Chin. Ann. Math. Ser. B
41(4), 2020, 645–656
DOI: 10.1007/s11401-020-0223-8

Chinese Annals of
Mathematics, Series B


# The Dimension Paradox in Parameter Space of Cosine Family*

Xiaojie HUANG[1] Weiyuan QIU[2]

**Abstract** It is proved in this paper that the union of escaping parameter rays without endpoints for the cosine family $S_\kappa(z) = \mathrm{e}^\kappa(\mathrm{e}^z + \mathrm{e}^{-z})$, where $\kappa \in \mathbb{C}$ is a parameter, has Hausdorff dimension 1, which implies that the ray endpoints alone have Hausdorff dimension 2. This shows that Karpińska's dimension paradox occurs also in the parameter plane of the cosine family.

**Keywords** Dimension paradox, Hausdorff dimension, Parameter rays, Dynamic rays
**2000 MR Subject Classification** 37F35, 37F10, 37F45

## 1 Introduction

The Julia set $J(f)$ of an entire function $f$ is the set of points at which the family of iterates of the entire function $f$ fails to be a normal family. Equivalently, the Julia set $J(f)$ is the closure of set of expanding periodic orbits of $f$. Its complement $F(f)$ is called the Fatou set of $f$. These sets are the main objects studied in complex dynamics of entire functions. The Julia set is often a rather complicated and interesting set (see [1]).

Such as Devaney and Krych [2] showed that for $0 < \lambda < \frac{1}{\mathrm{e}}$, the Julia set of $f_\lambda(z) = \lambda \mathrm{e}^z$ is an uncountable union of pairwise disjoint curves, which connect finite points, called the endpoints, with $\infty$. McMullen [3] showed that the Hausdorff dimension of this set is 2, and Karpińska [4] proved that the set of curves without endpoints has Hausdorff dimension 1. The results reveal a "dimension paradox": From the point of view in topology, one might think that "most" points in the Julia set are in the union of curves, but nonetheless the entire Hausdorff dimension sits in the set of endpoints. Such a phenomenon was first observed in exponential dynamic plane by Karpińska [4], so it is also called the "Karpińska paradox" (see [5]).

The escaping set

$$I(f) := \{z : \lim_{n\to\infty} f^{\circ n}(z) = \infty\}$$

for an entire function $f$ has a close relationship with the Julia set, where $f^{\circ n}$ denotes the $n$-th iterate of $f$. Eremenko [6] proved that $J(f) = \partial I(f)$, while Eremenko and Lyubich [7]

Manuscript received January 21, 2017. Revised November 30, 2017.
[1]School of Mathematical Science, Fudan University, Shanghai 200433, China; School of Science, Nanchang Institute of Technology, Nanchang 330099, Jiangxi, China. E-mail: xjhuang14@fudan.edu.cn
[2]School of Mathematical Science, Fudan University, Shanghai 200433, China.
E-mail: wyqiu@fudan.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 11671091, 11731003, 11771090), the Natural Science Foundation of Shanghai (No. 17ZR1402900) and the Science and Technology Foundation of Jiangxi Education Department (No. GJJ180944).

proved that $I(f) \subseteq J(f) = \overline{I(f)}$ for a large class of functions $f$ including the exponential family $f_\lambda(z) = \lambda e^z$ ($\lambda \in \mathbb{C} \setminus \{0\}$) and the cosine family $g_{a,b}(z) = ae^z + be^{-z}$ ($a, b \in \mathbb{C} \setminus \{0\}$). McMullen in fact proved that the escaping set of $f_\lambda$ has Hausdorff dimension 2 while the escaping set of $g_{a,b}$ has infinite planer Lebesgue measure (see [3]). In both cases, it was proved that the escaping set consists of uncountably many pairwise disjoint curves started at $\infty$, called dynamic rays, together with endpoints of certain (but not all) of these rays (see [8–10]). Furthermore, the dimension paradox of Karpińska for $f_\lambda$ with specific choices of $\lambda$ was generalized to the following theorem.

**Theorem A** (see [8]) (Dimension Paradox in Dynamic Plane for Exponential Functions) *For $f_\lambda(z) = \lambda e^z$, where $\lambda \in \mathbb{C} \setminus \{0\}$, the union of all dynamic rays has Hausdorff dimension* 1*, while the set of endpoints has Hausdorff dimension* 2*.*

**Theorem B** (see [9–10]) (Dimension Paradox in Dynamic Plane for Cosine Functions) *For $g_{a,b}(z) = ae^z + be^{-z}$, where $a, b \in \mathbb{C} \setminus \{0\}$, the union of all dynamic rays has Hausdorff dimension* 1*, while the set of the endpoints has Hausdorff dimension* 2 *and even infinite planar Lebesgue measure.*

The orbit behaviour of singularities of an entire function plays a crucial rule in the complex dynamics. The exponential function $\lambda e^z$ for every $\lambda \in \mathbb{C} \setminus \{0\}$ has unique singularity 0, which is an asymptotic value of $\lambda e^z$. Write $\lambda = e^\kappa$ and denote $E_\kappa(z) = e^{z+\kappa}$ with parameter $\kappa \in \mathbb{C}$. We have an escaping set in the parameter $\kappa$-plane

$$\mathcal{I}(E_\kappa) = \{\kappa \in \mathbb{C} : \lim_{n\to\infty} E_\kappa^{\circ n}(0) = \infty\}.$$

Forster, Rempe and Schleicher studied the set $\mathcal{I}(E_\kappa)$ in detail in [11–12]. They proved that, similar to the escaping set in the dynamical plane, $\mathcal{I}(E_\kappa)$ consists of uncountably many pairwise disjoint curves in parameter space, called the parameter rays, together with endpoints of certain parameter rays. Qiu [13] showed that the Hausdorff dimension of $\mathcal{I}(E_\kappa)$ is equal to 2. Bailesteanu, Balan and Schleicher [14] further proved that the Hausdorff dimension of the union of parameter rays is 1, which shows the phenomenon of dimension paradox also occurs in the parameter space of exponential family.

**Theorem C** (see [14]) (Dimension Paradox in Parameter Plane for Exponential Family) *For $E_\kappa(z) = e^{z+\kappa}$, the union of all parameter rays has Hausdorff dimension* 1*, while the set of the endpoints has Hausdorff dimension* 2*.*

Let us consider the cosine family $S_\kappa(z) = e^\kappa(e^z + e^{-z})$ with one parameter $\kappa \in \mathbb{C}$. The singularities of $S_\kappa$ are $k\pi i$, $k = 0, \pm 1, \pm 2, \cdots$. Since they have the same orbit, the orbit of 0 plays the key role in the dynamics of $S_\kappa$. We again have the escaping set in the parameter plane of $S_\kappa$:

$$\mathcal{I}(S_\kappa) = \{\kappa \in \mathbb{C} : \lim_{n\to\infty} S_\kappa^{\circ n}(0) = \infty\}.$$

A natural problem arises: what is the structure of $\mathcal{I}(S_\kappa)$ and whether the dimension paradox occurs in the parameter plane of $S_\kappa$? Tian [15] discussed the structure of the set $\mathcal{I}(S_\kappa)$ and obtained an analogous as the exponential family, that is, $\mathcal{I}(S_\kappa)$ can be divided into the union of all parameter rays and the set of endpoints of certain parameter rays. However, Tian had not showed the dimension paradox in the parameter plane for $S_\kappa$. In this paper, we will prove that the dimension paradox occurs in the parameter plane of the cosine family $S_\kappa$. Let $\mathcal{I}_R \subset \mathcal{I}(S_\kappa)$ be the union of all parameter rays, and $\mathcal{I}_E := \mathcal{I}(S_\kappa) \setminus \mathcal{I}_R$ be the set of endpoints of parameter rays. Our main result is the following theorem.

