm{d}x = -\|\Delta u\|_2^2 &- \int_\Omega f(\nabla u)\cdot\nabla u\mathrm{d}x - \int_0^\infty \mu(s)(\Delta\eta^t(s), \Delta u(t))\mathrm{d}s \\ &- \int_\Omega \nabla u_t\cdot\nabla u\mathrm{d}x - \int_\Omega (g(u)u - hu)\mathrm{d}x. \end{aligned} \tag{4.12}$$

Combining (4.11)–(4.12) with (3.13), we get

$$\Psi'(t) = -E(t) + \frac{3}{2}\|u_t\|_2^2 + \frac{3}{2}\|\nabla u_t\|_2^2 - \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{2}\|\eta^t\|_{\mu,2}^2$$

$$+ \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4, \tag{4.13}$$

where

$$\begin{aligned}
\mathrm{I}_1 &= \int_\Omega [G(u) - g(u)u]\mathrm{d}x,\\
\mathrm{I}_2 &= \int_\Omega [F(\nabla u) - f(\nabla u)\cdot\nabla u]\mathrm{d}x,\\
\mathrm{I}_3 &= -\int_0^\infty \mu(s)(\Delta\eta^t(s), \Delta u(t))\mathrm{d}s,\\
\mathrm{I}_4 &= -\int_\Omega \nabla u_t\cdot\nabla u\mathrm{d}x.
\end{aligned}$$

Now let us estimate $\mathrm{I}_i (i = 1, 2, 3, 4)$. From (3.8), we have $\mathrm{I}_1 \le 0$ promptly.

Combining (3.5) with (3.1), we see that

$$\mathrm{I}_2 \le \int_\Omega \frac{\beta}{2}|\nabla u|^2\mathrm{d}x \le \frac{\beta}{2\lambda_2}\|\Delta u\|_2^2. \tag{4.14}$$

By Young inequality, given $\nu > 0$, we get

$$\begin{aligned}
|\mathrm{I}_3| &\le \int_0^\infty \mu(s)\Big(\frac{1}{4\nu}\|\Delta\eta^t\|_2^2 + \nu\|\Delta u\|_2^2\Big)\mathrm{d}s\\
&= \nu\Big(\int_0^\infty \mu(s)\mathrm{d}s\Big)\|\Delta u\|_2^2 + \frac{1}{4\nu}\int_0^\infty \mu(s)\|\Delta\eta^t\|_2^2\mathrm{d}s\\
&= \nu\mu_0\|\Delta u\|_2^2 + \frac{1}{4\nu}\|\eta^t\|_{\mu,2}^2.
\end{aligned} \tag{4.15}$$

Using Young inequality and (3.1), we obtain

$$|\mathrm{I}_4| \le \nu\|\nabla u\|_2^2 + \frac{1}{4\nu}\|\nabla u_t\|_2^2 \le \frac{\nu}{\lambda_2}\|\Delta u\|_2^2 + \frac{1}{4\nu}\|\nabla u_t\|_2^2. \tag{4.16}$$

Then from (4.13)–(4.16) we obtain

$$\begin{aligned}
\Psi'(t) \le{}& -E(t) + \Big(\frac{3}{2\lambda_0} + \frac{3}{2} + \frac{1}{4\nu}\Big)\|\nabla u_t\|_2^2 + \Big(\frac{1}{2} + \frac{1}{4\nu}\Big)\|\eta^t\|_{\mu,2}^2\\
&- \Big[\frac{1}{2}\Big(1 - \frac{\beta}{\lambda_2}\Big) - \nu\Big(\frac{1}{\lambda_2} + \mu_0\Big)\Big]\|\Delta u\|_2^2.
\end{aligned} \tag{4.17}$$

Choose ν small enough such that

$$\frac{1}{2}\Big(1 - \frac{\beta}{\lambda_2}\Big) - \nu\Big(\frac{1}{\lambda_2} + \mu_0\Big) > 0.$$

Therefore, we get (4.10) with $C_5 = \frac{3}{2\lambda_0} + \frac{3}{2} + \frac{1}{4\nu}$ and $C_6 = \frac{1}{2} + \frac{1}{4\nu}$.