**Theorem 1.1** (Dimension Paradox in Parameter Plane for Cosine Family) *For $S_\kappa(z) = \mathrm{e}^\kappa(\mathrm{e}^z + \mathrm{e}^{-z})$, where $\kappa \in \mathbb{C}$, the set of the union of all parameter rays $\mathcal{I}_R$ has Hausdorff dimension 1, while the set of endpoints $\mathcal{I}_E$ has Hausdorff dimension 2 and even positive planar Lebesgue-measure.*

As Qiu proved that $\mathcal{I}(S_\kappa)$ has Hausdorff dimension 2 and positive planar Lebesgue measure (see [13]), it is sufficient for us to show that the set $\mathcal{I}_R$ has Hausdorff dimension 1. The method of the proof mainly comes from the works of Karpińska [4] and Bailesteanu, Balan and Schleicher [14] for the exponential family. However, we need to overcome some technical difficulties since the appearance of the term $\mathrm{e}^{-z}$ in $S_\kappa(z) = \mathrm{e}^\kappa(\mathrm{e}^z + \mathrm{e}^{-z})$.

## 2 Lemmas

For $\kappa \in \mathbb{C}$, write $S^n(\kappa) := S_\kappa^{\circ(n+1)}(0) = S_\kappa^{\circ n}(2\mathrm{e}^\kappa) = S_\kappa^{\circ n}(-2\mathrm{e}^\kappa)$ for integers $n \geq 1$. Then every $S^n$ is a transcendental entire function and the parameter escaping set

$$\mathcal{I}(S_\kappa) = \{\kappa \in \mathbb{C} : \lim_{n\to\infty} S^n(\kappa) = \infty\}.$$

For simplicity, we denote $\mathcal{I} = \mathcal{I}(S_\kappa)$.

If $\Lambda \subset \mathbb{C}$ is a domain such that $S^n : \Lambda \to V := S^n(\Lambda)$ is a conformal isomorphism, then for every integer $k \geq 0$, this defines a holomorphic map $S^{n,n+k} = S^{n+k} \circ (S^n)^{-1} : V \to \mathbb{C}$.

In the following, $Q$ always denotes an open square of side length $\frac{\pi}{2}$ with sides parallel to the axes, $\widetilde{Q} \supset Q$ denotes the open square of side length $\pi$ with sides parallel to $Q$ and center coincident with $Q$. We call $Q$ a standard square and $\widetilde{Q}$ a double square with respect to $Q$. We also denote $D_r(z)$ the open disk of radius $r$ around $z \in \mathbb{C}$.

**Lemma 2.1** *Let $M > 1$ and $\Lambda$ be a domain in $D_M(0)$ such that $S^n : \Lambda \to V := S^n(\Lambda)$ is a conformal isomorphism and satisfies*:

(1) $|(S^n)'(\kappa)| > 20$ *for all* $\kappa \in \Lambda$;

(2) *$V$ is convex and $V \subset \widetilde{Q}$ for some double square $\widetilde{Q} \subset \{z \in \mathbb{C} : |\operatorname{Re} z| > \xi\}$, where $\xi$ is real with $\mathrm{e}^\xi > 8\mathrm{e}^M$.*

*Then*

(1′) *$S^{n+1} : \Lambda \to S^{n+1}(\Lambda)$ is a conformal isomorphism with*

$$|(S^{n+1})'(\kappa)| \geq \frac{\mathrm{e}^{-M}\mathrm{e}^\xi}{4}|(S^n)'(\kappa)| > 2|(S^n)'(\kappa)|$$

*for all* $\kappa \in \Lambda$;

(2′) *$S^{n,n+1} : V \to S^{n+1}(\Lambda)$ is a conformal isomorphism with*

$$|(S^{n,n+1})'(z)| \geq \frac{\mathrm{e}^{-M}\mathrm{e}^\xi}{4}$$

*for all* $z \in V$.

**Proof** Note that $S^{n+1}(\kappa) = S_\kappa \circ S_\kappa^{\circ n+1}(0) = S_\kappa(S^n(\kappa))$, we can write $S^{n+1}(\kappa) = S_\kappa(z) = \mathrm{e}^\kappa(\mathrm{e}^z + \mathrm{e}^{-z})$ with $z = S^n(\kappa) \in V$ .

Without lose of generality, we assume $\mathrm{Re}\, z > \xi$ for $z \in V$. The discussion for the case $\mathrm{Re}\, z < -\xi$ for $z \in V$ is completely the same.

In order to show that $S^{n+1}$ restricted to $\Lambda$ is a conformal isomorphism onto its image, we need only to check that $S^{n+1}$ is injective on $\Lambda$. Suppose that there are $\kappa_1, \kappa_2 \in \Lambda$ with $S^{n+1}(\kappa_1) = S^{n+1}(\kappa_2)$. Set $z_j = S^n(\kappa_j) \in V \subset \widetilde{Q}$ for $j = 1, 2$. Then $|z_1 - z_2| < \sqrt{2}\pi$ and $\mathrm{e}^{\kappa_1}(\mathrm{e}^{z_1} + \mathrm{e}^{-z_1}) = \mathrm{e}^{\kappa_2}(\mathrm{e}^{z_2} + \mathrm{e}^{-z_2})$.

$$\begin{aligned}\kappa_1 - \kappa_2 &= z_2 - z_1 + (\ln(1 + \mathrm{e}^{-2z_2}) - \ln(1 + \mathrm{e}^{-2z_1})) + 2m\pi\mathrm{i} \\ &= z_2 - z_1 + \int_{z_1}^{z_2} \frac{-2\mathrm{e}^{-2z}\mathrm{d}z}{1 + 2\mathrm{e}^{-2z}} + 2m\pi\mathrm{i}\end{aligned}$$

for some integer $m$, where the integral path is taken to be the line segment $[z_1, z_2]$ since $V$ is convex.

Since $|z_2 - z_1| < \sqrt{2}\pi$ and $|(S^n)'(\kappa)| > 20$ for all $\kappa \in \Lambda$, we have

$$|\kappa_2 - \kappa_1| \le \frac{1}{20}|z_2 - z_1| < \frac{1}{10}\pi.$$

If $m \neq 0$, then since $\xi > \ln 8$,

$$\begin{aligned}|\kappa_1 - \kappa_2| &\ge 2|m|\pi - \Big\{|z_2 - z_1| + \Big|\int_{z_1}^{z_2} \frac{-2\mathrm{e}^{-2z}\mathrm{d}z}{1 + 2\mathrm{e}^{-2z}}\Big|\Big\} \\ &\ge 2|m|\pi - |z_2 - z_1|\Big(1 + \max_{z\in V} \frac{|-2\mathrm{e}^{-2z}|}{1 - |2\mathrm{e}^{-2z}|}\Big) \\ &\ge 2\pi - \sqrt{2}\pi\Big(1 + \frac{2\mathrm{e}^{-2\xi}}{1 - 2\mathrm{e}^{-2\xi}}\Big) > \frac{1}{10}\pi,\end{aligned}$$

which is a contradiction. So $m = 0$.

If $\kappa_1 \neq \kappa_2$, then $z_1 \neq z_2$ since $S^n$ is a conformal isomorphism.

$$|\kappa_1 - \kappa_2| \ge |z_2 - z_1| - \Big|\int_{z_1}^{z_2} \frac{-2\mathrm{e}^{-2z}\mathrm{d}z}{1 + 2\mathrm{e}^{-2z}}\Big| \ge |z_2 - z_1|\Big(1 - \frac{2\mathrm{e}^{-2\xi}}{1 - 2\mathrm{e}^{-2\xi}}\Big) > \frac{1}{2}|z_2 - z_1|,$$

which is also a contradiction. So $\kappa_1 = \kappa_2$. This proves the injectivity of $S^{n+1}$ on $\Lambda$.

Now we estimate the derivative of $S^{n+1}$. For $\kappa \in \Lambda$, set $z = S^n(\kappa) \in V$.

$$\begin{aligned}|(S^{n+1})'(\kappa)| &= |\mathrm{e}^\kappa(\mathrm{e}^z + \mathrm{e}^{-z}) + \mathrm{e}^\kappa(\mathrm{e}^z(S^n)'(\kappa) - \mathrm{e}^{-z}(S^n)'(\kappa))| \\ &\ge |\mathrm{e}^\kappa||(S^n)'(\kappa)|(\mathrm{e}^{\mathrm{Re}\, z} - \mathrm{e}^{-\mathrm{Re}\, z}) - |\mathrm{e}^\kappa|(\mathrm{e}^{\mathrm{Re}\, z} + \mathrm{e}^{-\mathrm{Re}\, z}) \\ &\ge \frac{1}{2}|\mathrm{e}^\kappa||(S^n)'(\kappa)|\mathrm{e}^{\mathrm{Re}\, z} - 2|\mathrm{e}^\kappa|\mathrm{e}^{\mathrm{Re}\, z} = \frac{|\mathrm{e}^\kappa|\mathrm{e}^{\mathrm{Re}\, z}}{2}(|(S^n)'(\kappa)| - 4) \\ &\ge \frac{|\mathrm{e}^\kappa|\mathrm{e}^{\mathrm{Re}\, z}}{2}\frac{|(S^n)'(\kappa)|}{2} \ge \frac{\mathrm{e}^{-M}\mathrm{e}^\xi}{4}|(S^n)'(\kappa)| > 2|(S^n)'(\kappa)|.\end{aligned}$$

The conformality of $S^{n,n+1}$ comes immediately from $S^{n,n+1} = S^{n+1} \circ (S^n)^{-1}$ and the conformality of $S^n$ and $S^{n+1}$. Since

$$|(S^{n+1})'(\kappa)| = |(S^{n,n+1} \circ (S^n))'(\kappa)| = |(S^{n,n+1})'(z)| \cdot |(S^n)'(\kappa)| \ge \frac{\mathrm{e}^{-M}\mathrm{e}^\xi}{4}|(S^n)'(\kappa)|,$$

we get $|(S^{n,n+1})'(z)| \geq \frac{\mathrm{e}^{-M}\mathrm{e}^{\xi}}{4}$.

For $p > 1$ and $\xi \geq 0$, define the doubly truncated parabola open set

$$P_{p,\xi} := \{z = x + \mathrm{i}y \in \mathbb{C} : |x| > \xi \text{ and } |y| < |x|^{\frac{1}{p}}\}.$$

For a bounded domain $\Lambda \subset \mathbb{C}$, define the sets

$$\mathcal{I}_{p,\Lambda} := \big\{\kappa \in \Lambda \cap \mathcal{I} : |(S^n)'(\kappa)| \to \infty \text{ as } n \to \infty \text{ and } S^n(\kappa) \in P_{p,0} \text{ for sufficiently large } n\big\}$$

and

$$\mathcal{I}^N_{p,\xi,\Lambda} := \big\{\kappa \in \mathcal{I}_{p,\Lambda} : S^n(\kappa) \in P_{p,\xi} \text{ for all } n \geq N\big\}.$$

**Lemma 2.2** *Fix $p > 1, M > 1, \xi > 0$ such that $\mathrm{e}^{\xi} > 128\mathrm{e}^M$. Let $\Lambda \subset D_M(0)$ be a domain. Then for any $\kappa \in \mathcal{I}_{p,\Lambda}$, there exist an integer $N \in \mathbb{N}$, a neighbourhood $U \subset \Lambda$ of $\kappa$ and a standard square $Q \subset \{\kappa \in \mathbb{C} : |\operatorname{Re}\kappa| > \xi + \pi\}$ with center at $S^N(\kappa)$ and a double square $\widetilde{Q}$ with respect to $Q$ such that*

(1) *$S^N : U \to \widetilde{Q}$ is a conformal isomorphism*;

(2) *$S^n(\kappa) \in P_{p,\xi}$ for all integers $n \geq N$*;

(3) *$|(S^N)'(\kappa)| > 20$ for all $\kappa \in U$.*

**Proof** Let $\kappa \in \mathcal{I}_{p,\Lambda}$. Note that $S^n(\kappa) \in P_{p,0}$ and $S^n(\kappa) \to \infty$ imply $|\operatorname{Re} S^n(\kappa)| \to \infty$ as $n \to \infty$. There exists $N_0 \in \mathbb{N}$ such that $S^n(\kappa) \in P_{p,\xi}$, $|\operatorname{Re} S^n(\kappa)| > \xi + 2\pi$, and $|(S^n)'(\kappa)| > 20$ for all $n \geq N_0$. Then there exists a neighbourhood $U_0 \subset \Lambda$ of $\kappa$ such that $S^{N_0} : U_0 \to V_0 := S^{N_0}(U_0)$ is a conformal isomorphism, and $V_0$ is a disk of radius $r_0 > 0$ with center at $z_0 := S^{N_0}(\kappa)$.

If $r_0 \geq \frac{\sqrt{2}\pi}{2}$, there is a standard square $Q$ with center at $z_0 = S^{N_0}(\kappa)$ such that its double square $\widetilde{Q} \subset V_0$. Then (1) follows immediately by taking $U$ be the preimage of $\widetilde{Q}$ under the map $S^{N_0}$ and setting $N = N_0$, and (2), (3) hold obviously by the choice of $N_0$.

If $r_0 < \frac{\sqrt{2}\pi}{2}$, we restrict $r_0$ if necessary so that $V_0$ is contained in a double square $\widetilde{Q}$ in $\{\kappa \in \mathbb{C} : |\operatorname{Re}\kappa| > \xi\}$. By Lemma 2.1, the maps $S^{N_0+1} : U_0 \to S^{N_0+1}(U_0)$ and $S^{N_0,N_0+1} : S^{N_0}(U_0) \to S^{N_0+1}(U_0)$ are conformal isomorphisms, and we have

$$|(S^{N_0,N_0+1})'(S^{N_0}(\kappa))| \geq \frac{\mathrm{e}^{-M}\mathrm{e}^{\xi}}{4} > 32.$$

By the Koebe $\frac{1}{4}$-theorem, there is a neighbourhood $U_1 \subset U_0$ of $\kappa_0$ so that $S^{N_0+1}(U_1)$ is a disk of radius $r_1 \geq \frac{32r_0}{4} = 8r_0$. Repeating this argument finitely many times, we obtain an index $N \in \mathbb{N}$ and a neighborhood $U$ of $\kappa$ such that $S^N : U \to \widetilde{Q}$ is a conformal isomorphism, where $\widetilde{Q}$ is a double square. So (1) follows. Again (2), (3) hold obviously by the choice of $N_0$ and Lemma 2.1 (1).

Now, we turn to estimate the Hausdorff dimension of $\mathcal{I}^N_{p,\xi,\Lambda}$.

**Lemma 2.3** *Fix $p > 1$ and an integer $N \geq 0$. Suppose $Q \subset \mathbb{C}$ is a standard square with double square $\widetilde{Q}$ and $\widetilde{\Lambda} \subset \mathbb{C}$ is a domain such that $S^N : \widetilde{\Lambda} \to \widetilde{Q}$ is a conformal isomorphism. Set $\Lambda := (S^N)^{-1}(Q) \cap \widetilde{\Lambda}$, and let $M > 1$ such that $\widetilde{\Lambda} \subset D_M(0)$. Suppose also that $|(S^N)'(\kappa)| > 20$ on $\widetilde{\Lambda}$. Then the Hausdorff dimension $\dim_H(\mathcal{I}^N_{p,\xi,\Lambda}) \leq 1 + \frac{1}{p}$ provided $\xi$ is sufficiently large depending only on $p$ and $M$.*

**Proof** Let $\xi_0 > 0$ such that $|\operatorname{Re} z| \in \left(\xi_0, \xi_0 + \frac{\pi}{2}\right)$ for all $z \in Q$. We can assume $\mathcal{I}^N_{p,\xi,\Lambda} \neq \varnothing$, otherwise $\dim_H(\mathcal{I}^N_{p,\xi,\Lambda}) \le 1 + \frac{1}{p}$ is obvious.