Let us choose $\varepsilon_1 = \min\{\frac{1}{C_5}, \frac{\delta}{2C_6}\}$, which is positive since we have assumed $\delta > 0$. Then combining (4.3) with (4.17), we infer that

$$\begin{aligned}
E_\varepsilon'(t) &= E'(t) + \varepsilon\Psi'(t)\\
&\le -\varepsilon E(t) - (1 - \varepsilon C_5)\|\nabla u_t\|_2^2 - \Big(\frac{\delta}{2} - \varepsilon C_6\Big)\|\eta^t\|_{\mu,2}^2\\
&\le -\varepsilon E(t), \quad \varepsilon \in (0, \varepsilon_1].
\end{aligned} \tag{4.18}$$

Let us take $\varepsilon_2 = \min\{\frac{1}{2C_1}, \varepsilon_1\}$. Then for all $\varepsilon \leq \varepsilon_2$, it follows from (4.5) that

$$\frac{1}{2}E(t) - \frac{1}{2}(\|h\|_2^2 + |\Omega|) \leq E_\varepsilon(t) \leq \frac{3}{2}E(t) + \frac{1}{2}(\|h\|_2^2 + |\Omega|). \tag{4.19}$$

Using (4.19) we see that

$$E'_\varepsilon(t) \leq -\frac{2\varepsilon}{3}E_\varepsilon(t) + \frac{\varepsilon}{3}(\|h\|_2^2 + |\Omega|),$$

which implies that

$$\begin{aligned} E_\varepsilon(t) &\leq E_\varepsilon(0)\mathrm{e}^{-\frac{2\varepsilon}{3}t} + \frac{1}{2}(1 - \mathrm{e}^{-\frac{2\varepsilon}{3}t})(\|h\|_2^2 + |\Omega|) \\ &= \Big[E_\varepsilon(0) - \frac{1}{2}(\|h\|_2^2 + |\Omega|)\Big]\mathrm{e}^{-\frac{2\varepsilon}{3}t} + \frac{1}{2}(\|h\|_2^2 + |\Omega|). \end{aligned}$$

Using again (4.19) we obtain

$$E(t) \leq 3E(0)\mathrm{e}^{-\frac{2\varepsilon}{3}t} + 2(\|h\|_2^2 + |\Omega|).$$

Therefore from (4.8) we conclude that

$$\|(u(t), u_t(t), \eta^t)\|_{\mathcal{H}}^2 \leq CE(0)\mathrm{e}^{-\frac{2\varepsilon}{3}t} + C(\|h\|_2^2 + |\Omega|), \tag{4.20}$$

where $C = \frac{2}{C_2}\max\{3, (2 + C_4)\}$.

Hence, taking the closed ball $\mathcal{B} = \overline{B}_{\mathcal{H}}(0, R)$ with $R = \sqrt{2C(\|h\|_2^2 + |\Omega|)}$ we infer from (4.20) that $\mathcal{B}$ is a bounded absorbing set for $S(t)$. The proof is complete.

As a straight consequence of Lemma 4.1, we have that the solutions of problem (1.11)–(1.14) are globally bounded provided initial data lying in bounded sets $B \subset \mathcal{H}$. Namely, let (u, u_t, η^t) be a solution of (1.11)–(1.14) with initial data (u_0, u_1, η_0) in a bounded set B. Then one has

$$\|(u(t), u_t(t), \eta^t)\|_{\mathcal{H}} \leq C_B, \quad \forall t \geq 0, \tag{4.21}$$

where C_B is a constant depending on B. Lemma 4.1 also ensures the existence of bounded positively invariant sets.