For any $\kappa \in \mathcal{I}^N_{p,\xi,\Lambda}$ such that $S^N(\kappa) \in Q$, we have $\xi < |\operatorname{Re} S^N(\kappa)| < \xi_0 + \frac{\pi}{2}$, then $\xi_0 \ge \xi - \frac{\pi}{2}$. By Lemma 2.1, when $\mathrm{e}^{\xi - \frac{\pi}{2}} > 8\mathrm{e}^M$, $S^{N,N+1} : Q \to S^{N,N+1}(Q)$ is a conformal isomorphism. We can write $S^{N,N+1}(z) = \mathrm{e}^{\kappa}(\mathrm{e}^z + \mathrm{e}^{-z})$ for $z \in Q$ with $\kappa = (S^N)^{-1}(z) \in \Lambda$. Note that the set $\exp(Q)$ is contained in an annulus between radii $\mathrm{e}^{\xi_0}$ and $\mathrm{e}^{\frac{\pi}{2}}\mathrm{e}^{\xi_0}$, and $S^{N,N+1}(z) = \mathrm{e}^{\kappa}(\mathrm{e}^z + \mathrm{e}^{-z}) \approx \mathrm{e}^{\kappa}\mathrm{e}^z$ or $\mathrm{e}^{\kappa}\mathrm{e}^{-z}$ when $\xi$ is sufficiently large. We have that $S^{N,N+1}(Q) \cap P_{p,\xi}$ (is not empty, for $\kappa \in \mathcal{I}^N_{p,\xi,\Lambda}, S^n(\kappa) \in P_{p,\xi}$ as $n \ge N$) has the absolute values of real parts between $\frac{\mathrm{e}^{-M}\mathrm{e}^{\xi_0}}{2}$ and $2\mathrm{e}^M\mathrm{e}^{\frac{\pi}{2}}\mathrm{e}^{\xi_0}$ (provided $\xi$ is sufficiently large). Consequently the imaginary parts in $S^{N,N+1}(Q) \cap P_{p,\xi}$ have absolute values at most $(2\mathrm{e}^M\mathrm{e}^{\frac{\pi}{2}}\mathrm{e}^{\xi_0})^{\frac{1}{p}}$ (again for sufficiently large $\xi$). Let $N(\xi_0)$ be the smallest number of standard squares such that $S^{N,N+1}(Q) \cap P_{p,\xi}$ is covered by these squares. Then

$$N(\xi_0) \le \frac{2 \cdot \left(2\mathrm{e}^M\mathrm{e}^{\frac{\pi}{2}}\mathrm{e}^{\xi_0} - \frac{\mathrm{e}^{-M}\mathrm{e}^{\xi_0}}{2}\right) \cdot 2(2\mathrm{e}^M\mathrm{e}^{\frac{\pi}{2}}\mathrm{e}^{\xi_0})^{\frac{1}{p}}}{\left(\frac{\pi}{2}\right)^2} \le C\mathrm{e}^{\xi_0(1+\frac{1}{p})},$$

where $C > 0$ is a constant depending only on $p$ and $M$. Denote these $N(\xi_0)$ standard squares by $Q_{1,i}$ for $i = 1, 2, \cdots, N(\xi_0)$.

Let $\widetilde{Q}_{1,i}$ be the double squares with respect to $Q_{1,i}$. Note that $|(S^{N,N+1})'(z)|$ is large on $\widetilde{Q}$ by Lemma 2.1 (2), then we have $S^{N,N+1}(\widetilde{Q}) \supset \bigcup_{i=1}^{N(\xi_0)} \widetilde{Q}_{1,i}$. We can thus pull back the squares $Q_{1,i}$, $i = 1, 2, \cdots, N(\xi_0)$, under $S^{N,N+1}$ and obtain a covering $\{U_{1,i}\}$ of the set $\widehat{Q}_1 := Q \cap (S^{N,N+1})^{-1}(P_{p,\xi})$ with $N(\xi_0)$ open sets $U_{1,i} =: W_{1,i}$, $i = 1, 2, \cdots, N(\xi_0)$, such that each $U_{1,i}$ has a neighbourhood $\widetilde{U}_{1,i}$ for which the restriction $S^{N,N+1} : \widetilde{U}_{1,i} \to \widetilde{Q}_{1,i}$ is a conformal isomorphism. By the Koebe distortion theorem, the restrictions $S^{N,N+1} : U_{1,i} \to Q_{1,i}$ have uniformly bounded distortions. By Lemma 2.1 (2), the derivatives of $S^{N,N+1}$ on $U_{1,i}$ are at least $\frac{\mathrm{e}^{-M}\mathrm{e}^{\xi_0}}{4K}$ where $K > 0$ is a universal constant which measures the distortion. We have for any $d$,

$$\sum_{i=1}^{N(\xi_0)} (\operatorname{diam} U_{1,i})^d \le N(\xi_0)\Big(\frac{4K}{\mathrm{e}^{-M}\mathrm{e}^{\xi_0}}\Big)^d \Big(\frac{\sqrt{2}\pi}{2}\Big)^d \le C_1\mathrm{e}^{\xi_0(1+\frac{1}{p}-d)}\Big(\frac{\sqrt{2}\pi}{2}\Big)^d,$$

where $C_1 > 0$ is a constant depending only on $p$ and $M$. If $d > 1 + \frac{1}{p}$ is fixed, then when $\xi$ is sufficiently large, we have $C_1\mathrm{e}^{\xi_0(1+\frac{1}{p}-d)} \le \frac{1}{K^{2d}}$, so that

$$\sum_{i=1}^{N(\xi_0)} (\operatorname{diam} U_{1,i})^d \le \frac{1}{K^{2d}}(\operatorname{diam} Q)^d. \tag{2.1}$$

This argument can be repeated: each standard square $Q_{1,i}$ has the absolute values of real parts at least $\xi_{1,i}$, where $2\mathrm{e}^M\mathrm{e}^{\frac{\pi}{2}}\mathrm{e}^{\xi_0} \ge \xi_{1,i} := O(\mathrm{e}^{\xi_0}) \ge \frac{\mathrm{e}^{-M}\mathrm{e}^{\xi_0}}{2} \gg \xi_0 \ge \xi - \frac{\pi}{2}$, so we can obtain a covering $\{W_{2,j_i}\}$ of the set $Q_{1,i} \cap (S^{N+1,N+2})^{-1}(P_{p,\xi})$ with $N(\xi_{1,i})$ open sets $W_{2,j_i}$, $j_i = 1, 2, \cdots, N(\xi_{1,i})$, such that for each $W_{2,j_i}$, $S^{N+1,N+2} : W_{2,j_i} \to Q_{2,j_i}$ is a conformal isomorphism, where $Q_{2,j_i}$ is a standard square which has the absolute values of real parts at least $\xi_{2,j_i} := O(\mathrm{e}^{\xi_{1,i}}) \gg \xi_{1,i}$, and so on. The union

$$\bigcup_{i=1}^{N(\xi_0)} Q_{1,i} \cap (S^{N+1,N+2})^{-1}(P_{p,\xi})$$

can be covered by $\bigcup_{i=1}^{N(\xi_0)}\{W_{2,j_i}\}$ which contains $N(\xi_1) := \sum_{i=1}^{N(\xi_0)} N(\xi_{1,i})$ open sets. We rewrite these open sets by $W_{2,i}$, $i = 1, 2, \cdots, N(\xi_1)$, where $\xi_1 := O(\mathrm{e}^{\xi_0})$. The covering $\{W_{2,i}\}$ can be pulled back under $S^{N,N+1}$ and yield a covering $\{U_{2,i}\}$ of the set

$$\widehat{Q}_2 := \{z \in Q : S^{N,N+1}(z) \in P_{p,\xi} \text{ and } S^{N,N+2}(z) \in P_{p,\xi}\}.$$

The conformal isomorphism $S^{N,N+2} : U_{2,i} \to Q_{2,i}$ for each $i$ can be extended to be a conformal isomorphism $\widetilde{U}_{2,i} \to \widetilde{Q}_{2,i}$, where $\widetilde{Q}_{2,i}$ is the double square with respect to $Q_{2,i}$.

Inductively, we obtain a family of coverings $\{U_{n,i}\}$ for every $n \geq 1$, each of them covers the set

$$\widehat{Q}_n := \{z \in Q : S^{N,N+k}(z) \in P_{p,\xi} \text{ for all } 1 \leq k \leq n\}.$$

Each $U_{n,i} \subset \widetilde{Q}$ is an open set such that $S^{N,N+n} : U_{n,i} \to Q_{n,i}$ is a conformal isomorphism, where $Q_{n,i}$ is a standard square which has absolute values of real parts at least $\xi_n := O(\mathrm{e}^{\xi_{n-1}}) \gg \xi_{n-1} \gg \xi_0$, and $\{U_{n,i}\}$ contains at most $N(\xi_{n-1})$ elements. The map $S^{N,N+n} : U_{n,i} \to Q_{n,i}$ can be extended to be a conformal isomorphism $\widetilde{U}_{n,i} \to \widetilde{Q}_{n,i}$, where $\widetilde{Q}_{n,i}$ is the double square with respect to $Q_{n,i}$.

Set

$$\widehat{Q} = \bigcap_{n\geq 1} \widehat{Q}_n = \{z \in Q : S^{N,N+n}(z) \in P_{p,\xi} \text{ for all } n \geq 1\}.$$

Then $\{U_{n,i}\}$ is a covering of $\widehat{Q}$ for every $n \geq 1$.

Then $\dim_H(\widehat{Q}) \leq 1 + \frac{1}{p}$. Indeed, for $d > 1 + \frac{1}{p}$ and $n$ is large enough (so $\xi_n$ is sufficiently large), like the proof of (1), note that the sets $W_{n+1,j_i} := S^{N,N+n}(U_{n+1,j_i})$ cover $Q_{n,i} \bigcap (S^{N+n,N+n+1}(z))^{-1}(P_{p,\xi})$ and the maps $S^{N+n,N+n+1} : W_{n+1,j_i} \to Q_{n+1,j_i}$ are conformal isomorphisms with uniformly bounded distortions, it follows that

$$\sum_{j_i} (\operatorname{diam} W_{n+1,j_i})^d \leq \frac{1}{K^{2d}} (\operatorname{diam} Q_{n,i})^d.$$

Since $S^{N,N+n} : U_{n,i} \to Q_{n,i}$ are conformal isomorphisms with uniformly bounded distortions, then $\operatorname{diam} Q_{n,i} \leq K(S^{N,N+n})'(\kappa_0) \operatorname{diam} U_{n,i}$ and $\frac{1}{K}(S^{N,N+n})'(\kappa_0) \operatorname{diam} U_{n+1,j_i} \leq \operatorname{diam} W_{n+1,j_i}$ for some fixed $\kappa_0$ in $U_{n,i}$, and so

$$\sum_{j_i} (\operatorname{diam} U_{n+1,j_i})^d \leq (\operatorname{diam} U_{n,i})^d.$$

Then

$$\sum_{i,j_i} (\operatorname{diam} U_{n+1,j_i})^d \leq \sum_i (\operatorname{diam} U_{n,i})^d.$$

Rewrite $\sum_{i,j_i}$ to $\sum_{i'}$.

$$\sum_{i'} (\operatorname{diam} U_{n+1,i'})^d \leq \sum_i (\operatorname{diam} U_{n,i})^d \leq \cdots \leq \frac{1}{K^{2d}} (\operatorname{diam} Q)^d < \infty. \tag{2.2}$$

Since $S^{N,N+n} : U_{n,i} \to Q_{n,i}$ are conformal isomorphisms with bounded distortions and derivatives tending to $\infty$ as $n \to \infty$ (by $(S^{N,N+n})' = (S^{N,N+1})' \cdots (S^{N+n-1,N+n})'$ and Lemma

2.1 (2)), it follows that $\sup_i \operatorname{diam} U_{n,i} \to 0$ as $n \to \infty$. Hence, (2.2) shows that $\dim_H(\widehat{Q}) \le d$. Since $d > 1 + \frac{1}{p}$ is arbitrary, we get $\dim_H(\widehat{Q}) \le 1 + \frac{1}{p}$. Finally, $S^N : \Lambda \to Q$ is a conformal isomorphism with $S^N(\mathcal{I}^N_{p,\xi,\Lambda}) \subset \widehat{Q}$. Therefore, $\dim_H(\mathcal{I}^N_{p,\xi,\Lambda}) \le 1 + \frac{1}{p}$ as well.

## 3 Proof of Theorem

To prove the main theorem, we need some results given in [9, 15]. So we turn to review them roughly.

Define the index sets

$$\mathbb{Z}_L := \{\cdots, -2_L, -1_L, 0_L, 1_L, 2_L, \cdots\},$$
$$\mathbb{Z}_R := \{\cdots, -2_R, -1_R, 0_R, 1_R, 2_R, \cdots\}.$$

Let $v_1 = 2\mathrm{e}^\kappa, v_2 = -2\mathrm{e}^\kappa$ be critical values of $S_\kappa(z) = \mathrm{e}^\kappa(\mathrm{e}^z + \mathrm{e}^{-z})$. Without lose of generality, we can assume that $\operatorname{Im}(v_1) \ge \operatorname{Im}(v_2)$. Let

$$\mathcal{A} := \{z \in \mathbb{C} : z = \lambda v_1 + (1-\lambda)v_2; \lambda \in [0,1]\} \cup \{z \in \mathbb{C} : \operatorname{Re}(z) = \operatorname{Re}(v_1), \operatorname{Im}(z) \ge \operatorname{Im}(v_1)\},$$

and set $\mathbb{C}' := \mathbb{C} \setminus \mathcal{A}$. Then define the strips $R_j$ as connected components of $S_\kappa^{-1}(\mathbb{C}')$, so that $S_\kappa : R_j \to \mathbb{C}'$ is a conformal isomorphism for all $j \in \mathbb{Z}_L \bigcup \mathbb{Z}_R$. The index of $R_j$ is taken as follows: if $j = j_R \in \mathbb{Z}_R$, $R_j$ is located in the right half plane; if $j = j_L \in \mathbb{Z}_L$, $R_j$ is located in the left half plane. $\{R_j\}$ gives an appropriate partition of the complex plane. Set

$$\mathbb{S} := (\mathbb{Z}_L \cup \mathbb{Z}_R)^{\mathbb{N}} = \{(s_1 s_2 s_3 \cdots) : s_k \in \mathbb{Z}_L \cup \mathbb{Z}_R\}.$$

Let $\sigma : \mathbb{S} \to \mathbb{S}$, $(s_1 s_2 s_3 s_4 \cdots) \mapsto (s_2 s_3 s_4 \cdots)$ be the shift map on $\mathbb{S}$. Denote $F(t) := \mathrm{e}^t - 1$. A sequence $\underline{s} = (s_1 s_2 s_3 \cdots) \in \mathbb{S}$ is called exponentially bounded if there is an $x \in \mathbb{R}^+$ such that $|s_k| \le F^{\circ(k-1)}(x)$ for all $k$. For every exponentially bounded sequence $\underline{s} = (s_1 s_2 s_3 \cdots) \in \mathbb{S}$, define

$$t_{\underline{s}} := \inf\Big\{t > 0 : \lim_{k\to\infty} \frac{|s_k|}{F^{\circ k}(t)} = 0\Big\}.$$