4.2 Stability inequality

Lemma 4.2 (see [15]) *Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be a C^1-vector field given by $f = (f_1, \cdots, f_N)$ and satisfy* (3.3). *Then, there exists a constant $K = K(k_j, p_j, N) > 0, j = 1, \cdots, N$, such that*

$$|f(x) - f(y)| \leq K\sum_{j=1}^{N}(1 + |x|^{\frac{p_j - 1}{2}} + |y|^{\frac{p_j-1}{2}})|x - y|, \quad \forall x, y \in \mathbb{R}^N. \tag{4.22}$$

Lemma 4.3 *Under the hypotheses of Theorem* 3.2*, given a bounded set $B \subset \mathcal{B}$, let $z_1 = (u, u_t, \eta^t)$ and $z_2 = (v, v_t, \xi)$ be two weak solutions of problem* (1.11)–(1.14) *such that $z_0^1 = (u_0, u_1, \eta_0)$ and $z_0^2 = (v_0, v_1, \xi_0)$ are in B. Then*

$$\begin{aligned} &\|z_1(t) - z_2(t)\|_{\mathcal{H}}^2 \\ &\leq \kappa\mathrm{e}^{-\gamma t}\|z_0^1 - z_0^2\|_{\mathcal{H}}^2 \\ &\quad + K_B\int_0^t \mathrm{e}^{-\gamma(t-s)}\Big(\sum_{j=1}^{N}\|\nabla(u(s) - v(s))\|_{p_j+1}^2 + \|u(s) - v(s)\|_{2(q+1)}^2\Big)\mathrm{d}s, \quad t \geq 0, \end{aligned} \tag{4.23}$$

where $\kappa, \gamma > 0$ and $K_B > 0$ are constants.

Proof Let us fix a bounded set $B \subset \mathcal{H}$. Put $w = u - v$ and $\zeta = \eta^t - \xi^t$. Then (w, ζ^t) satisfies

$$\begin{aligned} & w_{tt} - \Delta w_{tt} + \Delta^2 w - \operatorname{div}(f(\nabla u) - f(\nabla v)) + \int_0^\infty \mu(s)\Delta^2\zeta^t(s)\mathrm{d}s \\ & - \Delta w_t + g(u) - g(v) = 0, \qquad (4.24) \\ & \zeta^t = -\zeta^t_s + w_t, \qquad (4.25) \end{aligned}$$

with initial condition

$$w(0) = u_0 - v_0, \quad w_t(0) = u_1 - v_1, \quad \zeta^0 = \eta_0 - \xi_0.$$

Now we consider the functional

$$H(t) = \|\Delta w(t)\|_2^2 + \|\nabla w_t(t)\|_2^2 + \|w_t(t)\|_2^2 + \|\zeta^t\|^2_{\mu,2} \qquad (4.26)$$

and its perturbation

$$H_\varepsilon(t) = H(t) + \varepsilon\Phi(t),$$

where

$$\Phi(t) = \int_\Omega w_t(t)w(t)\mathrm{d}x - \int_\Omega \triangle w_t(t)w(t)\mathrm{d}x. \qquad (4.27)$$

Owing to (3.1), we get

$$\|z_1(t) - z_2(t)\|^2_{\mathcal{H}} \le H(t) \le \Big(1 + \frac{1}{\lambda_0}\Big)\|z_1(t) - z_2(t)\|^2_{\mathcal{H}}. \qquad (4.28)$$

Multiplying (4.24) by w_t in V_0 and (4.25) by ζ^t in $\mathcal{M}_2$, and integrating over Ω, we deduce that

$$\begin{aligned} & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}H(t) + \|\nabla w_t(t)\|_2^2 \\ & = (\operatorname{div}(f(\nabla u) - f(\nabla v)), w_t(t)) - (g(u) - g(v), w_t(t)) - (\zeta^t, \zeta^t_s)_{\mu,2}. \qquad (4.29) \end{aligned}$$

Let us estimate the right side of the above identity. Hereafter, C_B will denote several positive constants.