It is proved in [9] that every path component of the dynamical escaping set $I(S_\kappa)$ is a curve starting at infinity, which is called dynamic ray, possibly together with the escaping endpoint of the ray. Every dynamical ray is associated with an exponentially bounded sequence $\underline{s} = (s_1 s_2 s_3 \cdots) \in \mathbb{S}$. Denote the inverse mapping of $S_\kappa : R_j \to \mathbb{C}'$ by $L_j : \mathbb{C}' \to R_j$. According to [9], the dynamic ray associated with the exponentially bounded sequence $\underline{s} = (s_1 s_2 s_3 \cdots)$ is obtained by constructing a family of maps

$$g^n_{\kappa,\underline{s}} : \mathbb{R}^+ \to \mathbb{C} \quad \text{for } n \in \mathbb{N},$$
$$g^n_{\kappa,\underline{s}}(t) = L_{s_1} \circ L_{s_2} \circ \cdots \circ L_{s_n}(\pm F^{\circ n}(t) + 2\pi \mathrm{i} s_{n+1})$$

(the sign $\pm$ depends on $s_{n+1}$: it is $+$ if $s_{n+1} \in \mathbb{Z}_R$ and $-$ if $s_{n+1} \in \mathbb{Z}_L$) with the following properties:

(a) There exists a positive number $T$, such that $g^n_{\kappa,\underline{s}}(t)$ is well defined for all $t \ge T$ independent of $\underline{s}$ and $n$, and converges uniformly to a function $g_{\kappa,\underline{s}}(t)$ which is the tail of dynamic ray. The convergence is locally uniform for $\kappa$. Moreover, $S_\kappa(g_{\kappa,\underline{s}}(t)) = g_{\kappa,\sigma(\underline{s})}(F(t))$.

(b) There exists a positive number $B$, such that for every $\underline{s}$ and $n$, $|\mathrm{Re}(g^n_{\kappa,\underline{s}}(t))| > t - B$ when $t > T$.

If no critical orbit escapes, by using the relation $S_\kappa(g_{\kappa,\underline{s}}(t)) = g_{\kappa,\sigma(\underline{s})}(F(t))$, the ray tail can be extended to a dynamic ray $g_{\kappa,\underline{s}} : (t_{\underline{s}}, \infty) \mapsto I(S_\kappa)$ with the following properties:

(c) $S_\kappa(g_{\kappa,\underline{s}}(t)) = g_{\kappa,\sigma(\underline{s})}(F(t))$ for all $t > t_{\underline{s}}$.

(d) $S^{\circ n}_\kappa(g_{\kappa,\underline{s}}(t)) = \pm F^{\circ n}(t) \mp \kappa + 2\pi \mathrm{i} s_{n+1} + o(1)$ as $n \to \infty$, with sign $+$ if $s_{n+1} \in \mathbb{Z}_R$ and $-$ if $s_{n+1} \in \mathbb{Z}_L$, respectively. In particular, for every real $p > 1$,

$$\frac{|\mathrm{Im}(S^{\circ n}_\kappa(g_{\kappa,\underline{s}}(t)))|^p}{|\mathrm{Re}(S^{\circ n}_\kappa(g_{\kappa,\underline{s}}(t)))|} \to 0 \quad \text{as } n \to \infty.$$

In [15], Tian proved the differentiability of dynamic rays:

(e) For every $\kappa$, the dynamic ray $g_{\kappa,\underline{s}}(t)$ is continuously differentiable with respect to $t$ and satisfies

$$g'_{\kappa,\underline{s}}(t) = \varepsilon_1 \prod_{m=1}^{\infty} \frac{F^{\circ m}(t) + 1}{\varepsilon_{m+1} g_{\kappa,\sigma^m(\underline{s})}(F^{\circ m}(t))} \neq 0, \tag{3.1}$$

where $\varepsilon_m = 1$ or $\varepsilon_m = -1$ according to $s_m \in \mathbb{Z}_R$ or $s_m \in \mathbb{Z}_L$, respectively. Moreover, on ray tails, $|g'_{\kappa,\underline{s}}(t) \pm 1| < \mathrm{e}^{-\frac{1}{2}t}$ with sign $+$ if $s_1 \in \mathbb{Z}_R$ and $-$ if $s_1 \in \mathbb{Z}_L$, respectively.

In addition, it was shown in [15] that every path component of the parameter escaping set $\mathcal{I} = \mathcal{I}(S_\kappa)$ is a curve, which is called parameter ray, possibly together with its unique endpoint. The parameter ray associated with the exponentially bounded sequence $\underline{s} = (s_1 s_2 s_3 \cdots) \in \mathbb{S}$ is defined by $\kappa = G_{\underline{s}}(t)$ which is the unique solution of $g_{\kappa,\underline{s}}(t) = 2\mathrm{e}^\kappa$.

(f) Every parameter ray is a $C^1$-curve $G_{\underline{s}} : (t_{\underline{s}}, \infty) \to \mathcal{I}$ with $G'_{\underline{s}}(t) \neq 0$ for all $t > t_{\underline{s}}$. All the parameter rays are injective and disjoint curves. For $\kappa_0 = G_{\underline{s}}(t)$ there is a neighbourhood $\Lambda$ of $\kappa_0$ in parameter space such that $g_{\kappa,\underline{s}}(t)$ is defined for all $\kappa \in \Lambda$.

**Proposition 3.1** *The parameter ray $\kappa = G_{\underline{s}}(t)$ for $\underline{s} = (s_1 s_2 s_3 \cdots) \in \mathbb{S}$ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} S^n(G_{\underline{s}}(t)) \to \infty \quad \textit{and} \quad \frac{\mathrm{d}}{\mathrm{d}\kappa} S^n(\kappa) \to \infty$$

*as $n \to \infty$.*

**Proof** For the parameter ray $\kappa = G_{\underline{s}}(t)$, i.e., $g_{\kappa,\underline{s}}(t) = 2\mathrm{e}^\kappa$,

$$S^n(G_{\underline{s}}(t)) = S^n(\kappa) = S^{\circ n}_\kappa(2\mathrm{e}^\kappa) = S^{\circ n}_\kappa(g_{\kappa,\underline{s}}(t)).$$

By (c), the dynamic ray satisfies $S^{\circ n}_\kappa(g_{\kappa,\underline{s}}(t)) = g_{\kappa,\sigma^n(\underline{s})}(F^{\circ n}(t))$, where $\sigma^n$ denotes the $n$-th iterate of the shift map $\sigma$ on $\mathbb{S}$. Thus, with $\kappa = G_{\underline{s}}(t)$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} S^n(G_{\underline{s}}(t)) &= \frac{\mathrm{d}}{\mathrm{d}t} S^{\circ n}_\kappa(g_{\kappa,\underline{s}}(t)) = \frac{\mathrm{d}}{\mathrm{d}t} g_{\kappa,\sigma^n(\underline{s})}(F^{\circ n}(t)) \\ &= g'_{\kappa,\sigma^n(\underline{s})}(F^{\circ n}(t)) \cdot \frac{\mathrm{d}F^{\circ n}(t)}{\mathrm{d}t} + \frac{\partial}{\partial \kappa} g_{\kappa,\sigma^n(\underline{s})}(F^{\circ n}(t)) \cdot G'_{\underline{s}}(t). \end{aligned}$$

It is clearly that $\frac{\mathrm{d}F^{\circ n}(t)}{\mathrm{d}t} \to \infty$ as $n \to \infty$. We need only to show the following claims (i) and (ii).

(i) $|g'_{\kappa,\sigma^n(\underline{s})}(F^{\circ n}(t))| \to 1$ as $n \to \infty$.