Using generalized Höld inequality with $\frac{p_j-1}{2(p_j+1)} + \frac{1}{p_j+1} + \frac{1}{2} = 1$ and (4.22), we have

$$\begin{aligned} & |(\operatorname{div}(f(\nabla u) - f(\nabla v)), w_t)| \\ & = |(f(\nabla u) - f(\nabla v), \nabla w_t)| \\ & \le \int_\Omega |f(\nabla u) - f(\nabla v)||\nabla w_t(t)|\mathrm{d}x \\ & \le K\sum_{j=1}^N \int_\Omega (1 + |\nabla u|^{\frac{p_j-1}{2}} + |\nabla v|^{\frac{p_j-1}{2}})|\nabla w||\nabla w_t|\mathrm{d}x \\ & \le K\sum_{j=1}^N (|\Omega|^{\frac{p_j-1}{2(p_j+1)}} + \|\nabla u\|^{\frac{p_j-1}{2}}_{p_j+1} + \|\nabla v\|^{\frac{p_j-1}{2}}_{p_j+1})\|\nabla w\|_{p_j+1}\|\nabla w_t\|_2. \end{aligned}$$

From (3.12) and (4.21) we obtain

$$K(|\Omega|^{\frac{p_j-1}{2(p_j+1)}} + \|\nabla u\|_{p_j+1}^{\frac{p_j-1}{2}} + \|\nabla v\|_{p_j+1}^{\frac{p_j-1}{2}}) \le C_B < \infty, \quad j=1,\cdots,N.$$

Making use of Young inequality, there exists a constant $C_B > 0$ such that

$$\begin{aligned}|(\mathrm{div}(f(\nabla u) - f(\nabla v)), w_t)| &\le C_B \sum_{j=1}^{N} \|\nabla w\|_{p_j+1} \|\nabla w_t\|_2 \\ &\le \frac{C_B}{2} \sum_{j=1}^{N} \|\nabla w\|_{p_j+1}^2 + \frac{1}{4}\|\nabla w_t\|_2^2. \end{aligned} \tag{4.30}$$

Further, since $\frac{q}{2(q+1)} + \frac{1}{2(q+1)} + \frac{1}{2} = 1$, again by generalized Höld inequality, (3.6)–(3.7), (4.21), and (3.1), it follows that

$$\begin{aligned}&|(g(u) - g(v), w_t)| \\ &\le \sigma_0 \int_\Omega (1 + |u|^q + |v|^q)|w||w_t| \mathrm{d}x \\ &\le \sigma_0(|\Omega|^{\frac{q}{2(q+1)}} + \|u\|_{2(q+1)}^q + \|v\|_{2(q+1)}^q)\|w\|_{2(q+1)}\|w_t\|_2 \\ &\le C_B\|w\|_{2(q+1)}\|\nabla w_t\|_2.\end{aligned}$$

Using again Young inequality, there exists a constant $C_B > 0$ such that

$$|(g(u) - g(v), w_t)| \le \frac{C_B}{2}\|w\|_{2(q+1)}^2 + \frac{1}{4}\|\nabla w_t\|_2^2. \tag{4.31}$$

As in (4.2), we conclude that

$$-(\zeta^t, \zeta_s^t)_{\mu,2} \le -\frac{\delta}{2}\|\zeta^t\|_{\mu,2}^2. \tag{4.32}$$

Thus combining (4.29) with (4.30)–(4.32) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t) \le -\|\nabla w_t(t)\|_2^2 - \delta\|\zeta^t\|_{\mu,2}^2 + C_B\Big(\sum_{j=1}^{N}\|\nabla w\|_{p_j+1}^2 + \|w\|_{2(q+1)}^2\Big). \tag{4.33}$$

It follows promptly from the definition of $H(t)$ and $\Phi(t)$ that there exists a constant $C_7 > 0$ such that

$$|H_\varepsilon(t) - H(t)| \le \varepsilon C_7 H(t), \quad \forall\, t \ge 0, \quad \forall\, \varepsilon > 0. \tag{4.34}$$

As in the proof of Lemma 4.1, we claim that there exist constants $\varepsilon_3 > 0$, and $C_B > 0$ such that

$$H_\varepsilon'(t) \le -\frac{\varepsilon}{2}H(t) + C_B\Big(\sum_{j=1}^{N}\|\nabla w\|_{p_j+1}^2 + \|w\|_{2(q+1)}^2\Big), \quad \forall\, t \ge 0,\ \forall\, \varepsilon \in (0, \varepsilon_3]. \tag{4.35}$$

To prove this it suffices to prove that there exist constants $C_8, C_9 > 0$ and $C_B > 0$ such that

$$\Phi'(t) \le -\frac{1}{2}H(t) + C_8\|\nabla w_t(t)\|_2^2 + C_9\|\zeta^t\|_{\mu,2}^2$$

$$+ C_B\Big(\sum_{j=1}^{N}\|\nabla w\|_{p_j+1}^2 + \|w\|_{2(q+1)}^2\Big). \tag{4.36}$$

In fact, combining (4.33) with (4.36) and choosing $\varepsilon_3 = \min\big\{\frac{1}{C_8}, \frac{\delta}{C_9}, 1\big\}$, the inequality (4.35) holds for every $\varepsilon \in (0, \varepsilon_1]$.

In what follows, we prove that (4.36) holds. By differentiating (4.27), using (4.24) and (4.26), we get

$$\Phi'(t) = -\frac{1}{2}H(t) - \frac{1}{2}\|\Delta w(t)\|_2^2 + \frac{3}{2}\|w_t(t)\|_2^2 + \frac{3}{2}\|\nabla w_t(t)\|_2^2 + \frac{1}{2}\|\zeta^t\|_{\mu,2}^2 + \sum_{i=1}^{4} L_i, \tag{4.37}$$

where

$$\begin{aligned} L_1 &= -\int_0^\infty \mu(s)(\Delta\zeta^t(s), \Delta w(t))\mathrm{d}s,\\ L_2 &= -\int_\Omega \nabla w_t(t)\cdot\nabla w(t)\mathrm{d}x,\\ L_3 &= (\mathrm{div}(f(\nabla u(t)) - f(\nabla v(t))), w(t)),\\ L_4 &= -(g(u(t)) - g(v(t)), w(t)). \end{aligned}$$

Now we estimate the terms L_1, L_2, L_3, and L_4.

$$|L_1| \le \nu\mu_0\|\Delta w(t)\|_2^2 + \frac{1}{4\nu}\|\zeta^t\|_{\mu,2}^2$$

and

$$|L_2| \le \frac{\nu}{\lambda_2}\|\Delta w(t)\|_2^2 + \frac{1}{4\nu}\|\nabla w_t(t)\|_2^2,$$

where $\nu > 0$ is a small constant which will be chosen later.

$$|L_3| \le C_B\sum_{j=1}^{N}\|\nabla w(t)\|_{p_j+1}^2$$

and

$$|L_4| \le C_B\|w(t)\|_{2(q+1)}^2.$$

Going back to (4.37) and inserting these four last estimates we arrive at

$$\begin{aligned} \Phi'(t) \le& -\frac{1}{2}H(t) + \Big(\frac{3}{2\lambda_0} + \frac{3}{2} + \frac{1}{4\nu}\Big)\|\nabla w_t(t)\|_2^2 + \Big(\frac{1}{2} + \frac{1}{4\nu}\Big)\|\zeta^t\|_{\mu,2}^2\\ &+ C_B\Big(\sum_{j=1}^{N}\|\nabla w(t)\|_{p_j+1}^2 + \|w(t)\|_{2(q+1)}^2\Big) - \Big(\frac{1}{2} - \nu\Big(\mu_0 + \frac{1}{\lambda_2}\Big)\Big)\|\triangle w(t)\|_2^2. \end{aligned}$$

Therefore, taking $\nu > 0$ small enough the inequality (4.36) follows and consequently (4.35) holds.

Now we take $\varepsilon_4 = \min\big\{\frac{1}{2C_7}, \varepsilon_3\big\}$ and choose $\varepsilon \le \varepsilon_4$. Then (4.34) implies that

$$\frac{1}{2}H(t) \le H_\varepsilon(t) \le \frac{3}{2}H(t), \quad t \ge 0. \tag{4.38}$$

It follows from (4.35) and (4.38) that

$$H(t) \le 3\mathrm{e}^{-\gamma t} H(0) + K_B \int_0^t \mathrm{e}^{-\gamma(t-s)} \Big(\sum_{j=1}^{N} \|\nabla w(s)\|_{p_j+1}^2 + \|w(s)\|_{2(q+1)}^2 \Big) \mathrm{d}s, \quad t \ge 0,$$

where $\gamma = \frac{\varepsilon}{3} > 0$ is a small positive constant and K_B is a constant depending on bounded set B. From the above inequality and (4.28), we conclude that (4.23) holds. The proof is complete.