By (e), we have for $t > t_{\underline{s}}$,

$$|g'_{\kappa,\sigma^n(\underline{s})}(F^{\circ n}(t))| = \Big| \prod_{m=1}^{\infty} \frac{F^{\circ(m+n)}(t)+1}{g_{\kappa,\sigma^{(m+n)}(\underline{s})}(F^{\circ(m+n)}(t))} \Big| = \Big| \prod_{m=n+1}^{\infty} \frac{F^{\circ(m)}(t)+1}{g_{\kappa,\sigma^{(m)}(\underline{s})}(F^{\circ(m)}(t))} \Big|.$$

The claim $|g'_{\kappa,\sigma^n(\underline{s})}(F^{\circ n}(t))| \to 1$ follows directly from convergence of (3.1) (see (e)).

(ii) $\big|\frac{\partial}{\partial\kappa} g_{\kappa,\sigma^n(\underline{s})}(F^{\circ n}(t))\big|$ is bounded.

From the construction of dynamic rays, we have $g^0_{\kappa,\underline{s}}(t) := t$ and

$$g^{m+1}_{\kappa,\underline{s}}(t) := L_{s_1}(g^m_{\kappa,\sigma(\underline{s})}(F(t))) \tag{3.2}$$

for every $m \geq 0$, where $L_s : \mathbb{C}' \to R_s$ is an inverse branch of $S_\kappa(z) = \mathrm{e}^\kappa(\mathrm{e}^z + \mathrm{e}^{-z})$ for $s \in \mathbb{Z}_L \cup \mathbb{Z}_R$. The function $L_s(z)$ can be explicitly expressed as

$$L_s(z) = \log(z\mathrm{e}^{-\kappa} \pm (z^2\mathrm{e}^{-2\kappa} - 4)^{\frac{1}{2}}) - \log 2 + 2\pi\mathrm{i}s,$$

where the sign $+$ or $-$ is taken according to $s \in \mathbb{Z}_R$ or $s \in \mathbb{Z}_L$, respectively. The branch of log is taken the principal value and the branch of $(z^2\mathrm{e}^{-2\kappa} - 4)^{\frac{1}{2}}$ is taken such that $(z^2)^{\frac{1}{2}} = z$. Then the argument of $(z^2\mathrm{e}^{-2\kappa} - 4)^{\frac{1}{2}}$ and $z\mathrm{e}^{-\kappa}$ are close to each other as $|z| \to \infty$. We can reform $L_s(z) = \log(z\mathrm{e}^{-\kappa} - (z^2\mathrm{e}^{-2\kappa} - 4)^{\frac{1}{2}}) - \log 2 + 2\pi\mathrm{i}s$ to $L_s(z) = -\log(z\mathrm{e}^{-\kappa} + (z^2\mathrm{e}^{-2\kappa} - 4)^{\frac{1}{2}}) + \log 2 + 2\pi\mathrm{i}s$. Then we have

$$L_s(z) = \pm\log(z\mathrm{e}^{-\kappa} + (z^2\mathrm{e}^{-2\kappa} - 4)^{\frac{1}{2}}) \mp \log 2 + 2\pi\mathrm{i}s.$$

It is known from (a) that $g^m_{\kappa,\underline{s}}(t)$ converges uniformly to a limiting curve $g_{\kappa,\underline{s}}(t)$ for $t > T$ (the tail of dynamic ray), and the convergence is locally uniform for $\kappa$. So $g_{\kappa,\underline{s}}(t)$ depends holomorphically on $\kappa$ for fixed $t > T$, and

$$\frac{\partial}{\partial\kappa} g_{\kappa,\underline{s}}(t) = \lim_{m\to\infty} \frac{\partial}{\partial\kappa} g^m_{\kappa,\underline{s}}(t).$$

This construction is extended to entire dynamic rays: for all $t > t_{\underline{s}}$, a point $z$ is on a dynamic ray if $S^{\circ n}_\kappa(z)$ is on a ray tail for sufficiently large $n \geq 1$. Since we are interested in the limit $\frac{\partial}{\partial\kappa} g_{\kappa,\sigma^n(\underline{s})}(F^{\circ n}(t))$ as $n \to \infty$, we may restrict to sufficiently large $n$ such that $F^{\circ n}(t)$ are always on ray tails. We thereby need to prove

$$\overline{\lim_{n\to\infty}} \Big|\frac{\partial}{\partial\kappa} g_{\kappa,\sigma^n(\underline{s})}(F^{\circ n}(t))\Big| = \overline{\lim_{n\to\infty}} \Big|\lim_{m\to\infty} \frac{\partial}{\partial\kappa} g^m_{\kappa,\sigma^n(\underline{s})}(F^{\circ n}(t))\Big| \tag{3.3}$$

is bounded.

Denote

$$A^m_n := g^m_{\kappa,\sigma^n(\underline{s})}(F^{\circ n}(t)).$$

From (3.2), we have

$$\begin{aligned} A^{m+1}_n &= L_{s_{n+1}}(A^m_{n+1}) \\ &= \pm\log(\mathrm{e}^{-\kappa}A^m_{n+1} + ((\mathrm{e}^{-\kappa}A^m_{n+1})^2 - 4)^{\frac{1}{2}}) \mp \log 2 + 2\pi\mathrm{i}s_{n+1}. \end{aligned}$$

We obtain the recursive relation

$$\frac{\partial A_n^{m+1}}{\partial\kappa} = \pm\frac{\left(\mathrm{e}^{-\kappa}\frac{\partial A_{n+1}^m}{\partial\kappa} - \mathrm{e}^{-\kappa}A_{n+1}^m\right) + \frac{\mathrm{e}^{-2\kappa}A_{n+1}^m\frac{\partial A_{n+1}^m}{\partial\kappa} - (\mathrm{e}^{-\kappa}A_{n+1}^m)^2}{((\mathrm{e}^{-\kappa}A_{n+1}^m)^2-4)^{\frac12}}}{\mathrm{e}^{-\kappa}A_{n+1}^m + ((\mathrm{e}^{-\kappa}A_{n+1}^m)^2-4)^{\frac12}}$$

$$= \pm\frac{\mathrm{e}^{-\kappa}\frac{\partial A_{n+1}^m}{\partial\kappa}\left(1+\frac{\mathrm{e}^{-\kappa}A_{n+1}^m}{((\mathrm{e}^{-\kappa}A_{n+1}^m)^2-4)^{\frac12}}\right) - \mathrm{e}^{-\kappa}A_{n+1}^m\left(1+\frac{\mathrm{e}^{-\kappa}A_{n+1}^m}{((\mathrm{e}^{-\kappa}A_{n+1}^m)^2-4)^{\frac12}}\right)}{\mathrm{e}^{-\kappa}A_{n+1}^m + ((\mathrm{e}^{-\kappa}A_{n+1}^m)^2-4)^{\frac12}}$$

starting with

$$\frac{\partial A_{n+m+1}^0}{\partial\kappa} = \frac{\partial g^0_{\kappa,\sigma^{n+m+1}(\underline{s})}}{\partial\kappa}(F^{\circ(n+m+1)}(t)) = 0 < 4. \tag{3.4}$$

By (b), there is a constant $B>0$ so that $\left|\mathrm{Re}\, g^{m'}_{\kappa,\underline{s}'}(t')\right| > t'-B$ for all $m'$, $\underline{s}'$, $\kappa$ from a bounded domain, and $t'>T$. Choosing $n$ sufficiently large, we can be sure that

$$\mathrm{Re}\, A^m_{n+1} = \mathrm{Re}\, g^m_{\kappa,\sigma^{n+1}(\underline{s})}(F^{\circ(n+1)}(t)) > C$$

for a given $C>12$ and all $m$. Note that the argument of $((\mathrm{e}^{-\kappa}A^m_{n+1})^2-4)^{\frac12}$ and $\mathrm{e}^{-\kappa}A^m_{n+1}$ are close to each other, $\lim\limits_{n\to\infty}\frac{((A^m_{n+1})^2\mathrm{e}^{-2\kappa}-4)^{\frac12}}{A^m_{n+1}\mathrm{e}^{-\kappa}} = 1$, we get

$$\begin{aligned}\left|\frac{\partial A_n^{m+1}}{\partial\kappa}\right| &\le 3\frac{\left|\mathrm{e}^{-\kappa}\frac{\partial A^m_{n+1}}{\partial\kappa}\right| + |\mathrm{e}^{-\kappa}A^m_{n+1}|}{|\mathrm{e}^{-\kappa}A^m_{n+1}|}\\ &= \frac{3\left|\frac{\partial A^m_{n+1}}{\partial\kappa}\right|}{|A^m_{n+1}|}+3\\ &\le \frac14\left|\frac{\partial A^m_{n+1}}{\partial\kappa}\right| + 3.\end{aligned}$$

By induction, from (3.4) we have

$$\left|\frac{\partial g^{m+1}_{\kappa,\sigma^n(\underline{s})}}{\partial\kappa}(F^{\circ(n)}(t))\right| = \left|\frac{\partial A^{m+1}_n}{\partial\kappa}\right| \le \frac14\cdot 4 + 3 = 4$$

for all $m$. Thus the limit as $m\to\infty$ and then the $\overline{\lim}$ as $n\to\infty$ in (3.3) is bounded as claimed.

Since $G'_{\underline{s}}(t)$ is independent of $n$, we have proved that $\frac{\mathrm{d}}{\mathrm{d}t}S^n(G_{\underline{s}}(t))\to\infty$ as $n\to\infty$, which is the first part of the lemma.

Since $\kappa = G_{\underline{s}}(t)$, the second part of the lemma comes immediately from

$$\frac{\mathrm{d}}{\mathrm{d}\kappa}S^n(\kappa) = \frac{\mathrm{d}}{\mathrm{d}t}S^n(G_{\underline{s}}(t))\cdot\frac{1}{G'_{\underline{s}}(t)}$$

and the fact $G'_{\underline{s}}(t)\ne 0$ (see (f)).

Now we turn to prove Theorem 1.1.

**Proof of Theorem 1.1** It can be asserted that $\dim_H(\mathcal{I}_{p,\Lambda}) \le 1+\frac1p$. Choose $\xi>0$ depending on $\Lambda$ and $p$ as in Lemma 2.2. Pick any $\kappa\in\mathcal{I}_{p,\Lambda}$. By Lemma 2.2, there exist an integer $N\in\mathbb{N}$, a neighbourhood $U\subset\Lambda$ of $\kappa$ and a standard square $Q\subset\{\kappa\in\mathbb{C} : |\mathrm{Re}\,\kappa| > \xi+\pi\}$ with center at $S^N(\kappa)$ and a double square $\widetilde{Q}$ with respect to $Q$ such that $S^N : U\to\widetilde{Q}$ is a conformal isomorphism with $|(S^N)'(\kappa)| > 20$ and $S^n(\kappa) \in P_{p,\xi}$ for all $n \geq N$. This implies that $\kappa \in \mathcal{I}^N_{p,\xi,U}$ and by Lemma 2.3, $\dim_H(\mathcal{I}^N_{p,\xi,U}) \leq 1 + \frac{1}{p}$.

Since $\Lambda$ has countable topology and $N \in \mathbb{N}$ given in Lemma 2.2 forms a countable set, $\mathcal{I}_{p,\Lambda}$ is contained in the countable union of sets of dimension at most $1 + \frac{1}{p}$. So, for every $p > 1$ and every bounded open $\Lambda \subset \mathbb{C}$, we have $\dim_H(\mathcal{I}_{p,\Lambda}) \leq 1 + \frac{1}{p}$.

We now need to prove that for every open and bounded $\Lambda \subset \mathbb{C}$, we have $\mathcal{I}_R \cap \Lambda \subset \mathcal{I}_{p,\Lambda}$ for every $p > 1$. $\forall\ \kappa \in \mathcal{I}_R \cap \Lambda$, by (d),

$$\frac{|\operatorname{Im}(S_\kappa^{\circ(n+1)}(0))|^p}{\operatorname{Re}(S_\kappa^{\circ(n+1)}(0))} = \frac{|\operatorname{Im}(S^{\circ n}(\kappa))|^p}{\operatorname{Re}(S^{\circ n}(\kappa))} \to 0 \quad \text{as} \quad n \to \infty,$$

then $S^n(\kappa) \in P_{p,0}$ for sufficiently large $n$.

By Proposition 3.1, $|(S^n(\kappa))'| \to \infty$ as $n \to \infty$. So $\kappa \in \mathcal{I}_{p,\Lambda}$ and $\mathcal{I}_R \cap \Lambda \subset \mathcal{I}_{p,\Lambda}$ for every $p > 1$.

As $\dim_H(\mathcal{I}_{p,\Lambda}) \leq 1 + \frac{1}{p}$, then $\dim_H(\mathcal{I}_R \cap \Lambda) \leq 1 + \frac{1}{p}$. Because this holds for all $p > 1$, $\dim_H(\mathcal{I}_R) \leq 1$. On the other hand, since $\mathcal{I}_R$ contains curves, it could be concluded that $\dim_H(\mathcal{I}_R) = 1$. And because $\mathcal{I} = \mathcal{I}_R \bigcup \mathcal{I}_E$ and $\mathcal{I}$ has positive planar Lebesgue measure, $\mathcal{I}_E$ has positive planar Lebesgue measure.

**Acknowledgement** We would like to thank the referees for helpful comments.

## References

[1] Milnor, J., Dynamics in One Complex Variable, 3rd ed., Princeton University Press, Princeton, 2006.

[2] Devaney, R. L. and Krych, M., Dynamics of exp($z$), *Ergodic Theory and Dynamical Systems*, **4**, 1984, 35–52.

[3] McMullen, C., Area and Hausdorff dimension of Julia sets of entire functions, *Trans. Amer. Math. Soc.*, **300**, 1987, 329–342.

[4] Karpińska, B., Hausdorff dimension of the hairs without endpoints for $\lambda$exp($z$), *C. R. Acad. Sci. Paris Sér. I: math.*, **328**, 1999, 1039–1044.

[5] Bergweiler, W., Karpińska paradox in demension 3, *Duke Math.*, **3**, 2010, 599–630.

[6] Eremenko, A. E., On the iteration of entire functions, Dynamical Systems and Ergodic Theory (Warsaw, 1986), Banach Center Publ., **23**, Warsaw, 1989, 339–345.

[7] Eremenko, A. E. and Lyubich, M. Y., Dynamical properties of some classes of entire function, *Ann. Inst. Fourier(Grenoble)*, **42**, 1992, 989–1020.

[8] Schleicher, D. and Zimmer, J., Escaping points of exponential maps, *J. Lond. Math. Soc.*, **2**, 2003, 2380–2400.

[9] Rottenfusser, G. and Schleicher, D., Escaping points of the cosine family, Transcendental Dynamics and Complex Analysis, London Math. Soc. Lecture Note Ser., **348**, Cambridge Univ. Press, Cambridge, 2008, 396–424.

[10] Schleicher, D., The dynamical fine structure of iterated cosine maps and a dimension paradox, *Duke Math. J.*, **136**(2), 2007, 343–356.

[11] Forster, M., Rempe, L. and Schleicher, D., Classification of escaping exponential maps, *Proc. Amer. Math. Soc.*, **136**, 2008, 651–663.

[12] Forster, M. and Schleicher, D., Parameter rays in the space of exponential maps, *Ergodic Theory and Dynamical Systems*, **29**, 2009, 515–544.

[13] Qiu, W. Y., Hausdorff dimension of the $M$-set of $\lambda$exp($z$), *Acta. Math. Sinica* (*N.S.*), **10**, 1994, 362–368.

[14] Bailesteanu, M., Balan, H. V. and Schleicher, D., Hausdorff dimension of exponential parameter rays and their endpoints, *Nonlinearity*, **21**, 2008, 113–120.

[15] Tian, T., Parameter rays for the consine family, *Journal of Fudan University*, **50**, 2011, 10–22.