Lemma 4.4 (Asymptotic Smoothness) *Under the hypotheses of Theorem* 3.2*, the dynamical system* $(\mathcal{H}, S(t))$ *is asymptotic smooth.*

Proof We apply Lemma 4.3. Let B be a bounded subset of $\mathcal{H}$ positively invariant with respect to $S(t)$. Let $S(t)z_0^1 = (u, u_t, \eta^t)$ and $S(t)z_0^2 = (v, v_t, \xi^t)$ be two solutions for problem (1.11)–(1.14) corresponding to initial data $z_0^1, z_0^2 \in B$. Given $\varepsilon > 0$, from inequality (4.23), we can choose $T > 0$ such that

$$\begin{aligned} &\|S(T)z_0^1 - S(T)z_0^2\|_{\mathcal{H}} \\ &\le \varepsilon + C_B \Big\{ \int_0^T \Big(\sum_{j=1}^{N} \|\nabla(u(s) - v(s))\|_{p_j+1}^2 + \|u(s) - v(s)\|_{2(q+1)}^2 \Big) \mathrm{d}s \Big\}^{\frac{1}{2}}, \end{aligned} \tag{4.39}$$

where $C_B > 0$ is a constant which depends only on the size of B.

Let us estimate the right side of (4.39). Taking $\theta_j = \frac{1}{2} + \frac{N}{4}(1 - \frac{2}{p_j+1}), j = 1, \cdots, N$, for $N \ge 1$, then (3.4) implies that $\frac{1}{2} \le \theta_j \le 1$ and $\frac{N}{p_j+1} - 1 = \theta_j(\frac{N}{2} - 2) + \frac{N}{2}(1 - \theta_j), j = 1, \cdots, N$. Using Gagliardo-Nirenberg interpolation theorem we get

$$\begin{aligned} \|\nabla(u(t) - v(t))\|_{p_j+1} &\le C_{\theta_j} \|\Delta(u(t) - v(t))\|_2^{\theta_j} \|u(t) - v(t)\|_2^{1-\theta_j} \\ &\le C_B \|u(t) - v(t)\|_2^{1-\theta_j}. \end{aligned}$$

We observe that (3.7) implies that $2 < 2(q+1) < \infty$ if $1 \le N \le 4$ and $2 < 2(q+1) \le \frac{2N}{N-4}$ if $N \ge 5$. Taking $\lambda = \frac{N}{4}(1 - \frac{1}{q+1})$ it follows from Gagliardo-Nirenberg interpolation theorem that

$$\begin{aligned} \|(u(t) - v(t))\|_{p_j+1} &\le C_\lambda \|\Delta(u(t) - v(t))\|_2^{\lambda} \|u(t) - v(t)\|_2^{1-\lambda} \\ &\le C_B \|u(t) - v(t)\|_2^{1-\lambda}. \end{aligned}$$

Combining these two last estimates with (4.39), we conclude that there exists $C_B > 0$ such that

$$\|S(T)z_0^1 - S(T)z_0^2\|_{\mathcal{H}} \le \varepsilon + \phi_T(z_0^1, z_0^2),$$

where

$$\phi_T(z_0^1, z_0^2) = C_B \Big\{ \int_0^T \Big(\sum_{j=1}^{N} \|u(s) - v(s)\|_2^{2(1-\theta_j)} + \|u(s) - v(s)\|_2^{2(1-\lambda)} \Big) \mathrm{d}s \Big\}^{\frac{1}{2}}.$$

To conclude the proof of asymptotic smoothness, it remains to prove that the functional ϕ_T satisfies (2.1). Indeed, given a sequence of initial data $z_0^n = (u_0^n, u_1^n, \eta_0^n) \in B$, let us write $S(t)z_0^n = (u^n(t), u_t^n(t), \eta^{n,t})$. Since B is positively invariant with respect to $S(t)$, it follows that $(u^n(t), u_t^n(t), \eta^{n,t})$ is uniformly bounded in $\mathcal{H}$. In particular

$$(u^n(t), u_t^n(t)) \text{ is bounded in } C([0, T], V_2 \times V_1), \quad T > 0.$$

Then by compact embedding $V_2 \hookrightarrow V_0$, passing to a subsequence if necessary, we have

$$(u^n) \text{ converges strongly in } C([0,T],V_0).$$

Therefore one obtains

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\Big(\sum_{j=1}^N\|u^n(s)-u^m(s)\|_2^{2(1-\theta_j)}+\|u^n(s)-u^m(s)\|_2^{2(1-\lambda)}\Big)\mathrm{d}s=0,$$

which implies (2.1) holds. Then asymptotic smoothness follows from Theorem 2.2.

Proof of Theorem 3.2 part (i) We first note that Lemmas 4.1 and 4.4 imply that $(\mathcal{H}, S(t))$ is a dissipative dynamical system which is asymptotic smooth. Then the existence of a compact global attractor $\mathcal{A}$ to problem (1.11)–(1.14) in the phase space $\mathcal{H}$ follows from Theorem 2.1.

4.3 Finite-Dimensional attractor

Lemma 4.5 (Quasi-stability) *Suppose the assumptions of Theorem* 3.2 (ii) *hold. Then* $(\mathcal{H}, S(t))$ *is quasi-stable on any bounded positively invariant set* $B \subset \mathcal{H}$.

Proof Since $(\mathcal{H}, S(t))$ is defined as the solution operator of (1.11)–(1.14), it follows from Theorem 3.1 (i) that (2.2) and (2.3) hold with $X = V_2$, $Y = V_1$ and $Z = \mathcal{M}_2$. Also from Theorem 3.1 (iii) we see that condition (2.4) holds true. Then we only need to verify stability inequality (2.5).

Let $B \subset \mathcal{H}$ be a bounded set positively invariant with respect to $S(t)$. For $z_0^1, z_0^2 \in B$ we write $S(t)z_0^i = (u^i(t), u_t^i(t), \eta^{i,t}), i = 1, 2$. Let us define the seminorm

$$n_X(u)=\sum_{j=1}^N\|\nabla u\|_{p_j+1}+\|u\|_{2(q+1)}.$$

From assumption (3.15), we know that embeddings

$$V_2 \hookrightarrow W_0^{1,p_j+1}(\Omega) \quad \text{and} \quad V_2 \hookrightarrow L^{2(q+1)}(\Omega)$$

are compact. Then we conclude that $n_X(\cdot)$ is a compact seminorm on $X = V_2$. Hence from (4.23) we can see that

$$\|z_1(t)-z_2(t)\|_{\mathcal{H}}^2\le b(t)\|z_0^1-z_0^2\|_{\mathcal{H}}^2+c(t)\sup_{0<s<t}[n_X(u^1(s)-u^2(s))]^2,$$

where

$$b(t)=\kappa \mathrm{e}^{-\gamma t}\quad \text{and}\quad c(t)=K_B\int_0^t \mathrm{e}^{-\gamma(t-s)}\mathrm{d}s,\quad t\ge 0.$$

Now we note that $b \in L^1(\mathbb{R}^+)$ and $\lim_{t\to\infty} b(t) = 0$. Also, since B is bounded it follows that $c(t)$ is locally bounded on $[0, \infty)$. Hence, the assumptions of quasi-stability on bounded positively invariant sets are fulfilled.

Proof of Theorem 3.2 part (ii) From the proof of Theorem 3.2 part (i) we know that $(\mathcal{H}, S(t))$ has a compact global attractor $\mathcal{A}$, which is a bounded positively invariant set of $\mathcal{H}$. Then it follows from Lemma 4.4 that $(\mathcal{H}, S(t))$ is quasi-stable on $\mathcal{A}$. Based on Theorem 2.3, we conclude that the attractor $\mathcal{A}$ has finite fractal dimension.

Remark 4.1 In particular, the Hausdorff dimension of $\mathcal{A}$ is also finite since it is bounded by the fractal dimension.

Acknowledgement The authors are grateful to the referees for useful remarks and suggestions that have greatly improved the presentation.

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